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Cell-like resolutions in the strongly countable $\mathbb{Z}$-dimensional case

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Abstract

Suppose that X is a nonempty compact metrizable space and $X_1 \subset X_2 \subset \cdots$ is a sequence of nonempty closed subspaces such that for each $k \in \mathbb{N}$, $\dim_{\mathbb{Z}} X_k \leqslant k < \infty$. We show that there exists a compact metrizable space Z, having closed subspaces $Z_1 \subset Z_2 \subset \cdots$, and a surjective cell-like map $\pi : Z \to X$, such that for each $k \in \mathbb{N}$,

(a) $\dim Z_k \leqslant k$,
(b) $\pi(Z_k) = X_k$, and
(c) $\pi|Z_k : Z_k \to X_k$ is a cell-like map.

Moreover, there is a sequence $A_0 \subset A_1 \subset \cdots$ of closed subspaces of Z such that for each k, $Z_k \subset A_k$, $\dim A_k \leqslant k$, $\pi|A_k : A_k \to X$ is surjective, and for $k \in \mathbb{N}$, $\pi|A_k : A_k \to X$ is a UV^{k-1}-map.

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1. Introduction

The objective of this paper is to prove the following theorem.

Theorem 1.1. *Suppose that X is a nonempty compact metrizable space and $X_1 \subset X_2 \subset \cdots$ is a sequence of nonempty closed subspaces such that for each $k \in \mathbb{N}$, $\dim_{\mathbb{Z}} X_k \leqslant k < \infty$. Then there exists a compact metrizable space Z, having closed subspaces $Z_1 \subset Z_2 \subset \cdots$, and a surjective cell-like map $\pi : Z \to X$, such that for each $k \in \mathbb{N}$,*

(a) $\dim Z_k \leqslant k$,
(b) $\pi(Z_k) = X_k$, *and*
(c) $\pi|Z_k : Z_k \to X_k$ *is a cell-like map.*

Moreover, there is a sequence $A_0 \subset A_1 \subset \cdots$ of closed subspaces of Z such that for each k, $Z_k \subset A_k$, $\dim A_k \leqslant k$, $\pi|A_k : A_k \to X$ is surjective, and for $k \in \mathbb{N}$, $\pi|A_k : A_k \to X$ is a UV^{k-1}-map.

By $\dim_{\mathbb{Z}}$ we mean integral cohomological dimension [11,5]. This is sometimes called $\mathbb{Z}$-dimension. Our use of the term "strongly countable" in the title is meant to infer a parallel with the notion of strong countable dimension, which is standard in dimension theory.

Let us recall that a map $\pi : Z \to X$ is called cell-like if each of its fibers $\pi^{-1}(x)$ is a cell-like space [2], i.e., has the shape of a point [7]. On the other hand, π is called a UV^k map if each of its fibers has property UV^k. This means that each embedding $\pi^{-1}(x) \hookrightarrow A$ into an ANR A has property UV^k: for every $0 \leqslant r \leqslant k$ and every neighborhood U of $\pi^{-1}(x)$ in A, there exists a neighborhood V of $\pi^{-1}(x)$ in U such that every map of S^r into V is nullhomotopic in U. (We actually shall use an inverse sequence characterization of this property later in the paper.) It is well known that cell-like compacta have property UV^k for all k.

The Edwards–Walsh resolution theorem [4,11] was the first in the category of our Theorem 1.1.

Resolution Theorem 1.2. *If X is a metrizable compactum and $\dim_{\mathbb{Z}} X \leqslant m < \infty$, then there exists a metrizable compactum Z with $\dim Z \leqslant m$ and a cell-like map of Z onto X.*

Later resolution theorems extended this one, [10] to the case that X is a metrizable space, and [6] to the case that X is a Hausdorff compactum. An approach to extending the result to pairs was used in [8], and some of the ideas in the latter influenced our techniques. Finally, the work in [1] (see Section 7), which provided an alternative proof of Theorem 1.2, was an important inspiration for the current research.

