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A moving-wall boundary layer flow of a slightly rarefied gas free stream over a moving flat plate

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Abstract

In the current work, the boundary layer flow of a slightly rarefied gas free stream over a moving flat plate is presented and solved numerically. The first-order slip boundary condition is adopted in the derivation. The dimensionless velocity and shear stress profiles are plotted and discussed. A theoretical derivation of the estimated solution domain is developed, which will give a very close estimation to the exact solution domain obtained numerically. The influences of velocity slip at the wall on the velocity and shear stress are also addressed.
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Keywords: Moving flat plate; Blasius equation; Slightly rarefied gas

1. Introduction

It was Blasius who solved the boundary layer problem for a free stream past a fixed flat plate using a similarity transformation technique [1]. Klemp and Acrivos [2] studied the boundary layer flow for a free stream past a moving semi-infinite flat plate. The similarity differential equation for this problem was first derived as follows:

$$f'''(\eta) + f(\eta) f''(\eta) = 0, \tag{1}$$

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with the non-homogeneous boundary conditions

$$\eta = 0, \quad f = 0, \quad f' = -\lambda, \quad \eta = \infty, \quad f' = 1, \tag{2}$$

where f is the non-dimensional stream function $f = \frac{\Psi}{\sqrt{2U_\infty \nu x}}$, Ψ is the stream function, U_∞ is the free stream fluid velocity, ν is the fluid kinematic viscosity, η is the similarity variable defined as $\eta = y\sqrt{\frac{U_\infty}{2\nu x}}$, and λ is the ratio of the plate velocity to the free stream fluid velocity defined as $\lambda = \frac{U_w}{U_\infty}$ and it is assumed that the flat plate is moving in the reverse direction of the free stream. In the abovementioned derivation, the no-slip velocity condition is used at the wall. However, in many engineering applications in micro-scale such as in Micro-Electro-Mechanical Systems (MEMS), compared to the characteristic length of the micro-devices, the fluid behavior might be treated as a rarefied gas [3]. On the other hand, for large-scale problems with low density, the fluid is also modeled as a rarefied gas, for example, in outer space applications [4]. The behavior of a rarefied gas is determined by the Knudsen number, K_n, which is defined as the ratio of the mean free path of the fluid molecules to a characteristic length of the flow. The flow can be classified into four regimes according to the magnitude of the Knudsen number. If $K_n > 10$ it is the free molecule flow, if $10 > K_n > 0.1$ it is the transition flow, if $0.1 > K_n > 0.01$ it is the slip flow, and if $K_n < 0.01$ it is the conventional viscous flow. For the flow in the slip regime, the fluid motion still obeys the Navier–Stokes equations. The Blasius boundary layer flow with slip condition at the wall was discussed in a recent publication [5]. The boundary layer problem for a moving flat plate, however, has not been reported in the literature. Therefore, in this paper, the boundary layer flow of a slightly rarefied gas free stream over a moving flat plate will be solved and discussed.

2. Mathematical formulation

Following the same logic as previous researchers [1,2], the equation governing the laminar incompressible viscous boundary layer flow for a moving semi-infinite flat plate in a free stream of a slightly rarefied gas in the slip regime can be shown as follows using the same dimensionless variables f and η,

$$f''' + ff'' = 0 \tag{3}$$

with boundary conditions

$$f(0) = 0, \quad f'(0) = \lambda + f''(0)\gamma \quad \text{and} \quad f'(\infty) = 1 \tag{4}$$

where $\lambda = U_w/U_\infty$, which can be a positive number for $U_w > 0$ with the same direction as the free stream velocity and a negative number for $U_w < 0$ opposite to the free stream velocity, and $\gamma = \frac{U_{slip}}{U_\infty} = \left(\frac{2}{\sigma} - 1\right) K_{n,x} \mathrm{Re}_x^{1/2}$ is a dimensionless parameter with $K_{n,x} = \frac{l}{x}$, and $\mathrm{Re}_x = \frac{U_\infty x}{2\upsilon} \cdots$. In the current derivations, it is assumed that the positive x points in the direction of the free stream. The slip velocity at an isothermal wall can be obtained based on Maxell's first-order approximation as [1,3]

