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Large generalized books are p-good

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Abstract

Let $B_q^{(r)} = K_r + qK_1$ be the graph consisting of q distinct $(r+1)$-cliques sharing a common r-clique. We prove that if $p \geqslant 2$ and $r \geqslant 3$ are fixed, then

$$r(K_{p+1}, B_q^{(r)}) = p(q + r - 1) + 1$$

for all sufficiently large q.

Keywords: Ramsey numbers; p-Good; Generalized books; Szemerédi lemma

1. Introduction

The title of this paper refers to the notion of goodness introduced by Burr and Erdős [3] and subsequently studied by Burr and various collaborators. A connected graph H is p-good if the Ramsey number $r(K_p, H)$ is given by

$$r(K_p, H) = (p-1)(|V(H)| - 1) + 1.$$

In this paper we prove that for every $p \geqslant 3$ the generalized book $B_q^{(r)} = K_r + qK_1$ is p-good if q is sufficiently large.

As much as possible, standard notation is used; see, for example, [2]. A set of cardinality p is called a p-set. Unless explicitly stated, all graphs are defined on the vertex set $[n] = \{1, 2, \ldots, n\}$. Let u be any vertex; then $N_G(u)$ and $d_G(u) = |N_G(u)|$ denote its neighborhood and degree, respectively. A graph with n vertices and m

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edges will be designated by $G(n,m)$. By an *r-book* we shall mean some number of independent vertices that are each connected to every vertex of an r-clique. The given r-clique is called the *base* of the r-book and the additional vertices are called the *pages*. The number of pages of an r-book is called its *size*; the size of the largest r-book in a graph G is denoted by $bs^{(r)}(G)$. We shall denote the complete p-partite graph with each part having q vertices by $K_p(q)$. The *Ramsey number* $r(H_1, H_2)$ is the least number n such that for every graph G of order n either $H_1 \subset G$ or $H_2 \subset \overline{G}$.

2. The structure of subsaturated K_{p+1}-free graphs

We shall need the following theorem of Andrásfai et al. [1].

Theorem 1. *If G is a K_{p+1}-free graph of order n and*

$$\delta(G) > \left(1 - \frac{3}{3p-1}\right)n,$$

then G is p-chromatic.

The celebrated theorem of Turán gives a tight bound on the maximum size of a K_p-free graph of given order. In the following theorem, we show that if the size of a K_{p+1}-free graph is close to the maximum then we may delete a small portion of its vertices so that the remaining graph is p-chromatic. This is a particular stability theorem in extremal graph theory (see [8]).

Theorem 2. *For every $p \geqslant 2$ there exists $c = c(p) > 0$, such that for every α satisfying $0 < \alpha \leqslant c$, every K_{p+1}-free graph $G = G(n,m)$ satisfying*

$$m \geqslant \left(\frac{p-1}{2p} - \alpha\right)n^2$$

contains an induced p-chromatic graph G_0 of order at least $(1 - 2\alpha^{1/3})n$ and with minimum degree

$$\delta(G_0) \geqslant \left(1 - \frac{1}{p} - 4\alpha^{1/3}\right)n.$$

Proof. Let c_0 be the smallest positive root of the equation

$$x^3 + \left(1 + \frac{3}{3p-1}\left(\frac{p-1}{p}\right)^2\right)x - \frac{1}{2(3p-1)p} = 0 \tag{1}$$

and set $c(p) = c_0^3$; then, for every y satisfying $0 < y \leqslant c(p)$, we easily see that

$$y + \left(1 + \frac{3}{3p-1}\left(\frac{p-1}{p}\right)^2\right)y^{1/3} \leqslant \frac{1}{2(3p-1)p}. \tag{2}$$

A rough approximation of the function $c(p)$ is $c(p) \approx 6^{-3}p^{-6}$, obtained by neglecting the x^3 term in Eq. (1) and substituting the appropriate asymptotic (for large p) approximations for the remaining coefficients. This gives reasonable values even for small p. For all $p \geqslant 2$,

$$\frac{1}{(2p(3p+2))^3} < c(p) < \frac{1}{(2p(3p-1))^3}. \tag{3}$$

The upper bound is evident, and the lower bound follows from a simple computation.