Theorem 1.1 would be made stronger if one could prove:

Conjecture 1.3. *Suppose that X is a nonempty compact metrizable space and $X_1 \subset X_2 \subset \cdots$ is a sequence of nonempty closed subspaces such that for each $k \in \mathbb{N}$, $\dim_{\mathbb{Z}} X_k \leqslant k < \infty$. Then there exists a compact metrizable space Z, having closed subspaces $Z_1 \subset Z_2 \subset \cdots$, and a surjective cell-like map $\pi : Z \to X$, such that for each $k \in \mathbb{N}$,*

(a) $\dim Z_k \leqslant k$, *and*
(b) $\pi^{-1}(X_k) = Z_k$.

Moreover, there is a sequence $A_0 \subset A_1 \subset \cdots$ of closed subspaces of Z such that for each k, $Z_k \subset A_k$, $\dim A_k \leqslant k$, $\pi|A_k : A_k \to X$ is surjective, and for $k \in \mathbb{N}$, $\pi|A_k : A_k \to X$ is a UV^{k-1}*-map.*

2. Background

In this paper map will always mean continuous function, and Q will designate the Hilbert cube. We are going to provide the reader with some background material that will make our later work more easily understood.

The following lemma is a form of the homotopy extension theorem with control.

Lemma 2.1. *Let $f : X \to R$ be a map of a compact polyhedron X to a space R, X_0 be a closed subpolyhedron of X, and $\mathcal{U}$ be an open cover of R. Suppose that $F : X_0 \times I \to R$ is a $\mathcal{U}$-homotopy of $f|X_0$. Then there exists a $\mathcal{U}$-homotopy $H : X \times I \to R$ of f such that $H|X_0 \times I = F : X_0 \times I \to R$.*

There are also two facts from the theory of dimension dim and $\mathbb{Z}$-cohomological dimension, $\dim_{\mathbb{Z}}$ which will be used in the sequel. First is a formulation of the existence of j-invertible maps (due to Dranishnikov [3]) which will be sufficient for our needs. Note that the j-invertibility of D_j in this lemma implies that for any metrizable compactum Y with $\dim Y \leqslant j$, and any embedding $g : Y \to Q$, there exists a map $s : Y \to M_j$ such that $D_j \circ s = g$.

Lemma 2.2. *For each $j \geqslant 0$, there exists a j-invertible map $D_j : M_j \to Q$ where M_j is a metrizable compactum and* $\dim M_j \leqslant j$.

The other one goes as follows. A proof of it may be deduced from Theorem 7.3 of [9] or Theorem 5.1 of [11].

Theorem 2.3. *Let $m \in \mathbb{N}$. Suppose that X is a metrizable compactum,* $\dim_{\mathbb{Z}} X \leqslant m$, *and $g : X \to P$ is a map to a triangulated polyhedron P. Then for every finite-dimensional compactum Y and map $h : Y \to X$, there exists a map $f : Y \to P^{(m)}$ having the property that for each $x \in Y$, if $g(h(x))$ lies in a simplex σ of P, then $f(x) \in \sigma$ also.*

We state without proof the filtration version of Theorem 2.3, which is a straightforward corollary of the approximate lifting property of cell-like maps and our main result—Theorem 1.1.

Theorem 2.4. *Suppose that X is a metric compactum and $X_1 \subset X_2 \subset \cdots \subset X_m = X$ is a sequence of closed subspaces such that for each $k \leqslant m$, $\dim_{\mathbb{Z}} X_k \leqslant k$ and $g : X \to P$ is a map to a triangulated polyhedron (P, τ). Then for every finite-dimensional compactum Y, sequence $Y_1 \subset Y_2 \subset \cdots \subset Y_m = Y$ of closed subspaces, and map $h : Y \to X$ such that $h(Y_k) \subset X_k$ for every $k \leqslant m$, there exists a map $f : Y \to P^{(m)}$ having the property that for each $k \leqslant m$, $f(Y_k) \subset P^{(k)}$, and $\operatorname{dist}(f, g \circ h) < (m+1) \cdot \operatorname{mesh}(\tau)$.*

3. Preliminary results

Let the Hilbert cube $Q = \prod_{i=1}^{\infty} I$ be endowed with the metric ρ such that if $x = (x_i)$, $y = (y_i)$, then $\rho(x, y) = \sum_{i=1}^{\infty} 2^{-i} \cdot |x_i - y_i|$. As usual, $I = [0, 1]$. For any $i \in \mathbb{N}$ it will be convenient to write $Q = I^i \times Q_i$ in factored form. In this case, any subset E of I^i will always be treated as $E \times \{0\} \subset Q$. We shall use $p_i : Q \to I^i$ for coordinate projection.

