Sensitivity analysis for parametric generalized implicit quasi-variational-like inclusions involving $P$-$\eta$-accretive mappings

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Abstract


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Keywords: Parametric generalized implicit quasi-variational-like inclusion; Sensitivity analysis; $P$-$\eta$-accretive mapping; $P$-$\eta$-proximal-point mapping; Relaxed mixed Lipschitz mapping; Relaxed mixed accretive mapping; Generalized mixed pseudocontractive mapping; Mixed Lipschitz continuous

1. Introduction

Variational inequality theory has become very effective and powerful tool for studying a wide range of problems arising in mechanics, contact problems in elasticity, optimization and control problems, management science, opera-
tion research, general equilibrium problems in economics and transportation, unilateral, obstacle, moving boundary valued problems etc., see for example [4,12,19]. Variational inequalities have been generalized and extended in different directions using novel and innovative techniques.

In recent years, much attention has been given to develop general techniques for the sensitivity analysis of solution set of various classes of variational inequalities (inclusions). From the mathematical and engineering point of view, sensitivity properties of various classes of variational inequalities can provide new insight concerning the problem being studied and can stimulate ideas for solving problems. The sensitivity analysis of solution set for variational inequalities have been studied extensively by many authors using quite different techniques. By using the projection technique, Dafermos [7], Mukherjee and Verma [22], Noor [24] and Yen [28] studied the sensitivity analysis of solution of some classes of variational inequalities with single-valued mappings. By using the implicit function approach that makes use of so-called normal mappings, Robinson [27] studied the sensitivity analysis of solutions for variational inequalities in finite-dimensional spaces. By using resolvent operator technique, Adly [1], Noor [25], and Agarwal et al. [2] studied the sensitivity analysis of solution set for some classes of quasi-variational inclusions involving single-valued mappings.

Recently, by using projection and resolvent techniques, Ding and Luo [10], Liu et al. [21], Park and Jeong [26] and Ding [8,9], studied the behavior and sensitivity analysis of solution set for some classes of generalized variational inequalities (inclusions) involving set-valued mappings. It is worth mentioning that most of the results in this direction have been obtained in the setting of Hilbert space.

Inspired by recent research works in this area, in this paper, we consider a parametric generalized implicit quasi-variational-like inclusion problem involving $P$-$\eta$-accretive mapping (PGIQVLIP for short) in uniformly smooth Banach space. Further, using $P$-$\eta$-proximal mapping technique of $P$-$\eta$-accretive mapping given by Kazmi and Khan [18], and the property of the fixed point set of set-valued mapping, we study the behavior and sensitivity analysis of the solution set for PGIQVLIP. Further, the Lipschitz continuity of solution set of PGIQVLIP is proved under suitable conditions. The theorems presented in this paper generalize and improve the results given by many authors, see for example [2,7–10,21,22,24–26].

2. Preliminaries

We assume that $E$ is a real Banach space equipped with norm $\| \cdot \|; E^*$ is the topological dual space of $E$; $C(E)$ is the family of all nonempty compact subsets of $E$; $2^E$ is the power set of $E$; $H(\cdot, \cdot)$ is the Hausdorff metric on $C(E)$, defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y) \right\}, \quad A, B \in C(E);$$

$\langle \cdot, \cdot \rangle$ is the dual pair between $E$ and $E^*$, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = \| x \|^2, \| x \| = \| f \| \}, \quad x \in E.$$

We observe that if $E$ is smooth then $J$ is single-valued and if $E \equiv H$, a Hilbert space, then $J$ is the identity map on $H$. In sequel, we shall denote a selection of normalized duality mapping by $j$.

First, we recall and define the following concepts and results.