Let $0 < \alpha \leqslant c(p)$ and the graph $G = G(n,m)$ satisfy the hypothesis of the theorem. We shall prove first that

$$\sum_{u=1}^{n} d^2(u) \leqslant 2\left(\frac{p-1}{p}\right)mn. \tag{4}$$

Indeed, writing $k_3(G)$ for the number of triangles in G, we have

$$3k_3(G) = \sum_{uv \in E} |N(u) \cap N(v)| \geqslant \sum_{uv \in E} (d(u) + d(v) - n) = \sum_{u=1}^{n} d^2(u) - mn.$$

Applying Turán's theorem to the K_p-free neighborhoods of vertices of G, we deduce

$$3k_3(G) \leqslant \frac{p-2}{2(p-1)} \sum_{u=1}^{n} d^2(u).$$

Hence,

$$\sum_{u=1}^{n} d^2(u) - mn \leqslant \frac{p-2}{2(p-1)} \sum_{u=1}^{n} d^2(u)$$

and (4) follows.

Since $0 < \alpha \leqslant c(p)$, taking the upper bound in (3) for $p = 2$, we see that $\alpha \leqslant 20^{-3}$. Hence,

$$\begin{aligned}(1+8\alpha)\frac{4m^2}{n} &\geqslant 2(1+8\alpha)\left(\frac{p-1}{p} - 2\alpha\right)mn \\ &= 2\left(\frac{p-1}{p} + \left(6 - \frac{8}{p}\right)\alpha - 16\alpha^2\right)mn \\ &\geqslant 2\left(\frac{p-1}{p} + 2\alpha - 16\alpha^2\right)mn > 2\left(\frac{p-1}{p}\right)mn,\end{aligned}$$

and from (4) we deduce

$$\begin{aligned}\sum_{u=1}^{n}\left(d(u) - \frac{2m}{n}\right)^2 &= \sum_{u=1}^{n} d^2(u) - \frac{4m^2}{n} \leqslant 2\left(\frac{p-1}{p}\right)mn - \frac{4m^2}{n} \\ &< 8\alpha\frac{4m^2}{n} \leqslant 8\alpha\left(\frac{p-1}{p}\right)^2 n^3.\end{aligned} \tag{5}$$

Set $V = V(G)$ and let M_ε be the set of all vertices $u \in V$ satisfying $d(u) < 2m/n - \varepsilon n$. For every $\varepsilon > 0$, inequality (5) implies

$$|M_\varepsilon|\varepsilon^2 n^2 < \sum_{u\in M_\varepsilon}\left(d(u)-\frac{2m}{n}\right)^2 \leqslant \sum_{u\in V}\left(d(u)-\frac{2m}{n}\right)^2 \leqslant 8\alpha\left(\frac{p-1}{p}\right)^2 n^3,$$

and thus,

$$|M_\varepsilon| < 8\varepsilon^{-2}\alpha\left(\frac{p-1}{p}\right)^2 n. \tag{6}$$

Furthermore, setting $G_\varepsilon = G[V\backslash M_\varepsilon]$, for every $u \in V(G_\varepsilon)$, we obtain

$$d_{G_\varepsilon}(u) \geqslant d(u) - |M_\varepsilon| \geqslant \frac{2m}{n} - \varepsilon n - |M_\varepsilon| > \frac{p-1}{p}n - 2\alpha n - \varepsilon n - |M_\varepsilon|. \tag{7}$$

For $\varepsilon = 2\alpha^{1/3}$ we claim that

$$\frac{p-1}{p}n - 2\alpha n - \varepsilon n - |M_\varepsilon| > \frac{3p-4}{3p-1}(n - |M_\varepsilon|) = \frac{3p-4}{3p-1}v(G_\varepsilon). \tag{8}$$

Indeed, assuming the opposite and applying inequality (6) with $\varepsilon = 2\alpha^{1/3}$, we see that

$$\left(\frac{1}{(3p-1)p} - 2\alpha - 2\alpha^{1/3}\right)n \leqslant \frac{3}{3p-1}|M_{2\alpha^{1/3}}| < 2\frac{3}{3p-1}\left(\frac{p-1}{p}\right)^2\alpha^{1/3}n,$$

hence,

$$2\alpha + 2\left(1 + \frac{3}{3p-1}\left(\frac{p-1}{p}\right)^2\right)\alpha^{1/3} - \frac{1}{(3p-1)p} > 0,$$

contradicting (2).