The first type of result we want is a lemma which is technical, but which will help us find certain maps and to understand their fibers. Once the correct conditions are found on the construction of said maps, then our Theorem 1.1 will follow readily.

We will use the following notation. Let x belong to a metric space X and let $\delta > 0$. Then by $\overline{N}(x, \delta)$ we shall mean the closed δ-neighborhood of x in X. Whenever (P_i, g_i^{i+1}) is an inverse sequence, $T_i \subset P_i$ and $g_i^{i+1}(T_{i+1}) \subset T_i$ for each i, then we shall write (T_i, g_i^{i+1}) for the induced inverse sequence, using the same notation for the bonding maps as long as no confusion can arise.

Lemma 3.1. *Suppose that for each $i \in \mathbb{N}$ we have selected $n_i \in \mathbb{N}$, a compact subset $P_i \subset I^{n_i}$, $0 < \delta_i$, $0 < \varepsilon_i$, and a map $g_i^{i+1} : P_{i+1} \to P_i$ so that:*

(i) *if $u, v \in Q$ and $\rho(u, v) < \varepsilon_{i+1}$, then $\rho(p_{n_i}(u), p_{n_i}(v)) < \delta_i$,*
(ii) $n_i < n_{i+1}$,
(iii) $9 \cdot 2^{-n_i} < \varepsilon_i$,
(iv) $\rho(g_i^{i+1}(x), p_{n_i}(x)) < \delta_i$ *for all $x \in P_{i+1}$,*
(v) $\delta_i < 2^{1-n_i}$, *and*
(vi) $P_{i+1} \times Q_{n_{i+1}} \subset P_i \times Q_{n_i}$.

Put $X = \bigcap_{i=1}^{\infty} P_i \times Q_{n_i}$, $\mathbf{P} = (P_i, g_i^{i+1})$, and $Z = \lim \mathbf{P}$. Then for each $z = (a_1, a_2, \ldots) \in Z \subset \prod_{i=1}^{\infty} P_i$, and associated sequence (a_i) in Q,

(a) *(a_i) is a Cauchy sequence in Q whose limit lies in X, and*
(b) *the function $\pi : Z \to X$ given by $\pi(z) = \lim_{i\to\infty}(a_i)$ is continuous.*

Fix $x \in X$ and for each $i \in N$, let $B_{x,i} = \overline{N}(p_{n_i}(x), 2\delta_i) \cap P_i$, $B^{\#}_{x,i} = \overline{N}(p_{n_i}(x), \varepsilon_i) \cap P_i$. Then,

(c) $B_{x,i} \subset B^{\#}_{x,i}$ *and* $g_i^{i+1}(B^{\#}_{x,i+1}) \subset B_{x,i}$.

If we let $\mathbf{P}_x = (B_{x,i}, g_i^{i+1})$ *and* $\mathbf{P}^{\#}_x = (B^{\#}_{x,i}, g_i^{i+1})$, *then,*

(d) $\lim \mathbf{P}_x = \lim \mathbf{P}^{\#}_x$, *and*
(e) $\pi^{-1}(x) = \lim \mathbf{P}_x$.

In addition, suppose we are given, for each $i \in \mathbb{N}$, *a closed subspace* $T_i \subset P_i$ *in such a manner that* $g_i^{i+1}(T_{i+1}) \subset T_i$. *Put* $\mathbf{T} = (T_i, g_i^{i+1})$ *and* $Z' = \lim \mathbf{T} \subset Z$. *For* $x \in X$, *let* $S_{x,i} = B_{x,i} \cap T_i$, $\mathbf{T}_x = (S_{x,i}, g_i^{i+1})$; *set* $\tilde{\pi} = \pi|Z' \to X$. *Then,*

(f) $\tilde{\pi}^{-1}(x) = \lim \mathbf{T}_x$, *and*
(g) *if* $S_{x,i} \neq \emptyset$ *for each* i, *then* $\tilde{\pi}$ *is surjective.*

Proof. Observe that our choice of metric shows that for any $i \in \mathbb{N}$ and $x \in Q$,

(1) $\rho\big(p_{n_i}(x), x\big) \leqslant 2^{-n_i}$.