$$U_{slip} = \left(\frac{2}{\sigma} - 1\right) l \left.\frac{\mathrm{d}u}{\mathrm{d}y}\right|_w \tag{5}$$

where σ is the tangential momentum accommodation coefficient and l is the mean free path. The dimensionless parameter γ can also be arranged as

$$\gamma = \left(\frac{2}{\sigma} - 1\right) \eta_{99\%} \frac{l}{\delta} = \left(\frac{2}{\sigma} - 1\right) \eta_{99\%} K_{n,\delta} \tag{6}$$

where δ is the boundary layer thickness defined as $\delta = \eta_{99\%}\sqrt{\frac{2\nu x}{U_\infty}}$ in which $\eta_{99\%}$ is the value satisfying $f'(\eta_{99\%}) = 0.99$. It is seen from Eq. (6) that this non-dimensional parameter shows the relationship between the molecular mean free path to the boundary layer thickness. As pointed out by the previous researchers [5], because γ is dependent on x, the boundary layer flow is not self-similar any more. However, since the approach preserves the mass and momentum conservation, it is still valid to study the behavior of velocity and stress within the fluid. A general analytical solution of Eq. (3) is not available. In the following section, a shooting Runge–Kutta method [1] will be used to solve Eq. (3) with boundary conditions Eq. (4).

3. Results and discussions

3.1. Estimation of the solution domain for $\lambda < 1.0$

By using the Crocco variable formulation [6], which is in terms of dependent variables like the dimensionless shear stress $g(=f'')$ and non-dimensional velocity as independent variable $\theta(=f')$, Eq. (3) with the boundary conditions can be rewritten as

$$g(\theta)g''(\theta) + \theta = 0, \tag{7}$$

where $\lambda + f''(0)\gamma \le \theta \le 1$, $g'(\lambda + f''(0)\gamma) = 0$ and $g(1) = 0$. The following relationship can be simply derived as

$$\frac{-1-\lambda}{\gamma} < f''(0) < \frac{1-\lambda}{\gamma}. \tag{8}$$

From another point of view, by integrating Eq. (3) twice, substituting the boundary conditions, and rearranging the terms, we obtain

$$f' - f'(0) = f''(0)\int_0^\eta \mathrm{e}^{-\int_0^t f(\varsigma)\mathrm{d}\varsigma}\mathrm{d}t. \tag{9}$$

It is found from Eq. (9) that

$$1 - \lambda - f''(0)\gamma = f''(0)\int_0^\infty \mathrm{e}^{-\int_0^t f(\varsigma)\mathrm{d}\varsigma}\mathrm{d}t. \tag{10}$$

From Eqs. (8) and (10), we know $f''(0) > 0$. Because it is known that $f''(\eta) = f''(0)\mathrm{e}^{-\int_0^\eta f(t)\mathrm{d}t}$, we can find that $f''(\eta) > 0$. Thus, $f'(\eta)$ is a monotonically increasing function and $\lambda + f''(0)\gamma \le f'(\eta) \le 1$ for this flow configuration. Then we know that $f(\eta) < \eta$. By substituting this relation into Eq. (10), we can roughly estimate the value of $f''(0)$,

$$f''(0) < \frac{1-\lambda}{\sqrt{\pi/2}+\gamma} \tag{11}$$

which gives another estimation of $f''(0)$ resulting with further information more than Eq. (8).

It is also seen from Eq. (7) that the equation can be changed to another domain [6], say $0 \le \varepsilon \le 1 - [\lambda + f''(0)\gamma]$, by defining $\varepsilon = \theta - [\lambda + f''(0)\gamma]$. Through some manipulations, we can obtain

$$-\frac{\alpha^2}{2} + \frac{[1-(\lambda+\alpha\gamma)]^3}{6} + \frac{\lambda+\alpha\gamma}{2}[1-(\lambda+\alpha\gamma)]^2 > 0 \tag{12}$$

