Discrete Mathematics 34 (1981) 301–310
North-Holland Publishing Company

THE STRUCTURE AND GEOMETRY OF 4×4 PANDIAGONAL MATRICES

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Received 12 June 1979

The 4×4 pandiagonal matrices are tesselated by a group of transformations with two generators, which are analogous to a rotation and a rotatory inversion, acting on a single vector. These matrices have equivalence classes that are tesselated by a subgroup associated with a triangular tesselation of the sphere. The relationship of various subgroups to the permutation structure of pandiagonal matrices is studied.

The application of the techniques of linear algebra and group theory to elucidate the structure of 4×4 pandiagonal matrices gives a systematic development of their attractive properties. The usual definition of a 4×4 magic matrix is a square array of the numbers $1, 2, 3, \ldots, 16$ such that each row, each column, and the two main diagonals all have the same sum, namely the magic constant, which in this case is 34. The array can be represented as a matrix

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \tag{1}$$

where $a+b+c+d=34$, etc. A magic matrix is called a pandiagonal matrix if all the broken diagonals also have the same sum; that is, $e+b+o+l=34$, etc.

Evidently, the 4×4 pandiagonal matrices are a class of permutations on $\{1, \ldots, 16\}$ that satisfy certain linear arithmetic conditions. The 4×4 pandiagonal matrices are small enough so that they can be dealt with by hand. The use of linear algebra gives their coarse structure and their division into natural equivalence classes. The fine structure of the equivalence classes is then obtained by the use of methods of group theory. The 4×4 pandiagonal matrices will be seen to be generated by a group whose significant action is given essentially by a group of order 48 tesselating a single vector. This group of order 48 is isomorphic to the triangle group $T^*(2, 3, 4)$ used in tesselating the sphere with 48 congruent triangles. The geometrical nature of the group that tiles the pandiagonal matrices is indicated in that it may be generated by two generators which can be characterized as a "rotatory inversion" and a "rotation". Further, this group makes clear that certain pairs of numbers in these matrices are inseparably linked.

An elegant application of group theory to the enumeration of pandiagonal magic matrices is in the paper of Rosser and Walker [1].

1. Linear algebra

There are sixteen defining equations for the 4×4 pandiagonal matrix (1) such as

$$a+b+c+d=34, \qquad e+f+g+h=34, \quad \text{etc.}$$

Since the magic constant $34=2\cdot 17$, the defining equations of the pandiagonal matrix become homogeneous linear equations in the vector space $\{(a,\ldots,p)\}$ of 16-tuples mod 17 and mod 2. Thus, we have the sixteen homogeneous equations

$$a+b+c+d=0, \qquad e+f+g+h=0, \quad \text{etc. (mod 17 or mod 2).}$$

Let us call a 16-tuple vector solution written in the matrix form (1) an algebraic pandiagonal matrix (abbreviated to a.p.m.).

The coefficients of the equations of this homogeneous system may be written as a 16×16 matrix. The standard linear algebraic technique of finding the reduced row echelon form (mod 17) may be applied to this matrix. The result is that the a.p.m.'s are a four-dimensional solution space of the homogeneous system of equations in the space of 16-tuples (written as 4×4 matrices) of the form:

$$(l,n,o,p)=\begin{bmatrix} l+o+p & -l & l-n-o & -l+n-p \\ -o & -p & n+o+p & -n \\ -l+n+o & l-n+p & -l-o-p & l \\ -n-o-p & n & o & p \end{bmatrix} \pmod{17} \tag{2}$$

From this form, it is readily visible that every two alternate elements on any diagonal are additive inverses. Also, the sum of the elements in every 2×2 matrix equals zero (mod 17). The sum of the elements in opposite quadrants of any magic matrix are equal (eg, in that case, the sum of rows 1 and 2 equals the sum of columns 1 and 2, and the elements of the upper left quadrant cancel; thus, the sum of the elements of the lower left equals that of the upper right quadrant.) Therefore, in a 4×4 pandiagonal matrix an element and its inverse mod 17 are in opposite quadrants, so a quadrant sums to 34.

