Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Linear Algebra and its Applications 368 (2003) 159–169
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Total dilations

Jean-Christophe Bourin

Les Coteaux, 8 rue Henri Durel, 78510 Triel, France

Received 23 July 2002; accepted 1 November 2002

Submitted by T. Ando

Abstract

(1) Let A be an operator on a space $\mathscr{H}$ of even finite dimension. Then for some decomposition $\mathscr{H} = \mathscr{F} \oplus \mathscr{F}^{\perp}$, the compressions of A onto $\mathscr{F}$ and $\mathscr{F}^{\perp}$ are unitarily equivalent. (2) Let $\{A_j\}_{j=0}^{n}$ be a family of strictly positive operators on a space $\mathscr{H}$. Then, for some integer k, we can dilate each A_j into a positive operator B_j on $\oplus^k \mathscr{H}$ in such a way that: (i) The operator diagonal of B_j consists of a repetition of A_j. (ii) There exist a positive operator B on $\oplus^k \mathscr{H}$ and an increasing function $f_j : (0, \infty) \to (0, \infty)$ such that $B_j = f_j(B)$.

AMS classification: 47A20

Keywords: Dilation; Positive operators

0. Introduction

This paper is a continuation of a subsection of [2] entitled “commuting dilations”. We recall our definitions and notations. A pair of positive (semi-definite) operators A and B on a finite dimensional Hilbert space $\mathscr{H}$, dim $\mathscr{H} = d$, is said to be a monotone pair of positive operators, or a positive monotone pair, if there exists an orthonormal basis $\{e_k\}_{k=1}^{d}$ such that

$$A = \sum_{k=1}^{d} \mu_k(A) e_k \otimes e_k \quad \text{and} \quad B = \sum_{k=1}^{d} \mu_k(B) e_k \otimes e_k,$$

E-mail address: bourinjc@club-internet.fr (J.-C. Bourin).

0024-3795/03/$ - see front matter
doi:10.1016/S0024-3795(02)00666-3

where the numbers $\mu_k(\cdot)$ are the singular values arranged in decreasing order and counted with their multiplicities and $e_k \otimes e_k$ is the rank one projection associated with e_k. On the other hand, if

$$A = \sum_{k=1}^{d} \mu_k(A) e_k \otimes e_k \quad \text{and} \quad B = \sum_{k=1}^{d} \mu_{d+1-k}(B) e_k \otimes e_k,$$

we say that (A, B) is an antimonotone pair of positive operators. It is easy to define the notion of a monotone family $\{A_j\}_{j=0}^n$ of positive operators. Furthermore, this notion can be extended to the notion of a monotone family of hermitian operators $\{A_j\}_{j=0}^n$ by requiring that there is a (hilbertian) basis $\{e_k\}$ for which

$$A_j = \sum_{k \geqslant 1} \lambda_k(A_j) e_k \otimes e_k, \quad 0 \leqslant j \leqslant n,$$

where $\lambda_k(\cdot)$ are the eigenvalues arranged in decreasing order and counted with their multiplicities. Setting $A = \sum_{k=1}^{d} (d-k) e_k \otimes e_k$, we note that $A_j = f_j(A)$ for some increasing functions f_j.

Positive, monotone pairs (A, B) well behave in respect to the compression to a subspace $\mathscr{E}$ of $\mathscr{H}$ (we recall this classical notion in Section 1). For instance we proved [2, Corollaries 2.2 and 2.3] that

$$\lambda_k(A_{\mathscr{E}} B_{\mathscr{E}}) \leqslant \lambda_k((AB)_{\mathscr{E}})$$

and

$$\lambda_k(A_{\mathscr{E}} B_{\mathscr{E}} A_{\mathscr{E}}) \leqslant \lambda_k((ABA)_{\mathscr{E}})$$

for all k. From the first inequality we derived

$$\det A_{\mathscr{E}} \det B_{\mathscr{E}} \leqslant \det(AB)_{\mathscr{E}},$$

while we showed that, in case of an antimonotone pair (A, B) and a hyperplane $\mathscr{E}$, we have the opposite inequality

$$\det A_{\mathscr{E}} \det B_{\mathscr{E}} \geqslant \det(AB)_{\mathscr{E}}.$$

These results suggest the following question: Given a finite family of positive operators, how can we dilate them into a positive, monotone family? This paper precisely deals with the construction of such monotone dilations. However it appears that the dilations built up have the additional property to be *total* dilations. This notion is discussed in Section 1; the main result herein is the proof of the following fact:

Any $2n$-by-$2n$ matrix A is unitarily equivalent to a matrix of the form

$$\begin{pmatrix} B & \star \\ \star & B \end{pmatrix}$$

in which B is some n-by-n matrix and the stars hold for unspecified entries.

We devote Section 2 to the study of monotone dilations. This section is divided in two subsections; the first one presents results whose proofs have an algorithmic nature while the second one gives more theoretical facts.

1. Dilations and total dilations

Let B be an operator on a space $\mathscr{H}$ and let $\mathscr{E}$ be a subspace of $\mathscr{H}$. Denote by E the projection onto $\mathscr{E}$. The restriction of EB to $\mathscr{E}$, denoted by $B_{\mathscr{E}}$, is the compression of B to $\mathscr{E}$. Therefore, in respect to the decomposition $\mathscr{H} = \mathscr{E} \oplus \mathscr{E}^{\perp}$, we may write

$$B = \begin{pmatrix} B_{\mathscr{E}} & \star \\ \star & \star \end{pmatrix}.$$

The notion of compression has a natural extension: If A is an operator on a space $\mathscr{F}$ with $\dim \mathscr{F} \leqslant \dim \mathscr{H}$, we still say that A is a compression of B if there is an isometry $V : \mathscr{F} \to \mathscr{H}$ such that $A = V^*BV$. Thus, identifying A with VAV^* (equivalently, identifying $\mathscr{F}$ and $V(\mathscr{F})$), we can write

$$B = \begin{pmatrix} A & \star \\ \star & \star \end{pmatrix}.$$

One also says that B dilates A or that B is a dilation of A.

Denote by $\oplus^k \mathscr{H}$ the direct sum $\mathscr{H} \oplus \cdots \oplus \mathscr{H}$ with k terms. Given an operator A on $\mathscr{H}$ we say that an operator B on $\oplus^k \mathscr{H}$ is a *total* dilation of A, or that B *totally* dilates A, if we can write

$$B = \begin{pmatrix} A & \star & \cdots \\ \star & A & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

that is if the operator diagonal of B consists of a repetition of A. Clearly this notion has also a natural extension when A acts on any space $\mathscr{F}$ with $\dim \mathscr{F} = \dim \mathscr{H}$. Let $\{A_j\}_{j=0}^n$ be a family of operators on $\mathscr{H}$ and let $\{B_j\}_{j=0}^n$ be a family of operators on $\oplus^k \mathscr{H}$. We say that $\{B_j\}_{j=0}^n$ *totally* dilates $\{A_j\}_{j=0}^n$ if we can write, in respect to a (hilbertian) basis of $\mathscr{H}$,

$$B_0 = \begin{pmatrix} A_0 & \star & \cdots \\ \star & A_0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad \ldots, \quad B_n = \begin{pmatrix} A_n & \star & \cdots \\ \star & A_n & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

We give five examples of total dilations:

Example 1.1. A $2n \times 2n$ antisymmetric real matrix A totally dilates the n-dimensional zero operator: in respect to a suitable decomposition

$$A = \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}$$

for some symmetric real n-by-n matrix B.