The triangle inequality along with (1) and (iv) and (v) of the hypothesis show that $\rho(a_i, a_{i+1}) = \rho(g_i^{i+1}(a_{i+1}), a_{i+1}) \leqslant \rho(g_i^{i+1}(a_{i+1}), p_{n_i}(a_{i+1})) + \rho(p_{n_i}(a_{i+1}), a_{i+1}) < \delta_i + 2^{-n_i} < 2^{1-n_i} + 2^{-n_i} = 3 \cdot 2^{-n_i}$, so

(2) $\rho(a_i, a_{i+1}) < 2^{2-n_i}$,

independently of choice of $z = (a_1, a_2, \ldots) \in Z$.

From (2) and (ii), (a_i) is a Cauchy sequence. Since $X = \bigcap_{i=1}^{\infty} P_i \times Q_{n_i}$, $a_i \in P_i \subset P_i \times Q_{n_i}$ for each i, and (vi) is true, one concludes the validity of (a).

The function $\pi : Z \to X$ is continuous since (2) and (ii) show that it is the limit of the uniformly convergent sequence of maps $\pi_i|Z : Z \to I^{n_i} \subset Q$ where $\pi_i(z) = a_i$ whenever $z = (a_1, a_2, \ldots) \in Z$; this yields (b).

To prove (c), first note that (iii) and (v) imply that $2\delta_i < \varepsilon_i$ so that $B_{x,i} \subset B^{\#}_{x,i}$. Next let $u \in B^{\#}_{x,i+1}$. Observe that, $p_{n_i} \circ p_{n_{i+1}} = p_{n_i}$. Since $\rho(u, p_{n_{i+1}}(x)) < \varepsilon_{i+1}$, the triangle inequality, (iv), and (i) show that $\rho(g_i^{i+1}(u), p_{n_i}(x)) \leqslant \rho(g_i^{i+1}(u), p_{n_i}(u)) + \rho(p_{n_i}(u), p_{n_i} \circ p_{n_{i+1}}(x)) < \delta_i + \delta_i = 2\delta_i$. Hence $g_i^{i+1}(u) \in B_{x,i}$, giving us (c).

Item (d) is an immediate consequence of (c), so let us concentrate on (e). We now want to prove that $\pi^{-1}(x)$ is precisely $\lim \mathbf{P}_x$. If $(a_1, a_2, \ldots)$ is a thread of $\mathbf{P}_x$, then for $i \in \mathbb{N}$, $a_i \in B_{x,i}$, so, by applying (1) and (v), $\rho(a_i, x) \leqslant \rho(a_i, p_{n_i}(x)) + \rho(p_{n_i}(x), x) < 2\delta_i + 2^{-n_i} \leqslant 2^{2-n_i} + 2^{-n_i}$. Hence, $\lim(a_i) = x = \pi((a_i))$. Therefore, $\lim \mathbf{P}_x \subset \pi^{-1}(x)$.

Towards the opposite inclusion, suppose that a thread $(a_1, a_2, \ldots)$ of $\mathbf{P}$ lies in $\pi^{-1}(x)$. Apply the triangle inequality, the fact that (a_i) converges to x, (1), (2), (ii), and (iii) to see that when $i > 1$, $\rho(a_i, p_{n_i}(x)) \leqslant \rho(a_i, x) + \rho(x, p_{n_i}(x)) \leqslant \sum_{k=i}^{\infty} \rho(a_k, a_{k+1}) + 2^{-n_i} < \sum_{k=i}^{\infty} 2^{2-n_k} + 2^{-n_i} \leqslant 2 \cdot 2^{2-n_i} + 2^{-n_i} = 9 \cdot 2^{-n_i} < \varepsilon_i$. This puts $a_i \in B^{\#}_{x,i}$. So

$(a_1, a_2, \ldots) \in \lim \mathbf{P}_x^\# = \lim \mathbf{P}_x$, showing that $\pi^{-1}(x) \subset \lim \mathbf{P}_x$. Hence $\pi^{-1}(x) = \lim \mathbf{P}_x$ as we had proclaimed. We leave to the reader the routine proof of (f) and (g). □