We now find the patterns of the even and odd integers in a pandiagonal matrix (abbreviated as P.M.). For pandiagonal matrices A and B, we define $A \sim B$ to mean $A \equiv B$ (mod 2). It is easy to verify that '$\sim$' is an equivalence relation and so divides the P.M.'s into equivalence classes. Since $a+b=17$ becomes $a+b\equiv 1$

(mod 2), we have from (1), for a P.M. of the form of equation (2):

$$\begin{bmatrix} a & l+1 & c & d \\ o+1 & p+1 & g & n+1 \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \pmod 2$$

For a P.M., the rows and columns sum to zero mod 2, and each 2×2 quadrant sums to 34; hence, it sums to zero mod 2. Thus, we have that a P.M. of the form (2) is of the form:

$$[l, n, o, p] = \begin{bmatrix} l+o+p+1 & l+1 & l+n+o+1 & l+n+p+1 \\ o+1 & p+1 & n+o+p+1 & n+1 \\ l+n+o & l+n+p & l+o+p & l \\ n+o+p & n & o & p \end{bmatrix} \pmod 2 \tag{3}$$

By (3), the maximum number of equivalence classes is $2^4 = 16$. However, there are exactly 8 of these such that 2 rows or 2 columns have all zeros. In the corresponding P.M.'s, these two rows, say, would contain all the evens, whose sum, $2 + \cdots + 16 = 72$, is not a multiple of 34. Thus, these eight are impossible. Therefore, in every P.M., there are two even and two odd numbers in each row, column, and diagonal. More precisely,

Theorem 1. *For the pandiagonal matrices there are only eight* mod 2 *equivalence classes, given by* $[l, n, o, p] = [1, 1, 0, 0]$, $[1, 0, 1, 0]$, $[0, 1, 0, 1]$, $[0, 0, 1, 1]$, $[1, 1, 1, 0]$, $[1, 1, 0, 1]$, $[0, 0, 1, 0]$, *and* $[0, 0, 0, 1]$. *Each equivalence class contains the same number of elements.*

Proof. The transformation C of moving the left column to the right, and the reflection D across the main diagonal of a, f, k, p, preserve the pandiagonal property. For (3), these transformations can be written

$$C[l, n, o, p] = [l+n+o, o, p, n+o+p]$$

for the column motion (left to right), and

$$D[l, n, o, p] = [o, n+1, l, p]$$

for the reflection across the diagonal. We see that

$$\begin{aligned} C[1,1,0,0] &= [0,0,0,1], & D[1,1,0,0] &= [0,0,1,0], \\ C[0,0,0,1] &= [0,0,1,1], & D[0,0,0,1] &= [0,1,0,1], \\ C[0,0,1,1] &= [1,1,1,0], & D[0,0,1,1] &= [1,1,0,1], \\ C[1,1,1,0] &= [1,1,0,0], & D[1,1,1,0] &= [1,0,1,0]. \end{aligned}$$

Thus, using C and D we go from one equivalence class to another; furthermore, this implies that two equivalence classes have the same number of elements. That these classes occur can be shown by checking the following matrix.

$$\begin{bmatrix} 14 & 4 & 5 & 11 \\ 1 & 15 & 10 & 8 \\ 12 & 6 & 3 & 13 \\ 7 & 9 & 16 & 2 \end{bmatrix} \tag{4}$$

The other eight possibilities are given by

$$\begin{aligned} &C[1,0,0,1]=[1,0,1,1], &&D[1,0,0,1]=[0,1,1,1],\\ &C[1,0,1,1]=[0,1,1,0], &&D[1,0,1,1]=[1,1,1,1],\\ &C[0,1,1,0]=[0,1,0,0], &&D[0,1,1,0]=[1,0,0,0],\\ &C[0,1,0,0]=[1,0,0,1], &&D[0,1,0,0]=[0,0,0,0]. \end{aligned}$$

However, $[l, n, o, p]=[1,0,0,1]$ would represent a pandiagonal matrix with all zeros in columns 2 and 3. The sum of these columns would be $2+\cdots+16=72$, which is not a multiple of 34 or 17. This would contradict the assumption that the parent matrix should be pandiagonal.