Example 1.2. Any operator A on $\mathscr{H}$ can be totally dilated into a normal operator N on $\mathscr{H} \oplus \mathscr{H}$ by setting

$$N = \begin{pmatrix} A & A^* \\ A^* & A \end{pmatrix}.$$

Example 1.3. Denote by $\tau(A)$ the normalized trace $(1/n)\mathrm{Tr}\, A$ of an operator A on an n-dimensional space. Then the scalar $\tau(A)$ can be totally dilated into A. For an operator acting on a real space and for a hermitian operator the proof is easy. When A is a general operator on a complex space, this result follows from the Hausdorff–Toeplitz Theorem (see [4, p. 20]).

Example 1.4. Any contraction A on a finite dimensional space $\mathscr{H}$ can be totally dilated into a unitary operator U on $\oplus^k \mathscr{H}$ for any integer $k \geqslant 2$. Indeed by considering the polar decomposition $A = V|A|$, it suffices to construct a total unitary dilation W of $|A|$ and then to take $U = (\oplus^k V) \cdot W$. The construction of a total unitary dilation on $\oplus^k \mathscr{H}$ for a positive contraction X on $\mathscr{H}$ is easy: Let $\{x_j\}_{j=1}^n$ be the eigenvalues of X repeated according to their multiplicities and let $\{U_j\}_{j=1}^n$ be $k \times k$ unitary matrices such that $\tau(U_j) = x_j$. Example 1.3 and an obvious matrix manipulation show that $\oplus_{j=1}^n U_j$ totally dilates X.

Example 1.5. Let $\{A_k\}_{k=1}^n$ be a family of operators on $\mathscr{H}$ and let $\{B_k\}_{k=1}^n$ be the family of operators acting on $\oplus^n \mathscr{H}$ defined by

$$B_k = \begin{pmatrix} A_k & A_{k-1} & \cdots \\ A_{k+1} & A_k & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $\{B_k\}_{k=1}^n$ is a commuting family which totally dilates $\{A_k\}_{k=1}^n$. (We set $A_0 = A_n, A_{-1} = A_{n-1}, \ldots$)

In the last example above, the dilations do not preserve properties such as positivity, self-adjointness or normality. Using larger dilations we may preserve these properties:

Proposition 1.6. *Let $\{A_j\}_{j=0}^n$ be operators on a space $\mathscr{H}$. Then there exist operators $\{B_j\}_{j=0}^n$ on $\oplus^k \mathscr{H}$, where $k = 2^n$, such that*

(a) *For $i \neq j$, $B_i B_j = 0$.*
(b) *$\{B_j\}_{j=0}^n$ totally dilates $\{A_j\}_{j=0}^n$.*
(c) *If the A_j's are positive (respectively hermitian, normal) then the B_j's are of the same type.*

Proof. Given a pair A_0, A_1 of operators, construct

$$S = \begin{pmatrix} A_0 & A_0 \\ A_0 & A_0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} A_1 & -A_1 \\ -A_1 & A_1 \end{pmatrix}.$$

Then $ST = TS = 0$. We then proceed by induction. We have just proved the case of $n = 1$. Assume that the result holds for $n - 1$. Thus we have a family $\mathscr{C} = \{C_j\}_{j=0}^{n-1}$ which totally dilates $\{A_j\}_{j=0}^{n-1}$. Moreover $\mathscr{C}$ acts on a space $\mathscr{G}$, $\dim \mathscr{G} = 2^{n-1} \dim \mathscr{H}$. We dilate A_n to an operator C_n on $\mathscr{G}$ by setting $C_n = A_n \oplus \cdots \oplus A_n$, 2^{n-1} terms. We then consider the operators on $\mathscr{F} = \mathscr{G} \oplus \mathscr{G}$ defined by

$$B_j = \begin{pmatrix} C_j & C_j \\ C_j & C_j \end{pmatrix} \quad \text{for } 0 \leqslant j < n \quad \text{and} \quad B_n = \begin{pmatrix} C_n & -C_n \\ -C_n & C_n \end{pmatrix}.$$

The family $\{B_j\}_{j=0}^{n}$ has the required properties. □

If $\mathscr{H}$ is a space with an even finite dimension, we then say that the orthogonal decomposition $\mathscr{H} = \mathscr{F} \oplus \mathscr{F}^{\perp}$ is a *halving* decomposition whenever $\dim \mathscr{F} = \frac{1}{2} \dim \mathscr{H}$.