Definition 2.1. (See [15]; see also [14,16,17].) A mapping $\eta : E \times E \to E^*$ is said to be

(i) **monotone**, if

$$\langle x - y, \eta(x, y) \rangle \geq 0, \quad \forall x, y \in E;$$

(ii) **strictly monotone**, if

$$\langle x - y, \eta(x, y) \rangle > 0, \quad \forall x, y \in E,$$

and equality holds if and only if $x = y$;

(iii) **$\delta$-strongly monotone**, if $\exists \delta > 0$ such that

$$\langle x - y, \eta(x, y) \rangle \geq \delta \| x - y \|^2, \quad \forall x, y \in E;$$

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$$\langle x - y, \eta(x, y) \rangle \geq \delta \| x - y \|^2, \quad \forall x, y \in E;$$
(iv) $\tau$-Lipschitz continuous, if $\exists \tau > 0$ such that
\[
\|\eta(x, y)\| \leq \tau \|x - y\|, \quad \forall x, y \in E.
\]

**Definition 2.2.** (See [15]; see also [14,16,17].) Let $\eta : E \times E \to E^*$ be a single-valued mapping and $M : E \to 2^E$ be a set-valued mapping. Then $M$ is said to be

(i) $\eta$-accretive, if
\[
\langle u - v, \eta(x, y) \rangle \geq 0, \quad \forall x, y \in E \text{ and } \forall u \in Mx, \ v \in My;
\]

(ii) strictly $\eta$-accretive, if
\[
\langle u - v, \eta(x, y) \rangle > 0, \quad \forall x, y \in E \text{ and } \forall u \in Mx, \ v \in My,
\]
and equality holds if and only if $x = y$;

(iii) $\delta$-strongly $\eta$-accretive, if $\exists \delta > 0$ such that
\[
\langle u - v, \eta(x, y) \rangle \geq \delta \|x - y\|^2, \quad \forall u \in Mx, \ v \in My;
\]

(iv) generalized $m$-accretive, if $M$ is $\eta$-accretive and $(I + \rho M)(E) = E$ for all (equivalently, for some) $\rho > 0$, where $I$ stands for identity mapping.

**Theorem 2.1.** (See [15]; see also [14,16,17].) Let $\eta : E \times E \to E^*$ be a mapping and let $M : E \to 2^E$ be a generalized $m$-accretive mapping. Then

(a) $\langle u - v, \eta(x, y) \rangle \geq 0, \forall (u, y) \in \text{Graph}(M)$ implies $(u, x) \in \text{Graph}(M)$, where $\text{Graph}(M) := \{(u, x) \in E \times E : u \in Mx\}$;

(b) the mapping $(I + \rho M)^{-1}$ is single-valued for all $\rho > 0$.

**Theorem 2.2.** (See [15]; see also [14,16,17].) Let $\eta : E \times E \to E^*$ be a $\tau$-Lipschitz continuous mapping and $\delta$-strongly monotone mapping. Let $M : E \to 2^E$ be a generalized $m$-accretive mapping. Then proximal-point mapping (resolvent operator $J^M_\rho$ for $M$ is $\frac{\tau}{\delta}$-Lipschitz continuous, i.e.,
\[
\|J^M_\rho(x) - J^M_\rho(y)\| \leq \frac{\tau}{\delta} \|x - y\|, \quad \forall x, y \in E.
\]

**Definition 2.3.** Let $\eta : E \times E \to E$ be a mapping. Then a mapping $P : E \to E$ is said to be

(i) $\eta$-accretive, if $\exists j\eta(x, y) \in J\eta(x, y)$ such that
\[
\langle Px - Py, j\eta(x, y) \rangle \geq 0, \quad \forall x, y \in E;
\]

(ii) strictly $\eta$-accretive, if $\exists j\eta(x, y) \in J\eta(x, y)$ such that
\[
\langle Px - Py, j\eta(x, y) \rangle > 0, \quad \forall x, y \in E,
\]
and equality holds if and only if $x = y$;

(iii) $\delta$-strongly $\eta$-accretive, if $\exists j\eta(x, y) \in J\eta(x, y)$ and $\delta > 0$ such that
\[
\langle Px - Py, j\eta(x, y) \rangle \geq \delta \|x - y\|^2, \quad \forall x, y \in E.
\]