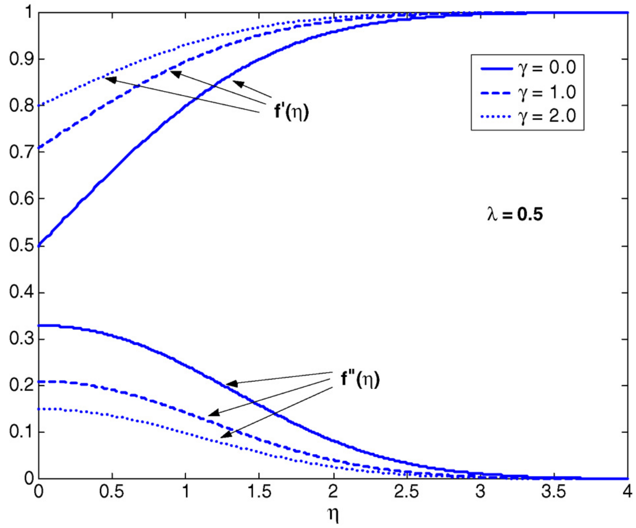

Fig. 1. $f'(\eta)$ and $f''(\eta)$ at different slip parameters for $\lambda = 0.5$.

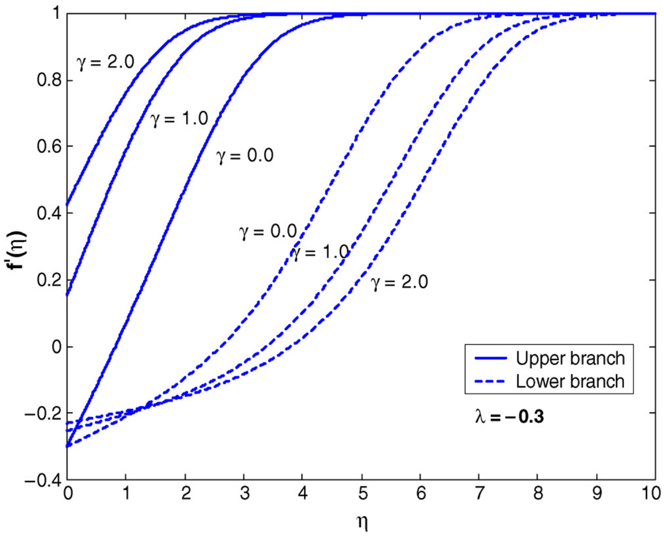

Fig. 2. $f'(\eta)$ at different slip parameters for $\lambda = -0.3$ for the two solution branches.

where $\alpha = f''(0)$. For each γ, the solution domain can be estimated from Eq. (12) combined with Eq. (11).

3.2. Numerical solutions

Since there is no analytical solution of Eq. (3) with the associated boundary conditions Eq. (4), a shooting method will be used to convert the boundary value problem into an initial value problem. A fourth-order Runge–Kutta integration scheme will be adopted to solve the relevant initial value problem.

The results of $f''(\eta)$ and $f'(\eta)$ at $\lambda = 0.5$ and $\lambda = -0.3$ under different γ are shown in Figs. 1–3. It is found that for $0 < \lambda < 1$, say $\lambda = 0.5$, there is only one solution. It is obvious that the slip velocity at

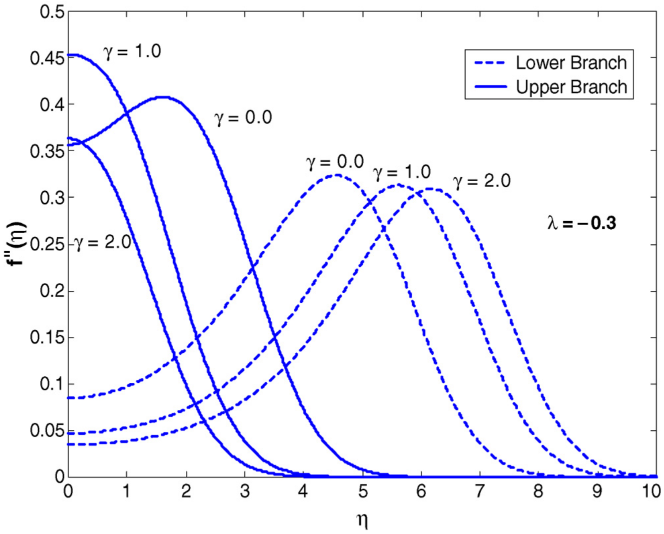

Fig. 3. $f''(\eta)$ at different slip parameters for $\lambda = -0.3$ for the two solution branches.