Corollary 2. *An a.p.m. whose entries are the numbers* $1,2,3,\ldots,16$ *whose corresponding* mod 2 *matrix is one of the equivalence classes for pandiagonal matrices is pandiagonal.*

Proof. Since the diagonals of a.p.m.'s sum to 34 because of inverse relations, it suffices to check the row and column sums. Each row or column has two odd and two even numbers. Hence, the sum is an even multiple of 17. But, for the numbers, $1,2,3,\ldots,16$ that multiple must be 34.

Corollary 3. *In a P.M., each* 2×2 *submatrix has sum* 34.

Proof. Each P.M. mod 2 is in one of the eight equivalence classes. Each 2×2 submatrix has 2 zeros and 2 ones (e.g. see (5) below). Therefore, each 2×2 submatrix of the P.M. has two even and two odd numbers. From (2), each 2×2 submatrix of a P.M. sums to zero mod 17. However, that sum is forced to be 34.

We now examine in detail the equivalence class denoted by $[1,1,0,0]$; that is, all P.M.'s that are equal mod 2 to the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \pmod 2 \tag{5}$$

The corresponding pandiagonal matrix must have the even numbers $2, \ldots, 16$ in the '0' positions. In (2), $o \equiv 2j$, $p \equiv 2k$, $l+o+p \equiv 2r$, $n+o+p \equiv 2s$ (mod 17) where $j, k, r, s \in \{1, \ldots, 8\}$. Thus, $l \equiv 2(r-j-k)$ and $n \equiv 2(s-j-k)$ (mod 17) and consequently, (2) becomes

$$(j, k, r, s) = 2 \begin{bmatrix} r & j+k-r & r-s-j & s-r-k \\ -j & -k & s & j+k-s \\ s+j-r & k+r-s & -r & r-j-k \\ -s & s-j-k & j & k \end{bmatrix} \pmod{17} \qquad (6)$$

which we denote by (j, k, r, s).

Theorem 4. *A magic matrix whose* mod 2 *form is given by* (5) *is a pandiagonal matrix in the class* $[1, 1, 0, 0]$ *if and only if it is of the form* (6) *and* $j, k, r, s \in \{1, 2, \ldots, 8\}$ *where* j, k, r, s *are distinct and satisfy*

$$s < j+k, \quad s < k+r, \quad r < j+k, \quad r < s+j \qquad (7)$$

and none *of the following equalities holds:*

$$\begin{aligned} &2r = s+j, \quad 2r = j+k, \quad 2s = j+k, \quad 2s = k+r, \\ &r+s = j+k, \quad s+j = r+k, \quad 2s+j = 2r+k. \end{aligned} \qquad (8)$$

Proof. The inequalities (7) reflect the fact that the even spots of a member of the class $[1, 1, 0, 0]$ must be in the form (6) and need to be positive; that these entries must be unequal if the matrix is pandiagonal is the condition that equalities (8) should not hold.

If (7) holds and none of the equalities in (8) holds in (6) for j, k, r, s, then the matrix is pandiagonal since the magic matrix has the entries $1, \ldots, 16$ and in each row and column there are two even and two odd numbers. Each row or column has a sum that is zero mod 17. However, that sum in this case must be an even number and a multiple of 17; so that 34 is the only possibility. That the diagonals sum to 34 follows from the fact that each diagonal consists of 2 numbers and their inverses mod 17. So, the matrix must be pandiagonal.

2. Group theory

The group of transformations that leave the equivalence class $[1, 1, 0, 0]$ invariant gives the fine permutation structure of these pandiagonal matrices. In fact, there is one pandiagonal tile that suffices to give a tiling to the class $[1, 1, 0, 0]$.