Corollary 3.2. *Suppose in Lemma* 3.1 *that for each $i \in \mathbb{N}$, P_i is a subpolyhedron of I^{n_i} having triangulation τ_i with* mesh $\tau_i < \delta_i$ *and that for all $k \geqslant 0$, $g_i^{i+1}(P_{i+1}^{(k)}) \subset P_i^{(k)}$. Let*

$$T_{k,i} = P_i^{(k)}, \qquad \mathbf{T}_k = \left(T_{k,i}, g_i^{i+1}\right), \quad \textit{and} \quad A_k = \lim \mathbf{T}_k.$$

Then $A_0 \subset A_1 \subset \cdots$, and for each $k \geqslant 0$,

(a) $\dim A_k \leqslant k$ *and $\pi|A_k : A_k \to X$ is surjective.*

Assume further that for each $x \in X$ and $i \in \mathbb{N}$, there is a subpolyhedron $P_{x,i}$ of P_i triangulated by a subset of τ_i so that

$$B_{x,i} \subset P_{x,i} \subset B_{x,i}^\#$$

and the inclusion $B_{x,i} \hookrightarrow P_{x,i}$ is null homotopic. Then

(b) *$\pi : Z \to X$ is a cell-like map and*
(c) *for each $k \in \mathbb{N}$, $\pi|A_k : A_k \to X$ is a* UV^{k-1}*-map.*

If all the above statements are true, $m \geqslant 0$, and $g_i^{i+1}(P_{i+1}^{(m+1)}) \subset P_i^{(m)}$ for infinitely many i, then

(d) *$\pi|A_m : A_m \to X$ is a cell-like map.*

Proof. Surely $\dim A_k \leqslant k$. Apply Lemma 3.1 with $T_i = T_{k,i}$ and $S_{x,i} = B_{x,i}$ for each $x \in X$. Then $\mathbf{T}$ becomes $\mathbf{T}_k$ and $Z' = A_k$. The facts: mesh $\tau_i < \delta_i$, and $p_{n_i}(X) \subset P_i$ easily can be used to check that $S_{x,i} \neq \emptyset$. So (g) of Lemma 3.1 shows that (a) is true.

Part (b) comes from (c) of Lemma 3.1 along with the fact that each $B_{x,i} \hookrightarrow P_{x,i}$ is null homotopic. This shows that the bonding maps in $\mathbf{P}_x$ are null homotopic. To get at (c), suppose that $0 \leqslant r \leqslant k-1$ and $h : S^r \to S_{x,i+1} \subset B_{x,i+1} \subset P_{x,i+1}$ is a map. Then there is a nullhomotopy H of h such that $\operatorname{im} H$ is contained in $P_{x,i+1}^{(k)} \subset B_{x,i+1}^\#$. Applying the fact that $g_i^{i+1}(P_{i+1}^{(k)}) \subset P_i^{(k)}$ and (c) of Lemma 3.1, one sees that the bonding map g_i^{i+1} carries $\operatorname{im} H$ into $S_{x,i}$ and therefore g_i^{i+1} induces a null-homomorphism of π_j, $j < k$. So all fibers of $\pi|A_k$ are UV^{k-1}.

The proof of (d) is not much different from this. Let $i \in \mathbb{N}$ be chosen such that $g_i^{i+1}(P_{i+1}^{(m+1)}) \subset P_i^{(m)}$. Using the fact that $\dim S_{x,i+1} \leqslant m$, we may assume that there is a nullhomotopy H of the inclusion of $S_{x,i+1}$ into $B_{x,i+1}$ such that $\operatorname{im} H \subset P_{x,i+1}^{(m+1)}$. But then, g_i^{i+1} carries $\operatorname{im} H$ into $S_{x,i}$, providing a nullhomotopy. Since this occurs infinitely often in the inverse sequence describing the fiber of $\pi|A_m : A_m \to X$ above x, the latter map is cell-like. □

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Choose a function $\nu:\mathbb{N}\to\mathbb{N}\cup\{0\}$ such that for each $i\in\mathbb{N}$,

(i) $\nu(i)\leqslant i$, and
(ii) $\nu^{-1}(i-1)$ is infinite.