the wall is increasing with the increase of slip parameter. The shear stress in the fluid is decreasing with increasing slip parameters. Also, the shear stress is monotonically decreasing with the distance from the wall increasing. However, different behavior appears when $\lambda < 0$. There are two solutions in this range as found in the literature for $\gamma = 0$ [2,6]. It is seen from Fig. 2 that the velocity varying behavior is greatly different for the two solution branches. For both branches, the wall slip velocity has a similar trend to the case for $0 < \lambda < 1$. However, the velocity variations in the fluid are very different for the two branches. For the upper branch, the velocity in the fluid is increasing with increment of γ for the whole domain. For the lower branch, however, there will be an intercept among the velocity profiles for different γ. The shear stress in the fluid, however, is totally different from the cases of $0 < \lambda < 1$. There is always a smaller shear stress near the wall for the lower solution branch. The maximum shear stress for the lower solution branch always occurs in the fluid at a certain distance from the wall. But for the upper solution branch, the maximum shear stress can appear on the wall, close to the wall, or at a certain distance from the wall, which is dependent on the slip parameter γ. As shown in Fig. 2 for the velocity and Fig. 3 for the shear stress, there will be two solutions for $\lambda_C < \lambda < 0$, where λ_C is a critical value having a single solution for this problem. For $\gamma = 0$, the estimated and the exact value of λ_C have been discussed [2,6]. A rough estimation of λ_C can be obtained through Eq. (12) by numerical method for arbitrary γ. Exact values of λ_C have to be determined by numerical methods. Some examples are shown in Fig. 4 for certain slip parameters. It is also seen that the value of α is always less than a certain number, say $\alpha_C \approx 0.4696$, which is the wall shear stress for the celebrated Blasius boundary layer solution for a fixed wall in a free stream [1]. The solution domain and the estimated domain, namely Eq. (12) and (11), are plotted side by side in Fig. 5. It is found that Eq. (12) does give a certain range of α. Further estimation of the range of α can be analyzed as follows. Defining $\beta = 1 - (\lambda + \alpha\gamma)$, and taking the equality of Eq. (12) yield

$$\frac{\beta^3}{3} - \frac{\beta^2}{2} + \frac{\alpha^2}{2} = 0. \tag{13}$$

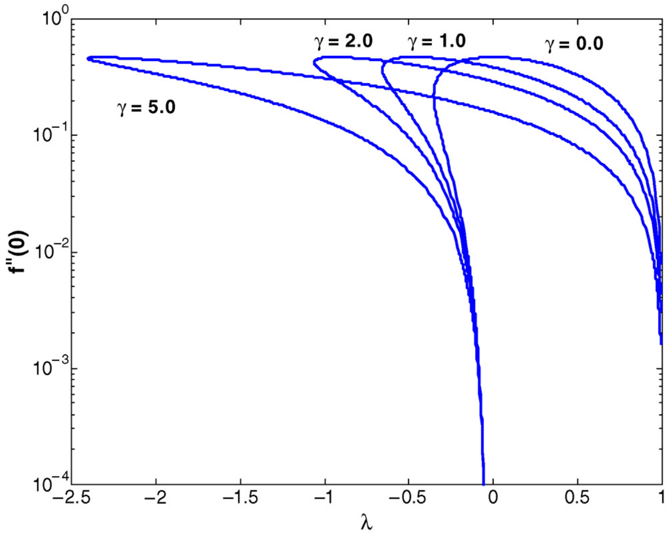

Fig. 4. Relationship between $f''(0)$ and λ for different slip parameters γ.