Theorem 1.7. *Let A be an operator on a space $\mathscr{H}$ with an even finite dimension. Then there exists a halving decomposition $\mathscr{H} = \mathscr{F} \oplus \mathscr{F}^{\perp}$ for which we have a total dilation*

$$A = \begin{pmatrix} B & \bigstar \\ \bigstar & B \end{pmatrix}.$$

Proof. Choose a halving decomposition of $\mathscr{H}$ for which we have a matrix representation of $\operatorname{Re} A$ of the following form:

$$\operatorname{Re} A = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}.$$

Consequently in respect to this decomposition we must have

$$A = \begin{pmatrix} Y & X \\ -X^* & Z \end{pmatrix}.$$

Let $X = U|X|$ and $Y_0 = U^*YU$. We have

$$\begin{pmatrix} U^* & 0 \\ 0 & I \end{pmatrix} A \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} U^* & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Y & U|X| \\ -|X|U^* & Z \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} Y_0 & |X| \\ -|X| & Z \end{pmatrix}.$$

Now observe that

$$\frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}\begin{pmatrix} Y_0 & |X| \\ -|X| & Z \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}$$
$$= \begin{pmatrix} (Y_0+Z)/2 & \bigstar \\ \bigstar & (Y_0+Z)/2 \end{pmatrix}.$$

Thus, using two unitary congruence we have exhibited an operator totally dilated into A. □

Remark 1.8. The proof of Theorem 1.7 is easy for a normal operator A: consider a representation

$$A = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$$

and use the unitary conjugation by

$$\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}.$$

Applying this to X^*X, for an operator X on an even dimensional space, we note that there exists a halving projection E such that XE and $XE^{\perp}$ have the same singular values (indeed EX^*XE and $E^{\perp}X^*XE^{\perp}$ are unitarily equivalent).

Problems 1.9. (a) Does Theorem 1.7 extend to infinite dimensional spaces? (b) Let $\mathscr{H}$, $\mathscr{F}$ be two finite dimensional spaces with $\dim \mathscr{H} = k \dim \mathscr{F}$ for an integer k. Is any operator A on $\mathscr{H}$ a total dilation of some operator B on $\mathscr{F}$?

The author has the feeling that the two questions above have a positive answer.

2. Constructions of monotone dilations

Recall that the notion of a monotone family of positive or hermitian operators has been discussed in the introduction.

2.1. Algorithmic constructions of monotone dilations

Given an operator A on $\mathscr{H}$ and an integer $k > 0$ we define the following total dilations of A on $\oplus^k \mathscr{H}$:

$$A(k) = \begin{pmatrix} A & 0 & \cdots \\ 0 & A & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad A[k] = \begin{pmatrix} A & A & \cdots \\ A & A & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Therefore, denoting by I_k the k-by-k identity matrix and by E_k the k-by-k matrix whose entries all equal to 1, we have $A(k) = A \otimes I_k$ and $A[k] = A \otimes E_k$. Note that $(1/k)E_k$ is a (rank one) projection, consequently, when A is positive so is $A[k]$. For $k > 1$ we introduce another total dilation of A on $\oplus^k \mathscr{H}$ by setting

$$A\langle k\rangle = \begin{pmatrix} A & \frac{I-A}{k-1} & \cdots \\ \frac{I-A}{k-1} & A & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus we have

$$A\langle k\rangle = \left(\frac{I-A}{k-1}\right)[k] + \left(\frac{kA-I}{k-1}\right)(k).$$

If A is a positive operator satisfying $I \geqslant A \geqslant (1/k)I$ the above relation shows that $A\langle k\rangle$ is a positive operator. Given two operators A, B on $\mathscr{H}$ one can check that $A[k]$ and $B\langle k\rangle$ commute, in fact

$$A[k]B\langle k\rangle = A[k] = B\langle k\rangle A[k].$$

If both A and B are positive, a more precise result holds.