One may assume that $X\subset Q$. We are going to prove the existence of a certain sequence $\mathfrak{S}_j=\{n_j,(P_j^k)_{k\in\mathbb{N}},\varepsilon_j,\delta_j,(\tau_j^k)_{k\in\mathbb{N}},g_{j-1}^j\}$, $j=1,2,3,\ldots,$ of elements of the following nature:

$n_j\in\mathbb{N}$; $P_j^1\subset P_j^2\subset\cdots\subset P_j^\infty$ are compact polyhedra of I^{n_j}; ε_j, and $\delta_j\in\mathbb{R}^+$;

τ_j^∞ is a triangulation of P_j^∞ and $\tau_j^k=\tau_j^\infty|_{P_j^k}$ is a triangulation of P_j^k;

$g_{j-1}^j:P_j^\infty\to P_{j-1}^\infty$ is a simplicial map relative to τ_j^∞ and τ_{j-1}^∞.

We shall require that for each $j\geqslant 1$ and $k\in\mathbb{N}$

$(1)_{j,j>1}$ $n_{j-1}<n_j$;
$(2)_{j,j\geqslant1}$ if $j<k<\infty$, then $P_j^k=P_j^\infty$ and $P_j^r\subset\operatorname{int}_{I^{n_j}}P_j^{r+1}$ whenever $r\leqslant j$;
$(3)_{j,j\geqslant1}$ $X\subset\operatorname{int}_Q(P_j^\infty\times Q_{n_j})\subset N(X,\frac{2}{j})$, $p_{n_{j-1}}(P_j^k)\subset\operatorname{int}_{I^{n_{j-1}}}P_{j-1}^k$, and, whenever $k\leqslant j$, $X_k\subset\operatorname{int}_Q(P_j^k\times Q_{n_j})\subset N(X_k,\frac{2}{j})$;
$(4)_{j,j>1}$ if $u,v\in Q$ and $\rho(u,v)<\varepsilon_j$, then $\rho(p_{n_{j-1}}(u),p_{n_{j-1}}(v))<\delta_{j-1}$;
$(5)_{j,j\geqslant1}$ $9\cdot2^{-n_j}<\varepsilon_j$;
$(6)_{j,j\geqslant1}$ $\delta_j<2^{1-n_j}$;
$(7)_{j,j\geqslant1}$ $\operatorname{mesh}\tau_j^\infty<\frac{\delta_j}{2}$;
$(8)_{j,j\geqslant1}$ if $x\in X_k$, then there exists a subpolyhedron $P_{x,j}^k$ of P_j^k, which is triangulated by τ_j^k, and so that $\overline{N}(p_{n_j}(x),2\delta_j)\subset P_{x,j}^k\subset\overline{N}(p_{n_j}(x),\varepsilon_j)\cap P_j^k$ and $\overline{N}(p_{n_j}(x),2\delta_j)$ is contractible in $P_{x,j}^k$;
$(9)_{j,j>1}$ whenever $x\in P_j^\infty$ and $g_{j-1}^j(x)\in\sigma$, where σ is a simplex of τ_{j-1}^∞, then $p_{n_{j-1}}(x)$ lies in $N(\sigma,\frac{\delta_{j-1}}{2})$ (and therefore, as it follows from here and $(7)_{j-1}$, $\rho(g_j^{j-1}(x),p_{n_{j-1}}(x))<\delta_{j-1}/2+\delta_{j-1}/2=\delta_{j-1}$ for all $x\in P_j^\infty$);
$(10)_{j,j>1}$ $g_{j-1}^j(P_j^k)\subset P_{j-1}^k$; and
$(11)_{j,j>1}$ $g_{j-1}^j((P_j^{\nu(j-1)})^{(\nu(j-1)+1)})\subset(P_{j-1}^{\nu(j-1)})^{(\nu(j-1))}$.

It is easy to check that the first step of the induction ($j=1$) will be accomplished if we choose $n_1=1$, $P_1^k=I^{n_1}$ for all $1\leqslant k\leqslant\infty$, $\varepsilon_1=5$, $\delta_1=\frac{1}{3}$. Select a triangulation τ_1^∞ of P_1^∞ with $\operatorname{mesh}\tau_1^\infty<\delta_1/2$ and put $\tau_1^k=\tau_1^\infty$ when $k<\infty$.

Before proving the existence of such data, let us see why they would imply the conclusion of Theorem 1.1. For each $i\in\mathbb{N}$, let $P_i^\infty=P_i$. The conditions (i)–(v) of Lemma 3.1 are clearly true. Condition $(3)_i$ implies (vi) and that $X=\bigcap_{i=1}^\infty P_i^\infty\times Q_{n_i}$. Surely $Z=\lim(P_i,g_i^{i+1})$ is a metrizable compactum, and we get the map $\pi:Z\to X$ defined by the formula given in Lemma 3.1(b).