Theorem 5. *The following transformations take a pandiagonal matrix in the class* $[1, 1, 0, 0]$ *into another pandiagonal matrix in the class* $[1, 1, 0, 0]$.

$$A(j, k, r, s) = (k, j, r+k-s, r),$$
$$B(j, k, r, s) = (j+k-s, s, k+r-s, r).$$

Furthermore, the elements of the 4×4 *pandiagonal matrices in this equivalence class are obtained by the action of the group generated by* A *and* B *on the single 4-tuple,* $(j, k, r, s) = (8, 1, 7, 5)$.

Proof. It is easy to verify that

$$A(j, k, r, s) = 2\begin{bmatrix} r+k-s & j-r+s & -s & s-j-k \\ -k & -j & r & j+k-4 \\ s & j+k-s & s-r-k & r-s-j \\ -r & r-k-j & k & j \end{bmatrix}$$

has the invariance property of mapping a P.M. in the class $[1, 1, 0, 0]$ into a P.M. in the class $[1, 1, 0, 0]$ since the conditions of Theorem 4 holding imply that the entries $\hat{j}, \hat{k}, \hat{r}, \hat{s}$ in the transformed matrix also satisfy the conditions, where $\hat{j} = k$, $\hat{k} = j$, $\hat{r} = r+k-s$, $\hat{s} = r$. Similarly, B can be seen to have the invariance property.

In a P.M. in the equivalence class $[1, 0, 0, 0]$ in the form (6), B permutes the pairs $\{a, b\}, \{g, h\}, \{o, p\}$, and $\{i, j\}$ cyclically; thus, we can move '1' into the j or k spot of (6). If 8 is not in the j or the k spot, then by the action of A, 8 can be brought into the r or s spot. However, in this case, it would force $j+k-r$ or $j+k-s$ to be zero or negative, which would be a contradiction. Now, by the use of A, 7 can be made to appear in the r spot. We may assume that $(j, k, r, s) = (1, 8, 7, s)$ or $(8, 1, 7, s)$. However, $(1, 8, 7, s)$ is impossible in view of the inequalities (7); we must have $s+1>4$, so $s>4$, but this is impossible since $r=7$. Consequently, we must have $(8, 1, 7, s)$. Since none of the equalities in (8) holds, the following choices for s are impossible; $s=2$, $s=4$, $s=6$. For example, since $r=7$, $s=2$ would imply $r+s=j+k=9$. The other values are excluded using inequalities $2s \neq k+r$ and $2r \neq s+j$.

Thus, there are two choices, $(8, 1, 7, 3)$ and $(8, 1, 7, 5)$. Consider $F = A^2B^2A^5B^3$, which is clearly in the group generated by A and B. Thus,

$$F(j, k, r, s) = (j, k, r, k+r-s);$$

hence, $F^2 = I$ and $F(8, 1, 7, 3) = (8, 1, 7, 5)$.

Consequently, every P.M. in the class $[1, 1, 0, 0]$ can be brought into the form $(8, 1, 7, 5)$ by the action of the group generated by A and B. The P.M. corresponding to $(8, 1, 7, 5)$ is exhibited above in (4).

Theorem 6. *The transformations* A *and* B *of Theorem 5 have the following properties*:

$$A^6 = B^4 = (BA^2)^2 = I \quad \text{and} \quad A^3B = BA^3. \tag{9}$$

The group generated by A and B has order 48. Furthermore, the group generated by $\{A^2, B\}$ is isomorphic to S_4, the group of permutations on 4 elements. Also, the group generated by A and B is isomorphic to the triangle group $T^*(2,3,4)$ (see [3, p. 75]). *The order of this group, and the number of P.M.'s in the class* [1, 1, 0, 0], *is* 48. *Hence, the number of P.M.'s is* 384.