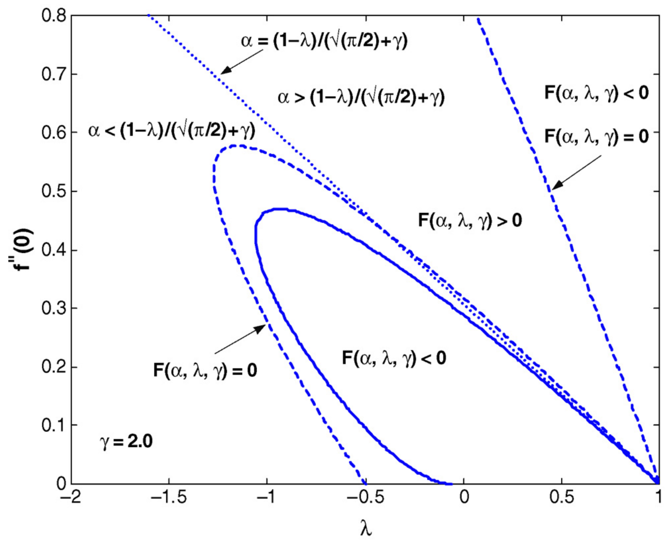

Fig. 5. The exact solution domain (the solid line) and the theoretical estimation (the dotted line and dashed line).

Following the rules in Ref. [7], Eq. (13) can be changed into

$$\phi^3 - \frac{3}{4}\phi + \frac{3\alpha^2}{2} - \frac{1}{4} = 0 \qquad (14)$$

where $\phi = \beta - \frac{1}{2}$. It is seen from Fig. 5 that when α is larger than a certain value, there is only one real solution for Eq. (14). Therefore the estimated critical value can be obtained based on the following condition [7]

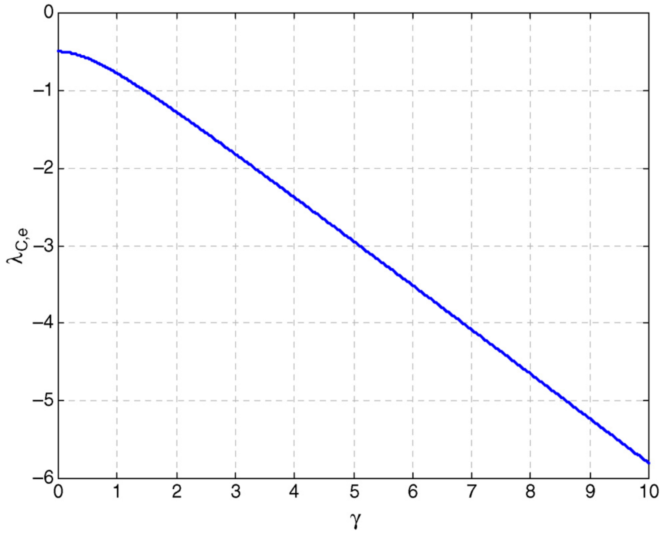

Fig. 6. The relationship between $\lambda_{C,e}$ and the slip parameter γ.

$$\left(\frac{3\alpha^2}{2} - \frac{1}{4}\right)^2 + \frac{4}{27}\left(-\frac{3}{4}\right)^3 = 0. \tag{15}$$

Solving the above equation yields

$$\alpha_{C,e} = \frac{\sqrt{3}}{3}. \tag{16}$$

The same logic can also be applied to obtain the relationship between $\lambda_{C,e}$ and γ, and the equation is given as follows

$$\lambda_{C,e} = \frac{1 - \sqrt{1 + 9\gamma^2 + 27\gamma^4 + 27\gamma^6}}{9\gamma^2}. \tag{17}$$

The results are shown in Fig. 6 for different γ. It is found that with the increase of γ the solution domain will be expanded in the negative λ direction. For sufficiently large γ, the relationship between $\lambda_{C,e}$ and γ will be asymptotic to linear dependence as

$$\lambda_{C,e} \approx -\frac{\sqrt{3}}{3}\gamma. \tag{18}$$

The solution also exists for $\lambda \geq 1$. When $\lambda = 1$, there is a trivial solution with $f'(\eta) = 1$ and $f''(\eta) = 0$ for arbitrary slip parameters. There is no slip velocity at the wall for this trivial solution. For $\lambda > 1$, the solution is, to some extent, similar to the solution of the continuously stretching surface problem with constant stretching speed as discussed before [8] for no-slip flow. A discussion on slip flow past a stretching surface can be found in a recent publication [9]. However, the situation for the continuously stretching surface is not exactly the same as the current situation. Some examples are illustrated in Fig. 7 for velocity and shear stress profiles at $\lambda = 1.5$. It is seen that the fluid velocity will decrease with the increase of the slip parameter for a certain wall velocity. The shear stresses in the