To see that π is surjective, let $T_i = P_i = P_i^\infty$. According to the notation of the last part of Lemma 3.1, one sees that for $x \in X$, $S_{x,i} = B_{x,i} = \overline{N}(p_{n_i}(x), 2\delta_i) \cap P_i$. From $(3)_i$ it is sure that $p_{n_i}(x) \in P_i$ and therefore $p_{n_i}(x) \in B_{x,i}$, showing that the latter is not empty. The map $\tilde{\pi}$ is the same as π in this setting, so (g) of Lemma 3.1 shows that π is surjective.

One then checks that all the hypotheses of Corollary 3.2 except for the very last one (which we do not need yet) are also satisfied. Thus (a)–(c) hold true, so π is a cell-like map, and we are assured of the existence of the subspaces $A_0 \subset A_1 \subset \cdots$, $A_k = \lim(P_i^{(k)}, g_i^{i+1})$, as required by Theorem 1.1 so that when $k \in \mathbb{N}$, $\dim A_k \leqslant k$, and π carries A_k in a UV^{k-1} manner onto X.

Fix $m \in \mathbb{N}$. In the last part of Lemma 3.1, instead of putting $T_i = P_i^\infty$, as we did previously, use $T_i = P_i^m$. It is an easy consequence of $(7)_i$ that for $x \in X_m$, $S_{x,i} \neq \emptyset$. Define Z'_m to be $\lim \mathbf{T}_m$ where $\mathbf{T}_m = (P_i^m, g_i^{i+1})$. Put $Z_m = A_m \cap Z'_m = \lim(T_{m,i}, g_i^{i+1}) = \lim((P_i^m)^{(m)}, g_i^{i+1})$. The ultimate condition of Corollary 3.2 is now operative because of (i) and (ii) of this section and (11). If we apply (d) of Corollary 3.2, then we find that $\pi|Z_m : Z_m \to X_m$ is a cell-like map. Of course, $\dim Z_m \leqslant m$ and $Z_1 \subset Z_2 \subset \cdots$, so our proof of Theorem 1.1 will be complete once we have obtained the information in conditions (1)–(11).

Assume that we have completed the construction of $\mathfrak{S}_i$ through index $i \in \mathbb{N}$. Choose an open cover $\mathcal{V}$ of P_i^∞ having the property that $\operatorname{mesh} \mathcal{V} < \frac{\delta_i}{2}$. Then select a finer open cover $\mathcal{W}$ such that any two $\mathcal{W}$-close maps of any space into P_i^∞ are $\mathcal{V}$-homotopic. Let τ be a subdivision of τ_i^∞ such that every simplex of τ lies in an element of $\mathcal{W}$. Hence,

(12) $\operatorname{mesh} \tau < \frac{\delta_i}{2}$.

If $i > 1$, choose a map $\mu : P_i^\infty \to P_i^\infty$ which is simplicial from τ to τ_i^∞ and which is a simplicial approximation to the identity on P_i^∞. Then the map $\lambda = g_{i-1}^i \circ \mu$ is simplicial from τ to τ_{i-1}^∞. If we replace g_{i-1}^i by λ and τ_i^∞ by τ, then all the conditions (1)–(11) for index i still prevail (the only ones affected being $(7)_i$–$(8)_i$ and $(11)_i$). So we assume that these replacements have been made, but continue to use g_{i-1}^i and τ_i^∞ to denote the respective bonding map and triangulation.

Using Lemma 2.2, find a $(\nu(i)+1)$-invertible map $D : M \to Q$, with $\dim M \leqslant \nu(i)+1$, and put

$$Y = D^{-1}(X_{\nu(i)}) \subset M.$$

Note that $p_{n_i} \circ D|Y$ factors through $X_{\nu(i)}$ and $\dim Y \leqslant \nu(i)+1$. Let

$$C_1 = \left(P_i^{\nu(i)}\right)^{(\nu(i))}.$$

Since $\dim_{\mathbb{Z}} X_{\nu(i)} \leqslant \nu(i)$, and $\dim Y < \infty$, Theorem 2.3, $(3)_i$, and the fact that τ_i^∞ refines $\mathcal{W}$ show that there exists a map $f : Y \to C_1$ such that f is $\mathcal{W}$-close to $p_{n_i} \circ D|Y$. We may assume that f is defined on a closed neighborhood N of Y in M, $f(N) \subset C_1$, and that

(13) f is $\mathcal{W}$-close to $p_{n_i} \circ D|N$.