Proposition 2.1. *Let (A, B) be a pair of positive operators on $\mathscr{H}$ and assume that $I \geqslant B \geqslant (1/k)I$ for some integer $k > 0$. Then $(A[k], B\langle k\rangle)$ is a monotone pair of positive operators which totally dilates (A, B).*

Proposition 2.1 is just a restatement of Theorem 2.11 in [2]. The next result is a generalization for more general families than pairs. It is convenient to introduce some notations. First an expression like $A(k)\langle l\rangle[m]$ should be understood in the following way: begin by constructing $B = A(k)$, then construct $C = B\langle l\rangle$ and finally construct $C[m]$. Second, given a sequence $\{k_j\}_{j=1}^n$ of integers, we complete it with $k_{-1} = k_0 = k_{n+1} = 1$ and we set, for $0 \leqslant j \leqslant n$:

$$k'_j = \prod_{l=0}^{j-1} k_l \quad \text{and} \quad k''_j = \prod_{l=j+1}^{n} k_l \quad (\text{consequently } k'_0 = k''_n = 1).$$

Theorem 2.2. *Let $\{A_j\}_{j=0}^n$ be positive operators on a space $\mathscr{H}$. Assume that for $j > 0$ we have integers $k_j > 0$ such that $I \geqslant A_j \geqslant (1/k_j)I$. Then there exist positive operators $\{B_j\}_{j=0}^n$ on $\oplus^k \mathscr{H}$, where $k = \prod_{j=1}^n k_j$, such that:*

(a) *$\{B_j\}_{j=0}^n$ is a monotone family of positive operators.*
(b) *$\{B_j\}_{j=0}^n$ totally dilates $\{A_j\}_{j=0}^n$.*

A suitable choice for each B_j is $A(k'_j)\langle k_j\rangle[k''_j]$.

Multiplying by appropriate scalars, we note that the assumptions $I \geqslant A_j \geqslant (1/k_j)I$ may be replaced by $\operatorname{cond}(A_j) = \|A_j\|\|A_j^{-1}\| \leqslant k_j$ $(j > 0)$.

Proof. We proceed by induction. For $n = 1$, this is Theorem 2.11 in [2]. Assume that the result holds for $n - 1$. Let $\mathscr{A}_0 = \{A_j\}_{j=0}^{n-1}$. By the induction assumption there is a monotone family $\mathscr{C} = \{C_j\}_{j=0}^{n-1}$ which totally dilates $\mathscr{A}_0$. Furthermore $\mathscr{C}$ acts on a space $\mathscr{G}$ with $\dim \mathscr{G} = \prod_{j=1}^{n-1} k_j \dim \mathscr{H} = k_n' \dim \mathscr{H}$. Next, we dilate A_n into an operator C_n on $\mathscr{G}$ by setting $C_n = A_n(k_n')$. To prove the theorem it now suffices to show that we can totally dilate the family $\mathscr{C}' = \{C_j\}_{j=0}^{n}$ on $\mathscr{G}$ into a monotone family $\mathscr{B} = \{B_j\}_{j=0}^{n}$ on a larger space $\mathscr{F}$ with $\dim \mathscr{F} = k_n \dim \mathscr{G}$.