Proof. That the group generated by A and B has 48 elements is a result of the generating properties (9). Let G be the group generated by $C = A^2$ and B. From (9), we have $C^3 = B^4 = (BC)^2 = I$. It is well-known that a group with these generator conditions is isomorphic to S_4 (eg., Lederman [2, p. 124]). The left cosets of G in the group generated by A and B are just G and AG because $A^{2n}G = G$ and $A^{2n+1}G = A(A^{2n}G) = AG$, together with $A^3B = BA^3$, implies that if α_n is odd,

$$B^{\beta_1}A^{\alpha_1}\cdots B^{\beta_n}A^{\alpha_n}G = B^{\beta_1}A^{\alpha_1}\cdots B^{\beta_n}A^3G = A^3(B^{\beta_1}A^{\alpha_1}\cdots A^{\alpha_{n-1}})G.$$

Therefore, the group generated by A and B has order 48 because its subgroup G has order 24 and index 2.

The number of P.M.'s in [1, 1, 0, 0] is 48, since each element of the group gives a different P.M. Since each of the eight equivalence classes has 48 P.M.'s, the total number of P.M.'s is 384 (see also Rosser and Walker [1]).

The triangle group $T^*(2,3,4)$ generated by repeated reflections in the sides of the spherical triangle Δ with angles $\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi$ produces a tesselation of the sphere by 48 congruent replicas of Δ. The reflections L, M, N in the sides of Δ generate $T^*(2,3,4)$ with the relations

$$L^2 = M^2 = N^2 = I, \qquad (LM)^4 = (MN)^3 = (NL)^2 = I.$$

(See Magnus [3, p. 75]). The following relationships are not difficult to verify. Taking $A = LMN, B = LM$, then $L = AB^2A^2, M = B^3AB^2A^2, N = B^3A$, and $A^6 = B^4 = I$ with $A^3B = BA^3$, which implies that A, B suffice to generate $T^*(2,3,4)$. Therefore, $T^*(2,3,4)$ is isomorphic to a factor group of the above group of invariant transformations on [1, 1, 0, 0]. Since the orders of these two groups are the same, they are isomorphic.

It is revealing to write the action of the transformations A and B in the form (1).

$$A = \begin{bmatrix} j & i & m & n \\ f & e & a & b \\ g & h & d & c \\ k & l & p & o \end{bmatrix} \quad B = \begin{bmatrix} j & i & e & f \\ n & m & a & b \\ o & p & d & c \\ k & l & h & g \end{bmatrix} \tag{10}$$

Note that A and B are invariant not only on [1, 1, 0, 0] but on all the pandiagonal matrices. Now, $(LMN)^3$ represents the central inversion of the sphere (identification of antipodal points) ([3] or [4]), whereas A^3, its isomorph, interchanges columns 1 and 2, and 3 and 4, respectively in (1). Also, LMN represents a glide-reflection [4, p. 91], and A "glides" and reflects cyclically (ij), (ab), (gh) and reflects (op). These analogies appear to be very close.

Inspection of (10) shows that if it sufficies to write A and B as the permutations

$$A=\begin{pmatrix} a & b & g & h & i & j & o & p \\ j & i & a & b & g & h & p & o \end{pmatrix}, \qquad B=\begin{pmatrix} a & b & g & h & i & j & o & p \\ j & i & a & b & o & p & h & g \end{pmatrix}$$

since 8 elements in non-inverse places determine the pandiagonal matrix. Writing these in the usual form for S_8, we have

$$A\cong\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 1 & 2 & 3 & 4 & 8 & 7 \end{pmatrix}, \qquad B\cong\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 1 & 2 & 7 & 8 & 4 & 3 \end{pmatrix}. \tag{11}$$

We now give a second interesting direct proof that the group generated by A and B is of order 48, using these permutations.