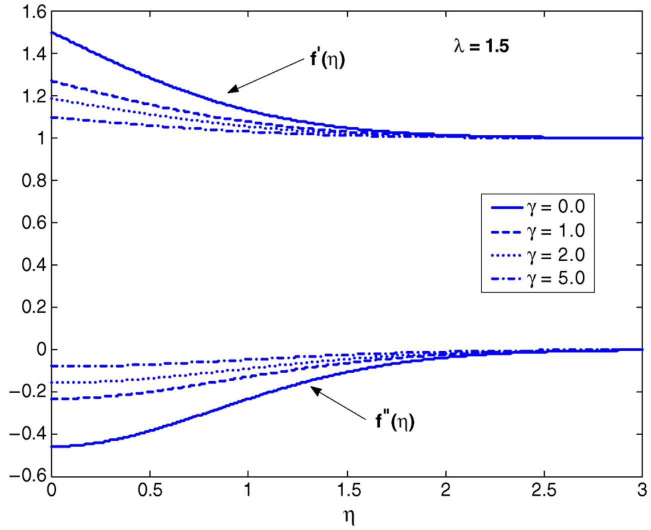

Fig. 7. $f'(\eta)$ and $f''(\eta)$ at different slip parameters for $\lambda = 1.5$.

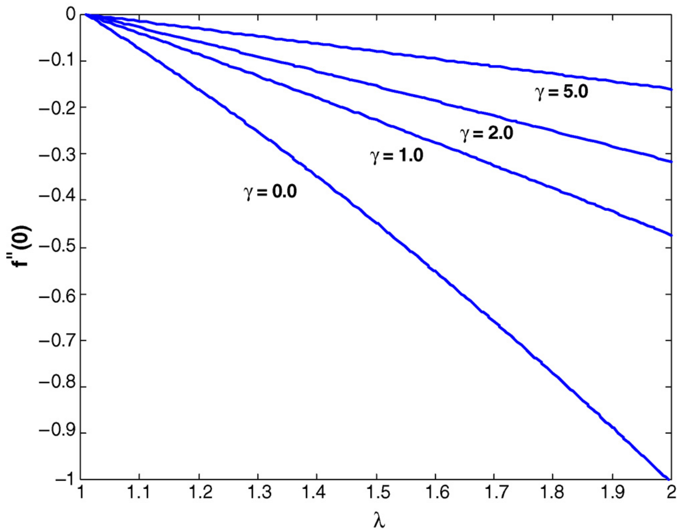

Fig. 8. Relationship between $f''(0)$ and λ for different slip parameters γ for $\lambda > 1.0$.

fluid also decrease with increment of the slip parameter. The solution domains are also computed and shown in Fig. 8. It is found that there is only one solution for $\lambda > 1$. The influence of slip parameters on the wall stresses is somehow similar to the cases of no-slip flow with blowing [8], but they are different in nature.

As stated in the literatures [3,4,10], under certain circumstances, slip condition will give more satisfactory results than no-slip condition for slightly rarefied gas with low density in outer space or micro-scale flow in micro-devices. Based on the above discussion, it is seen that the slip condition does change the velocity and stress field within the fluid. For positive wall motion, the wall slip will make the stress decrease, which is useful for both drag and heat loss reduction. However, for negative wall movement, slip condition will give greatly different results from no-slip conditions. Wall stress will change significantly with slip parameters. By choosing different slip parameters, the stress can be reduced or enhanced. Because of the analogy between the wall stress and heat flux, heat transfer can also be adjusted by changing slip parameter. Possible applications of the current results might be in the wall stress control and heat transfer optimization of MEMS for micro-scale problem [3] or the flow control for large-scale problems in the outer space [4,10].

4. Conclusion

In this paper, the governing equation for the momentum boundary layer of a slightly rarefied gas free stream over a moving flat plate is derived using the first-order slip condition at the wall. The equation is simplified by defining a dimensionless variable and solved numerically to obtain the velocity and shear stress profiles. The solution domain is estimated theoretically. The ranges are obtained in exact forms. It is found from the results that the wall slip velocity has a great influence on the velocity and shear stress distributions within the fluid and on the wall.

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