Set $G_0 = G_{2\alpha^{1/3}}$; from (8), we see that G_0 satisfies the conditions of Theorem 1, so it is p-chromatic.

Finally, from (6) and (7), we have

$$\begin{aligned}\delta(G_0) &\geqslant \frac{p-1}{p}n - 2\alpha n - 2\alpha^{1/3}n - \left(\frac{p-1}{p}\right)^2\alpha^{1/3}n > \frac{p-1}{p}n - 2\alpha n - 3\alpha^{1/3}n \\ &> \left(1 - \frac{1}{p} - 4\alpha^{1/3}\right)n,\end{aligned}$$

completing the proof. □

3. A Ramsey property of K_{p+1}-free graphs

The main result of this section is the following theorem.

Theorem 3. *Let $r \geqslant 2$, $p \geqslant 2$ be fixed. For every $\xi > 0$ there exists an $n_0 = n_0(p, r, \xi)$ such that every graph G of order $n \geqslant n_0$ that is K_{p+1}-free either satisfies $bs^{(r)}(\overline{G}) > n/p$,* *or contains an induced p-chromatic graph* G_1 *of order* $(1-\xi)n$ *and minimum degree*

$$\delta(G_1) \geqslant \left(1 - \frac{1}{p} - 2\xi\right)n.$$

Our main tool in the proof of Theorem 3 is the regularity lemma of Szemerédi (SRL); for expository matter on SRL see [2,7]. For the sake of completeness we formulate here the relevant basic notions.

Let G be a graph; if $A, B \subset V(G)$ are nonempty disjoint sets, we write $e(A, B)$ for the number of $A - B$ edges and call the value

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

the *density* of the pair (A, B).

Let $\varepsilon > 0$; a pair (A, B) of two nonempty disjoint sets $A, B \subset V(G)$ is called ε-*regular* if the inequality

$$|d(A, B) - d(X, Y)| < \varepsilon$$

holds whenever $X \subset A$, $Y \subset B$, $|X| \geqslant \varepsilon |A|$, and $|Y| \geqslant \varepsilon |B|$.

We shall use SRL in the following form.

Theorem 4 (Szemerédi's regularity lemma). *Let* $l \geqslant 1$, $\varepsilon > 0$. *There exists* $M = M(\varepsilon, l)$ *such that, for every graph G of sufficiently large order n, there exists a partition* $V(G) = \bigcup_{i=0}^{k} V_i$ *satisfying* $l \leqslant k \leqslant M$ *and*:

(i) $|V_0| < \varepsilon n$, $|V_1| = \cdots = |V_k|$;
(ii) *all but at most* εk^2 *pairs* (V_i, V_j), $(i, j \in [k])$, *are* ε-*uniform.*

We also need a few technical results; the first one is a basic property of ε-regular pairs (see [7, Fact 1.4]).

Lemma 1. *Suppose* $0 < \varepsilon < d \leqslant 1$ *and* (A, B) *is an* ε-*regular pair with* $e(A, B) = d|A||B|$. *If* $Y \subset B$ *and* $(d-\varepsilon)^{r-1}|Y| > \varepsilon |B|$ *where* $r > 1$, *then there are at most* $\varepsilon r |A|^r$ *r-sets* $R \subset A$ *with*

$$\left|\left(\bigcap_{u \in R} N(u)\right) \cap Y\right| \leqslant (d-\varepsilon)^r |Y|.$$

The next lemma gives a lower bound on the number of r-cliques in a graph consisting of several dense ε-regular pairs sharing a common part.

Lemma 2. *Suppose* $0 < \varepsilon < d \leqslant 1$ *and* $(d-\varepsilon)^{r-2} > \varepsilon$. *Suppose H is a graph and* $V(H) = A \cup B_1 \cup \cdots \cup B_t$ *is a partition with* $|A| = |B_1| = \cdots = |B_t|$ *and such that for every* $i \in [t]$ *the pair* (A, B_i) *is* ε-*regular with* $e(A, B_i) \geqslant d|A||B_i|$. *If m is the number of the r-cliques in A, then at least*

$$t|A|(m - \varepsilon r|A|^r)(d-\varepsilon)^r$$

$(r+1)$-*cliques of H have exactly r vertices in A.*

Proof. Set $a = |A| = |B_1| = \cdots = |B_t|$. For every $i \in [t]$, applying Lemma 1 to the pair (A, B_i) with $Y = B_i$ we conclude that there are at most $\varepsilon r a^{r-1}$ r-sets $R \subset A$ with

$$\left| \left(\bigcap_{u \in R} N(u) \right) \cap B_i \right| \leqslant (d - \varepsilon)^r a,$$

and therefore, at least $(m - \varepsilon r a^r)$ r-cliques $R \subset A$ satisfy

$$\left| \left(\bigcap_{u \in R} N(u) \right) \cap B_i \right| > (d - \varepsilon)^r a.$$