First, A and B generate a subgroup T of S_8. A and B are permutations on the pairs $\{1,2\},\ldots,\{7,8\}$ such that the number of pairs $\{a,b\}$ in the permutation where $a<b$ is even and the number of 'switched' pairs is even. Furthermore, A and B are in the subgroup of S_8 that leaves the polynomial

$$(x_1-x_2)(x_3-x_4)(x_5-x_6)(x_7-x_8)$$

invariant. Consequently, if a permutation switches an odd number of pairs around, it alters this polynomial and, therefore, cannot be in the subgroup generated by A and B.

If a permutation in T leaves 8 fixed, it also leaves 7 fixed. Let W_{78} be the subgroup of T that leaves 7 and 8 fixed. since

$$8\xrightarrow[A]{}7\xrightarrow[B]{}4\xrightarrow[A]{}2\xrightarrow[A]{}5\xrightarrow[A]{}3\xrightarrow[A]{}1\xrightarrow[A]{}6\xrightarrow[B]{}8,$$

we see that T is a transitive group, where, in the diagram, $i \xrightarrow[x]{} j$ means that the motion x takes i into j. Thus, we have the coset decomposition for T:

$$T=W_{78}+x_1W_{78}+\cdots x_7W_{78}. \tag{12}$$

Thus the index of W_{78} in T is 8.

The subgroup of W_{78} that leaves 1 fixed also leaves 2 fixed. Denote this subgroup by Z_{12}. Because of the fact that pairs and pair switches occur an even number of times, there are only four possibilities for membership in Z_{12}. Besides the identity, the candidates are $F=A^2B^2A^5B^3$ and P, Q given by

$$F=\begin{pmatrix} 12 & 34 & 56 & 78 \\ 12 & 65 & 43 & 78 \end{pmatrix}, \quad P=\begin{pmatrix} 12 & 34 & 56 & 78 \\ 12 & 56 & 34 & 78 \end{pmatrix}, \quad Q=\begin{pmatrix} 12 & 34 & 56 & 78 \\ 12 & 43 & 65 & 78 \end{pmatrix}$$

It is easy to verify that the motions on pandiagonal matrices that correspond to P and Q do not preserve pandiagonality (e.g. use matrix (4)). Hence, P and Q cannot be in Z_{12}, so $Z_{12}=\{I, F\}$. We now show that the only elements left in W_{78}

are:

$$A^2=\begin{pmatrix}1&2&3&4&5&6&7&8\\4&3&6&5&1&2&7&8\end{pmatrix},\qquad A^4=\begin{pmatrix}1&2&3&4&5&6&7&8\\5&6&2&1&4&3&7&8\end{pmatrix},$$

$$A^2F=\begin{pmatrix}1&2&3&4&5&6&7&8\\4&3&2&1&5&6&7&8\end{pmatrix},\qquad A^4F=\begin{pmatrix}1&2&3&4&5&6&7&8\\5&6&3&4&1&2&7&8\end{pmatrix}.$$

It can be verified that elements with the cycle (12) do not exist in this subgroup since they destroy pandiagonality. Next, there cannot be any element ε in the group such that $\varepsilon:1\to 3$ and $\varepsilon:2\to 4$, since $A^4\varepsilon$ will send $1\to 3\to 2$ and $2\to 4\to 1$; also, there cannot be any δ such that $\delta:1\to 6$ and $\delta:2\to 5$, since $A^2\delta$ takes $1\to 2$ and $2\to 1$. Consequently, there are only two possibilities: $1\to 4$ and $1\to 5$. An element that takes $1\to 4$ must be A^2F if it takes $4\to 1$ and A^2 if it takes $4\to 5$; if it takes $4\to 2$, $4\to 3$, or $4\to 6$, then the square of the element will map $1\to 2$, $1\to 3$, or $1\to 6$, respectively, so these cases are impossible. A similar arguement shows that A^4 and A^4F are the only possibilities sending $1\to 5$.

Therefore, W_{78} consists of $\{I, F, A^2, A^4, A^2F, A^4F\}$ and the order of W_{78} is 6. Hence, the order of T is 48.