To this purpose we consider on $\mathscr{F} = \mathscr{G} \oplus \cdots \oplus \mathscr{G}$, k_n terms, the following operators: for $0 \leqslant j \leqslant n-1$,

$$B_j = \begin{pmatrix} C_j & C_j & \cdots \\ C_j & C_j & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

and for $j = n$

$$B_n = \begin{pmatrix} C_n & \frac{I-C_n}{k_n-1} & \cdots \\ \frac{I-C_n}{k_n-1} & C_n & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Because $\{C_j\}_{j=0}^{n-1}$ is a monotone family, so is $\{B_j\}_{j=0}^{n-1}$ (recall that $B_j = C_j \otimes E_{k_n}$ for $j < n$ where E_p is, up to a scalar multiple, a rank one projection). Reasoning as in the proof of Theorem 2.11 in [2] we obtain that (B_j, B_n), $0 \leqslant j < n$, are monotone pairs. Consequently $\{B_j\}_{j=0}^{n}$ is a monotone family (if $\{X_j\}_{j=0}^{n-1}$ is a monotone family and (X_j, X_n) are monotone pairs, $j < n$, then $\{X_j\}_{j=0}^{n}$ is a monotone family). Finally a close look to our constructions reveals that the B_j's are given by the formulae of the last part of the theorem. □

Corollary 2.3. *Let $\{A_j\}_{j=0}^{n}$ be hermitian operators on a space $\mathscr{H}$. Then we can totally dilate them into a monotone family of hermitian operators on a larger space $\mathscr{F}$ with* $\dim \mathscr{F} = 2^n \dim \mathscr{H}$.

Proof. We set $A_j' = \alpha_j A_j + \frac{3}{4}I$ where $\alpha_j > 0$ is sufficiently small to have $\frac{1}{2}I \leqslant A_j' \leqslant I$. We apply Theorem 2.2 to dilate A_j' to B_j'. The operators $B_j = (1/\alpha_j)B_j' - (3/4\alpha_j)I$ are the wanted dilations. □

We may note that the proofs of the two preceding results have an algorithmic nature. More precisely, let us consider a sequence of hermitians $\{A_j\}_{j=0}^{n}$. The Frobenius norm $\|A_j\|_2$ is easily computed. Setting $\alpha_j = \frac{1}{4}\|A_j\|_2$ and applying Theorem 2.2 as in the proof of Corollary 2.3 we may easily construct a monotone family totally dilating $\{A_j\}_{j=0}^n$.

Remark 2.4. If A, B are positive noninvertible operators, it is not possible, in general, to dilate them into a positive monotone pair. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose that (S, T) is a positive, monotone dilation of (A, B). We should have the matrix representations respectively to a basis $(e_1, \dots, e_n)$ of some space

$$S = \begin{pmatrix} 1 & 0 & \star & \cdots \\ 0 & 0 & 0 & \cdots \\ \star & 0 & \star & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & \star & \cdots \\ 0 & \star & \star & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since (S, T) is supposed to be positive, monotone we would have one of the following relations: $\ker S \subset \ker T$ or $\ker T \subset \ker S$. Say $\ker S \subset \ker T$, we would deduce that $Te_1 = Te_2 = 0$ and we would reach a contradiction.

2.2. Theoretical constructions of monotone dilations

In the previous subsection we have constructed monotone dilations in a rather explicit way by using matrix manipulations. Now we give more theoretical constructions; the resulting dilations will act on more economical spaces but will not be total dilations. Our first construction uses a standard dilation argument in connection with the numerical range of an operator and we refer the reader to Chapter 1 of [4] for a detailed discussion of the numerical range.

Proposition 2.5. *Let A, B be two strictly positive operators on a space $\mathscr{H}$. Then we can dilate them into a monotone pair of strictly positive operators on a larger space $\mathscr{F}$ with* $\dim \mathscr{F} = 6 \dim \mathscr{H}$.