There exists a neighborhood B of $X_{\nu(i)}$ in Q such that $D^{-1}(B) \subset N$.

One may find $m_0 \in \mathbb{N}$ such that if $m \geqslant m_0$, then $X \subset p_m(X) \times Q_m \subset N(X, \frac{2}{i+1})$ and for all $k \leqslant i+1$, $X_k \subset p_m(X_k) \times Q_m \subset N(X_k, \frac{2}{i+1})$. We may therefore choose $n_{i+1} > \max\{n_i, m_0\}$ and compact subpolyhedra P^k_{i+1} of $I^{n_{i+1}}$, $k \leqslant i+1$, so that $(2)_{i+1}$–$(3)_{i+1}$ are true. We may also insure that $P^{\nu(i)}_{i+1} \subset B$, i.e., that

(14) $D^{-1}(P^{\nu(i)}_{i+1}) \subset N$.

Let

$$C_2 = \left(P^{\nu(i)}_{i+1}\right)^{(\nu(i)+1)}.$$

From (14) and the fact that D is $(\nu(i)+1)$-invertible, there is a map $s : C_2 \to M$ enjoying,

(15) $D \circ s(x) = x$ for all $x \in C_2$, and
(16) $s(C_2) \subset N$.

Consider the map $\varphi : C_2 \to C_1$ given by $\varphi(x) = f(s(x))$. For such x, $p_{n_i}(x) = p_{n_i}(D(s(x)))$. From this and (13), one sees that φ and $p_{n_i}|C_2$ are $\mathcal{W}$-close, so they are $\mathcal{V}$-homotopic. From Lemma 2.1 we see that $p_{n_i}|P^\infty_{i+1} \to P^\infty_i$ is $\mathcal{V}$-homotopic to a map, which we shall denote $\varphi : P^\infty_{i+1} \to P^\infty_i$ (an extension of the previously named φ). Let us note from this that,

(17) φ is $\mathcal{V}$-close to $p_{n_i}|P^\infty_{i+1}$.

With this, the part of $(3)_{i+1}$ indicating that $p_{n_i}(P^k_{i+1}) \subset \operatorname{int}_{I^{n_i}} P^k_i$ for each $k \leqslant i+1$, and the fact that we could have chosen $\mathcal{V}$ as fine as we wish, we may assume that,

(18) $\varphi(P^k_{i+1}) \subset P^k_i$ for all $1 \leqslant k \leqslant \infty$.

There exists ε_{i+1} such that $(4)_{i+1}$–$(5)_{i+1}$ hold. Select δ_{i+1} and a triangulation τ^∞_{i+1} so that $(6)_{i+1}$–$(8)_{i+1}$ are true. Making τ^∞_{i+1} finer if necessary, choose a map $g^{i+1}_i : P^\infty_{i+1} \to P^\infty_i$ which is simplicial from τ^∞_{i+1} to τ^∞_i and which is a simplicial approximation to φ. Now it is easy to check the validity of $(9)_{i+1}$ and $(10)_{i+1}$; item $(11)_{i+1}$ is a consequence of the fact that g^{i+1}_i is a simplicial approximation of φ, and $\varphi(C_2) \subset C_1$.

At the end of the proof, for the reader's convenience we formulate the simplicial approximation theorem used above. □

Theorem 4.1. *Suppose that (P, τ_P) and (Q, τ_Q) are compact polyhedra, $P_0 < P$ and $Q_0 < Q$ compact subpolyhedra. Then for every map $f : P \to Q$, $f(P_0) \subset Q_0$, there exist a triangulation $\tau'_P < \tau_P$ and a simplicial map $f' : (P, \tau'_P) \to (Q, \tau_Q)$ such that*

(19) $\operatorname{dist}(f', f) < 2 \cdot \operatorname{mesh}(\tau_Q)$, *and*
(20) $f'(P_0) \subset Q_0$.

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