Hence, at least $t(d - \varepsilon)^r (m - \varepsilon r a^r) a$ $(r+1)$-cliques of H have exactly r vertices in A and one vertex in $\bigcup_{i \in [t]} B_i$, completing the proof. □

The following consequence of Ramsey's theorem has been proved by Erdős [5].

Lemma 3. *Given integers $p \geqslant 2$, $r \geqslant 2$, there exist a $c_{p,r} > 0$ such that if G is a K_{p+1}-free graph of order n and $n \geqslant r(K_{p+1}, K_r)$ then G contains at least $c_{p,r} n^r$ independent r-sets.*

We need another result related to the regularity lemma of Szemerédi, the so-called Key Lemma (e.g. see [7, Theorem 2.1]). We shall use the following simplified version of the Key Lemma.

Theorem 5. *Suppose $0 < \varepsilon < d < 1$ and let m be a positive integer. Let G be a graph of order $(p+1)m$ and let $V(G) = V_1 \cup \cdots \cup V_{p+1}$ be a partition of $V(G)$ into $p+1$ sets of cardinality m so that each of the pairs (V_i, V_j) is ε-regular and has density at least d. If $\varepsilon \leqslant (d - \varepsilon)^p / (p+2)$ then $K_{p+1} \subset G$.*

Proof of Theorem 3. Our proof is straightforward but rather rich in technical details, so we shall briefly outline it first. For some properly selected ε, applying SRL, we partition all but εn vertices of G in k sets $V_1, \ldots, V_k$ of equal cardinality such that almost all pairs (V_i, V_j) are ε-regular. We may assume that the number of dense ε-regular pairs (V_i, V_j) is no more than $\frac{p-1}{2p} k^2$, since otherwise, from Theorem 5 and Turán's theorem, G will contain a K_{p+1}. Therefore, there are at least $(1/2p + o(1))k^2$ sparse ε-regular pairs (V_i, V_j). From Lemma 3 it follows that the number of independent r-sets in any of the sets $V_1, \ldots, V_k$ is $\Theta(n^r)$. Consider the size of the r-book in $\overline{G}$ having for its base the average independent r-set in V_i. For every sparse ε-regular pair (V_i, V_j) almost every vertex in V_j is a page of such a book. Also each ε-regular pair (V_i, V_j) whose density is not very close to 1 contributes substantially many additional pages to such books. Precise estimates show that either $bs^{(r)}(\overline{G}) > n/p$ or else the number of all ε-regular pairs (V_i, V_j) with density close to 1 is $(\frac{p-1}{2p} + o(1))k^2$. Thus the size of G is $(\frac{p-1}{2p} + o(1))n^2$ and therefore, according to Theorem 2, G contains the required induced p-chromatic subgraph with the required minimum degree.

Details of the proof: Let $c(p)$ be as in Theorem 2 and $c_{p,r}$ be as in Lemma 3. Select

$$\delta = \min\left\{\frac{\xi^3}{32}, \frac{c(p)}{4}\right\}, \tag{9}$$

set

$$d = \min\left\{\left(\frac{\delta}{2}\right)^{r+1}\left(\frac{r}{c_{p,r}} + 2r + 1 + 2p\right)^{-1},\ \frac{p\delta}{1+p\delta}\left(\frac{r}{c_{p,r}} + 2r + 1\right)^{-1}\right\}, \tag{10}$$

and let

$$\varepsilon = \min\left\{\delta, \frac{d^p}{2(p+1)}\right\}. \tag{11}$$

These definitions are justified at the later stages of the proof. Since $c_{p,r} < r!$ we easily see that $0 < 2\varepsilon < d < \delta < 1$. Hence, Bernoulli's inequality implies

$$(d-\varepsilon)^p \geqslant d^p - p\varepsilon d^{p-1} > d^p - p\varepsilon = 2(p+1)\varepsilon - p\varepsilon = (p+2)\varepsilon. \tag{12}$$