Proof. Invertibility of A and B ensures the existence of a real $r > 0$ such that

$$S = \begin{pmatrix} A & A - rI \\ A - rI & A \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} B & -B + rI \\ -B + rI & B \end{pmatrix}$$

are strictly positive operators. Moreover $ST = TS$. Hence $N = S + \mathrm{i}T$ is a normal operator acting on $\mathscr{G} = \mathscr{H} \oplus \mathscr{H}$. Because $S > 0$ and $T > 0$, the spectrum of N, $\mathrm{Sp}\, N$, lies in the open quadrant of $\mathbf{C}$,

$$Q = \{z = x + \mathrm{i}y \mid x > 0 \text{ and } y > 0\}.$$

We may then find a triangle $\Delta = \{x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2, x_3 + \mathrm{i}y_3\}$ in Q such that

$$x_1 < x_2 < x_3 \quad \text{and} \quad y_1 < y_2 < y_3 \tag{$*$}$$

and $\operatorname{conv} \Delta \supset \operatorname{Sp} N$. A standard dilation argument shows that there is a normal operator M acting on a space $\mathscr{F} \supset \mathscr{G}$, $\dim \mathscr{F} = 3 \dim \mathscr{G}$, such that $\operatorname{Sp} M = \Delta$ and $M_{\mathscr{G}} = N$. Therefore

$$(\operatorname{Re} M)_{\mathscr{H}} = (\operatorname{Re} N)_{\mathscr{H}} = A \quad \text{and} \quad (\operatorname{Im} M)_{\mathscr{H}} = (\operatorname{Im} N)_{\mathscr{H}} = B.$$

From $(*)$ we deduce that $(\operatorname{Re} M, \operatorname{Im} M)$ is a monotone pair dilating (A, B). □

At a time when it was not so clear to the author that a sequence of $n + 1$ hermitians could be dilated into a commuting family, Ando has pointed out to the author [1] the fact that it was a straightforward consequence of Naimark's Dilation Theorem. More precisely this theorem entails that the multiplicative constant 2^n in Proposition 1.6 can be replaced, in case of positive or hermitian operators, by $n + 2$ (but then the dilations are no longer total). We refer the reader to [3, p. 260] for a modern proof of Naimark's Theorem. Here the only thing we would need to know is the following particular case: Given positive operators $\{A_j\}_{j=0}^n$ on $\mathscr{H}$ satisfying $\sum A_j = I$, we can dilate them into a family $\{Q_j\}_{j=0}^n$ of mutually orthogonal projections on a larger space $\mathscr{F} = \mathscr{G} \otimes \mathscr{H}$ in which $\dim \mathscr{G} = n + 2$. Actually, rather than Naimark's Theorem, we only need the following much more elementary statement. Let us say that an operator B *essentially* acts on a subspace $\mathscr{E}$ if both the range and the corange of B are contained in $\mathscr{E}$ (equivalently, $\operatorname{ran} B \subset \mathscr{E}$ and $(\ker B)^\perp \subset \mathscr{E}$).

Lemma 2.6. *Fix an integer n and a space $\mathscr{H}$. Then there exist a larger space $\mathscr{F}$, $\dim \mathscr{F} = (n + 1) \dim \mathscr{H}$, and an orthogonal decomposition $\mathscr{F} = \mathscr{E}_0 \oplus \cdots \oplus \mathscr{E}_n$, in which $\dim \mathscr{E}_j = \dim \mathscr{H}$ for each j, such that: for every family of operators $\{A_j\}_{j=0}^n$ on $\mathscr{H}$ there is a family $\{B_j\}_{j=0}^n$ of operators on $\mathscr{F}$ with B_j essentially acting on $\mathscr{E}_j$ and $A_j = (B_j)_{\mathscr{H}}$, $0 \leqslant j \leqslant n$. Moreover when the A_j's are hermitian or positive, the B_j's can be taken of the same type.*