Theorems 1 and 6 imply that the tesselation of the P.M.'s from one basic pandiagonal tile is possible by the actions from the group generated by A, B, C, D. What is interesting is that A and C suffice to generate this group.

Theorem 7. *The group generated by the "rotatory inversion" A and the "cyclic rotation" C has order* 384 *and tesselates the* 4×4 *pandiagonal matrices.*

Proof. It is easy to verify that we may write the diagonal reflection $D=(AC)^2$ and the "rotation" $B=A^3C^3(AC)^2CAC$. Thus, the group generated by A, B, C, D is the same as that generated by A and C. Now, Theorems 1 and 6 imply that the order is 384.

However, a direct coset proof that the group generated by A and C has order 384 is interesing in that it shows something about the structure of the 4×4 P.M.'s. The proof is divided into steps, and the equalities involved are readily verified.

(i) The transformation $R=DCD$ cyclically moves the top row to the bottom row. Since $R^4=C^4=I$ and $RC=CR$, the subgroup generated by R and C consists of elements of the form $R^\alpha C^\beta$ and has order 16.

(ii) A^3 or D can be moved to the left in any product with R and C, in consequence of the equalities: (a) $CA^3=A^3C^3$, (b) $RA^3=A^3R$, (c) $CD=DR$, (d) $RD=DC$.

(iii) Let $\mathcal{B}$ be subgroup generated by A^3, D, R, C. From (i) and (ii), any product of these four elements may be written as $ZR^\alpha C^\beta$ where Z is in the subgroup generated by A^3 and D. Since $(A^3)^2=D^2=(A^3D)^4=I$, this subgroup is isomorphic to the dihedral group of order 8. Consequently, the order of $\mathcal{B}$ is $8\times 16=128$.

(iv) It is clear that the cosets $\mathcal{B}$, $A\mathcal{B}$, $A^2\mathcal{B}$ are all distinct, and multiplication by A produces no new cosets. The relationships (e) $CA = A^2(A^3DC^3)$ and (f) $CA^2 = A(A^3DR)$ imply that $CA\mathcal{B} = A^2\mathcal{B}$ and $CA^2\mathcal{B} = A\mathcal{B}$. These relationships imply that $\mathcal{B}$ has index 3 in the group generated by A and C; hence, the order of that group is 384.

The tesselating action of the group $\mathcal{B}$ is interesting in that it is the largest subgroup such that the P.M.'s that it tiles are regular in the sense that these P.M.'s are obtained by only simple permutations of the rows and or columns of the original matrix that preserve rows or columns except for order; e.g. the first row (a, b, c, d) subsequently must appear as a row or column except for order in a permutation of the group generated by $(bcda)$ and $(badc)$. All the other P.M.'s are obtained by breaking up the rows by the action of A or A^2.

We note that it may be shown that the group generated by A and C is the same as that in Rosser and Walker [1], which can be denoted as $[3^2, 4]$. The group isomorphic to $T^*(2, 3, 4)$ that tesselates $[1, 1, 0, 0]$ is of the same type, namely it can be denoted as $[3^1, 4]$. Thus, the group of transformations on the P.M.'s has these two successive groups of the type $[3^m, 4]$ appear in a natural hierarchy. These groups are called the hyper-octahedral groups in Coxeter and Moser [5, pp. 90 and 123].

We say that two elements are linked if they are always adjacent vertically or horizontally in a P.M. Because of the action of A and B on the class $[1, 1, 0, 0]$, we see that the pairs in (4) such as $(16, 2)$, $(14, 4)$, etc., are linked in $[1, 1, 0, 0]$. If a pair of elements are linked before the application of D, reflection across the main diagonal, they are linked after reflection. Furthermore, if we tile the plane with the P.M.'s, we see that C, which moves the left column to the right, also preserves linked elements. Consequently, the pairs $(16, 2)$, $(14, 4)$, etc. are always linked in a P.M.

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