Applying SRL we find a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$ so that $|V_0| < \varepsilon n$, $|V_1| = \cdots = |V_k|$ and all but εk^2 pairs (V_i, V_j) are ε-regular. Without loss of generality we may assume $|V_i| > r(K_{p+1}, K_r)$ and $k > 1/\varepsilon$. Consider the graphs H_{irr}, H_{lo}, H_{mid} and H_{hi} defined on the vertex set $[k]$ as follows:

(i) $(i,j) \in E(H_{\text{irr}})$ iff the pair (V_i, V_j) is not ε-regular,
(ii) $(i,j) \in E(H_{\text{lo}})$ iff the pair (V_i, V_j) is ε-regular and

$$d(V_i, V_j) \leqslant d,$$

(iii) $(i,j) \in E(H_{\text{mid}})$ iff the pair (V_i, V_j) is ε-regular and

$$d < d(V_i, V_j) \leqslant 1 - \delta,$$

(iv) $(i,j) \in E(H_{\text{hi}})$ iff the pair (V_i, V_j) is ε-regular and

$$d(V_i, V_j) > 1 - \delta.$$

Clearly, no two of these graphs have edges in common; thus

$$e(H_{\text{irr}}) + e(H_{\text{lo}}) + e(H_{\text{mid}}) + e(H_{\text{hi}}) = \binom{k}{2}.$$

Hence, from $d > 2\varepsilon$ and $k > 1/\varepsilon$, we see that

$$\begin{aligned} e(H_{\text{lo}}) + e(H_{\text{mid}}) + e(H_{\text{hi}}) &\geqslant \binom{k}{2} - \varepsilon k^2 = \frac{k^2}{2} - \frac{k}{2} - \varepsilon k^2 \\ &\geqslant \frac{k^2}{2} - \varepsilon k^2 - \varepsilon k^2 > \left(\frac{1}{2} - d\right)k^2. \end{aligned} \tag{13}$$

Since G is K_{p+1}-free, from (12), we have $\varepsilon \leqslant (d-\varepsilon)^p/(p+2)$; applying Theorem 5, we conclude that the graph $H_{\text{mid}} \cup H_{\text{hi}}$ is K_{p+1}-free. Therefore, from

Turán's theorem,

$$e(H_{\text{mid}}) + e(H_{\text{hi}}) \leqslant \left(\frac{p-1}{2p}\right)k^2,$$

and from inequality (13) we deduce

$$e(H_{\text{lo}}) > \left(\frac{1}{2p} - d\right)k^2. \tag{14}$$

Next we shall bound $bs^{(r)}(\overline{G})$ from below. To achieve this we shall count the independent $(r+1)$-sets having exactly r vertices in some V_i and one vertex outside V_i. Fix $i \in [k]$ and let m be the number of independent r-sets in V_i. Observe that Lemma 3 implies $m \geqslant c_{p,r}|V_i|^r$.

Set $L = N_{H_{\text{lo}}}(i)$ and apply Lemma 2 with $A = V_i$, $B_j = V_j$, for all $j \in L$, and

$$H = \overline{G}\left[A \cup \left(\bigcup_{j \in L} B_j\right)\right].$$

Since, for every $j \in L$, the pair (V_i, V_j) is ε-regular and

$$e_H(V_i, V_j) \geqslant (1-d)|V_i||V_j|,$$

we conclude that there are at least

$$d_{H_{\text{lo}}}(i)|V_i|(m - \varepsilon r|V_i|^r)(1-d-\varepsilon)^r$$

independent $(r+1)$-sets in G having exactly r vertices in V_i and one vertex in $\bigcup_{j \in L} B_j$.