Let us sketch the elementary proof of this lemma. First, choose subspaces $\{\mathscr{E}_j\}_{j=0}^n$ of $\mathscr{F} = \oplus^{n+1} \mathscr{H}$ in such a way that for each j (a) $\dim \mathscr{E}_j = \dim \mathscr{H}$, (b) the projection E_j from $\mathscr{F}$ onto $\mathscr{E}_j$ verifies: $(E_j)_{\mathscr{H}}$ is a strictly positive operator on $\mathscr{H}$. Now, fix an integer j and observe that any vector $h \in \mathscr{H}$ can be lifted to a unique vector $h_j \in \mathscr{E}_j$ such that $Hh_j = h$, where H is the projection onto $\mathscr{H}$. Consequently any rank one operator of the form $R = h \otimes h$, $h \in \mathscr{H}$, can be lifted into a positive rank one operator T essentially acting on $\mathscr{E}_j$ such that $T_{\mathscr{H}} = R$. This ensures that given a general (respectively hermitian, positive) operator A on $\mathscr{H}$ there exists a general (respectively hermitian, positive) operator B essentially acting on $\mathscr{E}_j$ such that $B_{\mathscr{H}} = A$.

Theorem 2.7. *Let $\{A_j\}_{j=0}^n$ be hermitian operators on a space $\mathscr{H}$. Then we can dilate them into a monotone family of hermitian operators on a larger space $\mathscr{F}$ with $\dim \mathscr{F} = 2(n + 1) \dim \mathscr{H} - 1$.*

Proof. By Lemma 2.6 we may dilate $\{A_j\}_{j=0}^n$ into a commuting family of hermitians $\{S_j\}_{j=0}^n$ on a larger space $\mathscr{G}$ with $\dim \mathscr{G} = (n+1)\dim \mathscr{H} = d$. Thus, there is a basis $\{g_k\}_{k=0}^d$ in $\mathscr{G}$ and real numbers $\{s_{j,k}\}$ such that

$$S_j = \sum_{k=0}^{d} s_{j,k} g_k \otimes g_k \quad (0 \leqslant j \leqslant n).$$

We take for $\mathscr{F}$ a space of the form

$$\mathscr{F} = \mathscr{E}_0 \oplus \mathscr{E}_1 \oplus \cdots \oplus \mathscr{E}_d$$

in which $\dim \mathscr{E}_0 = 1$ and $g_0 \in \mathscr{E}_0$; and for $k > 0$, $\dim \mathscr{E}_k = 2$ and $g_k \in \mathscr{E}_k$. Hence, we have $\dim \mathscr{F} = 2(n+1)\dim \mathscr{H} - 1$.

For $k > 0$, let $\{e_{1,k}; e_{2,k}\}$ be a basis of $\mathscr{E}_k$ and suppose that $g_k = (e_{1,k} + e_{2,k})/\sqrt{2}$ $(*)$. We set, for $0 \leqslant j \leqslant n$,

$$B_j = s_{j,0} g_0 \otimes g_0 + \sum_{k=1}^{d} (r_{j,k} e_{1,k} \otimes e_{1,k} + t_{j,k} e_{2,k} \otimes e_{2,k}),$$

where the reals $r_{j,k}$ and $t_{j,k}$ are chosen in such a way that:

(1) $s_{j,k} = (r_{j,k} + t_{j,k})/2$, $j = 0, \ldots, n$.
(2) $r_{j,d} < \cdots < r_{j,1} < s_{j,0} < t_{j,1} < \cdots < t_{j,d}$, $j = 0, \ldots, n$.

From (1) and $(*)$ we deduce that $S_j = (B_j)_{\mathscr{G}}$ so that $A_j = (B_j)_{\mathscr{H}}$. From (2) we infer that $\{B_j\}_{j=0}^n$ is a monotone family. □

We close this paper with the final observation:

Remark 2.8. The results of Section 2 still hold for infinite dimensional spaces (and then we simply have $\mathscr{F} = \mathscr{H} \oplus \mathscr{H}$). Also, we may consider real operators on real spaces as well as complex operators on complex spaces.

References

[1] T. Ando, private communication.
[2] J.-C. Bourin, Singular values of compressions, restrictions and dilations, Linear Algebra Appl. 360 (2003) 259–272.
[3] K.R. Davidson, C^*-algebras by example, Fieds Institute Monographs 6, American Mathematical Society, 1996.
[4] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, 1991.