Set now $M = N_{H_{\text{mid}}}(i)$, and apply Lemma 2 with $A = V_i$, $B_j = V_j$ for all $j \in M$ and

$$H = \overline{G}\left[A \cup \left(\bigcup_{j \in M} B_j\right)\right].$$

Since, for every $j \in M$, the pair (V_i, V_j) is ε-regular and

$$e_H(V_i, V_j) \geqslant \delta|V_i||V_j|,$$

we conclude that there are at least

$$d_{H_{\text{mid}}}(i)|V_i|(m - \varepsilon r|V_i|^r)(\delta - \varepsilon)^r$$

independent $(r+1)$-sets in G having exactly r vertices in V_i and one vertex in $\bigcup_{j \in L} B_j$. Since

$$\left(\bigcup_{j \in L} B_j\right) \bigcap \left(\bigcup_{j \in M} B_j\right) = \emptyset,$$

there are at least

$$d_{H_{\text{lo}}}(i)|V_i|(m - \varepsilon r|V_i|^r)(1-d-\varepsilon)^r + d_{H_{\text{mid}}}(i)|V_i|(m - \varepsilon r|V_i|^r)(\delta - \varepsilon)^r$$

independent $(r+1)$-sets in G having exactly r vertices in V_i and one vertex outside V_i. Thus, taking the average over all m independent r-sets in V_i, we conclude

$$\begin{aligned} bs^{(r)}(\overline{G}) &\geqslant |V_i|\left(1-\frac{\varepsilon r}{c_{p,r}}\right)(d_{H_{\mathrm{lo}}}(i)(1-d-\varepsilon)^r + d_{H_{\mathrm{mid}}}(i)(\delta-\varepsilon)^r) \\ &\geqslant n\left(\frac{1-\varepsilon}{k}\right)\left(1-\frac{\varepsilon r}{c_{p,r}}\right)(d_{H_{\mathrm{lo}}}(i)(1-d-\varepsilon)^r + d_{H_{\mathrm{mid}}}(i)(\delta-\varepsilon)^r). \end{aligned}$$

Summing this inequality for all $i = 1, \dots, k$ we obtain

$$\begin{aligned} \frac{bs^{(r)}(\overline{G})}{n} &\geqslant (1-\varepsilon)\left(1-\frac{\varepsilon r}{c_{p,r}}\right)\left(\frac{2e(H_{\mathrm{lo}})}{k^2}(1-d-\varepsilon)^r + \frac{2e(H_{\mathrm{mid}})}{k^2}(\delta-\varepsilon)^r\right) \\ &> \left(1-\left(\frac{r}{c_{p,r}}+1\right)\varepsilon\right)\left(\frac{2e(H_{\mathrm{lo}})}{k^2}(1-r(d+\varepsilon)) + \frac{2e(H_{\mathrm{mid}})}{k^2}(\delta-\varepsilon)^r\right) \\ &> \left(1-\left(\frac{r}{c_{p,r}}+1\right)d\right)\left(\frac{2e(H_{\mathrm{lo}})}{k^2}(1-2rd) + \frac{2e(H_{\mathrm{mid}})}{k^2}\left(\frac{\delta}{2}\right)^r\right) \\ &> \left(1-\left(\frac{r}{c_{p,r}}+2r+1\right)d\right)\frac{2e(H_{\mathrm{lo}})}{k^2} + \left(1-\left(\frac{r}{c_{p,r}}+1\right)d\right) \\ &\quad \times \left(\frac{\delta}{2}\right)^r \frac{2e(H_{\mathrm{mid}})}{k^2}. \end{aligned} \tag{15}$$

Assume the assertion of the theorem false and suppose

$$bs^{(r)}(\overline{G}) \leqslant \frac{n}{p}. \tag{16}$$

We shall prove that this assumption implies

$$e(H_{\mathrm{lo}}) < \left(\frac{1}{2p}+\frac{\delta}{2}\right)k^2, \tag{17}$$

$$e(H_{\mathrm{mid}}) < \delta k^2. \tag{18}$$

Disregarding the term $e(H_{\mathrm{mid}})$ in (15), in view of (16) and (10), we have

$$\begin{aligned} e(H_{\mathrm{lo}}) &< \left(1-\left(\frac{r}{c_{p,r}}+2r+1\right)d\right)^{-1}\frac{bs^{(r)}(\overline{G})}{2n}k^2 \\ &\leqslant \left(1-\left(\frac{r}{c_{p,r}}+2r+1\right)d\right)^{-1}\frac{k^2}{2p} \\ &\leqslant \left(1-\frac{p\delta}{1+p\delta}\right)^{-1}\frac{k^2}{2p} = \left(\frac{1}{2p}+\frac{\delta}{2}\right)k^2, \end{aligned}$$

and inequality (17) is proved.

Furthermore, observe that equality (10) implies

$$\left(\frac{r}{c_{p,r}}+1\right)d < \left(\frac{r}{c_{p,r}}+2r+1\right)d \leqslant \frac{p\delta}{1+p\delta} \leqslant p\delta < \frac{1}{2},$$

and consequently,

$$\left(1-\left(\frac{r}{c_{p,r}}+1\right)d\right)>\frac{1}{2}.$$

Hence, from (15), taking into account (16) and (14), we find that

$$\begin{aligned}\frac{e(H_{\text{mid}})}{2}\left(\frac{\delta}{2}\right)^{r} &< e(H_{\text{mid}})\left(\frac{\delta}{2}\right)^{r}\left(1-\left(\frac{r}{c_{p,r}}+1\right)d\right)\\ &\leqslant \frac{bs^{(r)}(\overline{G})k^2}{2n}-\left(1-\left(\frac{r}{c_{p,r}}+2r+1\right)d\right)e(H_{\text{lo}})\\ &< \left(\frac{1}{2p}-\left(1-\left(\frac{r}{c_{p,r}}+2r+1\right)d\right)\left(\frac{1}{2p}-d\right)\right)k^2\\ &= \left(1+\left(\frac{r}{c_{p,r}}+2r+1\right)\left(\frac{1}{2p}-d\right)\right)dk^2\\ &< \frac{1}{2p}\left(\frac{r}{c_{p,r}}+2r+1+2p\right)dk^2<\left(\frac{\delta}{2}\right)^{r+1}k^2.\end{aligned}$$

Therefore, inequality (18) holds also.

Furthermore, inequality (13), together with (17) and (18), implies

$$e(H_{\text{hi}})>\left(\frac{1}{2}-d\right)k^2-\left(\frac{1}{2p}+\frac{\delta}{2}\right)k^2-\delta k^2=\left(\frac{p-1}{2p}-\frac{5\delta}{2}\right)k^2,$$

and consequently, from the definition of H_{hi}, we obtain

$$\begin{aligned}e(G) &\geqslant e(H_{\text{hi}})\left(\frac{(1-\varepsilon)n}{k}\right)^2(1-\delta)>\left(\frac{p-1}{2p}-\frac{5\delta}{2}\right)(1-2\varepsilon)(1-\delta)n^2\\ &=\frac{p-1}{2p}\left(1-\frac{5p\delta}{p-1}\right)(1-2\varepsilon)(1-\delta)n^2>\\ &>\frac{p-1}{2p}\left(1-\left(\frac{5p}{p-1}+3\right)\delta\right)n^2>\left(\frac{p-1}{2p}-4\delta\right)n^2.\end{aligned}$$

Hence, by (9), applying Theorem 2, it follows that G contains an induced p-chromatic graph with the required properties. □

Following the basic idea of the proof of Theorem 3 but applying the complete Key Lemma instead of Theorem 5, we obtain a more general result, whose proof, however, is considerably easier than the proof of Theorem 3.

Theorem 6. *Suppose H is a fixed $(p+1)$-chromatic graph. For every H-free graph G of order n,*

$$bs^{(r)}(\overline{G})>\left(\frac{1}{p}+o(1)\right)n.$$

Note that the graph $K_p(q+r-1)$ is p-chromatic and its complement has no $B_q^{(r)}$, so for every $(p+1)$-chromatic graph H and every r, q we have

$$r(H, B_q^{(r)}) \geqslant p(q+r-1)+1.$$

Hence, from Theorem 6, we immediately obtain the following theorem.

Theorem 7. *For every fixed $(p+1)$-chromatic graph H and fixed integer $r>1$,*

$$r(H, B_q^{(r)}) = pq + o(q).$$

Note that it is not possible to avoid the $o(q)$ term in Theorem 7 without additional stipulations about H, since, as Faudree, Rousseau and Sheehan have shown in [6], the inequality

$$r(C_4, B_q^{(2)}) \geqslant q + 2\sqrt{q}$$

holds for infinitely many values of q. However, when $H = K_{p+1}$ and q is large we can prove a precise result.

4. Ramsey numbers $r(K_p, B_q^{(r)})$ for large q

In this section we determine $r(K_p, B_q^{(r)})$ for fixed $p \geqslant 3$, $r \geqslant 2$ and large q.

Theorem 8. *For fixed $p \geqslant 2$ and $r \geqslant 2$, $r(K_{p+1}, B_q^{(r)}) = p(q+r-1)+1$ for all sufficiently large q.*

Proof. Since $K_p(q+r-1)$ contains no K_{p+1} and its complement contains no $B_q^{(r)}$, we have

$$r(K_{p+1}, B_q^{(r)}) \geqslant p(q+r-1)+1.$$

Let G be a K_{p+1}-free graph of order $n = p(q+r-1)+1$. Since $n/p > q$, either we're done or else G contains an induced p-chromatic subgraph G_1 of order $pq + o(q)$ with minimum degree

$$\delta(G_1) \geqslant \left(1 - \frac{1}{p} + o(1)\right)n.$$

Using this bound on $\delta(G_1)$ we can easily prove by induction on p that G_1 contains a copy of $K_p(r)$. Fix a copy of $K_p(r)$ in G_1 and let $A_1, A_2, \ldots, A_p$ be its vertex classes. Let $A = A_1 \cup \cdots \cup A_p$ and $B = V(G) \backslash A$. If some vertex $i \in B$ is adjacent to at least one vertex in each of the parts $A_1, A_2, \ldots, A_p$ then G contains a K_{p+1}. Otherwise for each vertex $u \in B$ there is at least one v so that u is adjacent in $\overline{G}$ to all members of A_v. It follows by the pigeonhole principle that $bs^{(r)}(\overline{G}) = s$ where

$$s \geqslant \left\lceil \frac{n - p(r-1)}{p} \right\rceil = \left\lceil q - 1 + \frac{1}{p} \right\rceil = q,$$

and we really are done. □

The proof using the regularity lemma that $r(K_{p+1}, B_q^{(r)}) = p(q + r - 1) + 1$ if q is sufficiently large does indeed require that q increase quite rapidly as a function of the parameters p and r. This raises the question of what growth rate is actually required. The following simple calculation shows that polynomial growth in p is not sufficient.

Theorem 9. *For arbitrary fixed k and r,*

$$\frac{r(K_m, B_{m^k}^{(r)})}{m^{k+r-1}} \to \infty$$

as $m \to \infty$.

Proof. We shall prove that $r(K_m, B_{m^k}^{(r)}) > cm^{k+r}/(\log m)^r$ for all sufficiently large m. Let $N = \lfloor cm^{k+r}/(\log m)^r \rfloor$ where c is to be chosen, and set $p = (C/m)\log m$ where $C = 2(k + r - 1)$. Let G be the random graph $G = G(N, 1 - p)$. The probability that $K_m \subset G$

$$\mathbb{P}(K_m \subset G) \leqslant \binom{N}{m}(1-p)^{\binom{m}{2}} \leqslant \binom{N}{m} e^{-pm(m-1)/2} < \left(\frac{Ne}{m}\right)^m e^{pm/2} m^{-(k+r-1)m}$$
$$= \left(\frac{Ne^{1+p/2} m^{-(k+r-1)}}{m}\right)^m = o(1), \quad m \to \infty.$$

To bound the probability that $B_{m^k}^{(r)} \subset \overline{G}$, we use the following simple consequence of Chernoff's inequality [4]: if $X = X_1 + X_2 + \cdots + X_n$ where independently each $X_i = 1$ with probability $\mathfrak{p}$ and $X_i = 0$ with probability $1 - \mathfrak{p}$ then

$$\mathbb{P}(X \geqslant M) \leqslant \left(\frac{n\mathfrak{p}e}{M}\right)^M$$

for any $M \geqslant n\mathfrak{p}$. Thus we find

$$\mathbb{P}(B_{m^k}^{(r)} \subset \overline{G}) \leqslant \binom{N}{r} p^{r(r-1)/2} \left(\frac{(N-r)p^r e}{m^k}\right)^{m^k}.$$

Since the product of the first two factors has polynomial growth in m, to have $\mathbb{P}(B_{m^k}^{(r)}) = o(1)$ when $m \to \infty$, it suffices to take $c = 1/(3C^r)$, so that

$$\frac{(N-r)p^r e}{m^k} \leqslant \frac{(c\, m^{k+r}/(\log m)^r)((C/m)\log m)^r e}{m^k} = \frac{e}{3},$$

making the last factor approach 0 exponentially. □

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