**Definition 2.4.** (See [6].) Let $\eta : E \times E \to E$ be a single-valued mapping. Then a set-valued mapping $M : E \to 2^E$ is said to be

(i) $\eta$-accretive, if $\exists j\eta(x, y) \in J\eta(x, y)$ such that
\[
\langle u - v, j\eta(x, y) \rangle \geq 0, \quad \forall x, y \in E \text{ and } \forall u \in Mx, \ v \in My;
\]

(ii) $\eta$-$m$-accretive, if $M$ is $\eta$-accretive and $(I + \rho M)(E) = E$ for any $\rho > 0$, where $I$ stands for identity mapping.
Definition 2.5. (See [18].) Let \( \eta : E \times E \to E \) and \( P : E \to E \) be nonlinear mappings. Then a set-valued mapping \( M : E \to 2^E \) is said to be \( P, \eta \)-accretive, if \( M \) is \( \eta \)-accretive and \( (P + \rho M)(E) = E \) for any \( \rho > 0 \).

The following theorem give some properties of \( P, \eta \)-accretive mappings.

Theorem 2.3. (See [18].) Let \( \eta : E \times E \to E \) be a mapping; let \( P : E \to E \) be a strictly \( \eta \)-accretive mapping and let \( M : E \to 2^E \) be a \( P, \eta \)-accretive set-valued mapping. Then

(a) \( (u - v, j\eta(x, y)) \geq 0 \), \( \forall (v, y) \in \text{Graph}(M) \) implies \( (u, x) \in \text{Graph}(M) \), where \( \text{Graph}(M) := \{(u, x) \in E \times E : u \in Mx\} \);
(b) the mapping \( (P + \rho M)^{-1} \) is single-valued for all \( \rho > 0 \).

By Theorem 2.3, we can define \( P, \eta \)-proximal-point mapping for a \( P, \eta \)-accretive mapping \( M \) as follows:

\[
J^M_\rho(z) = (P + \rho M)^{-1}(z), \quad \forall z \in E,
\]

(2.1)

where \( \eta : E \times E \to E \) is a nonlinear mapping, \( P : E \to E \) is a strictly \( \eta \)-accretive mapping, and \( \rho > 0 \) is a constant.

Remark 2.1. \( P, \eta \)-proximal-point mapping generalize the corresponding concepts given by Chidume et al. [6] and Fang and Huang [11].

Next, the following theorem shows that \( P, \eta \)-proximal-point mapping is Lipschitz continuous.

Theorem 2.4. (See [18].) Let \( \eta : E \times E \to E \) be a \( \tau \)-Lipschitz continuous mapping and let \( P : E \to E \) be a \( \delta \)-strongly \( \eta \)-accretive mapping. Let \( M : E \to 2^E \) be a \( P, \eta \)-accretive mapping. Then \( P, \eta \)-proximal-point mapping \( J^M_\rho \) is \( \frac{\tau}{\delta} \)-Lipschitz continuous, i.e.,

\[
\|J^M_\rho(x) - J^M_\rho(y)\| \leq \frac{\tau}{\delta} \|x - y\|, \quad \forall x, y \in E.
\]

Remark 2.2. Theorems 2.3 and 2.4 generalize Lemma 2.6 in [6] and Lemma 2.8 in [6], respectively; generalize Theorems 2.1–2.2 [11] and Theorem 2.3 [11], respectively.

Throughout the rest of the paper unless otherwise stated, let \( E \) be a real uniformly smooth Banach space with \( \rho_E(t) \leq \varepsilon t^2 \) for some \( \varepsilon > 0 \), where \( \rho_E \) is called the modulus of smoothness defined below.

Lemma 2.1. (See [13].) Let \( E \) be a real uniformly smooth Banach space and let \( J : E \to E^* \) be the normalized duality mapping. Then, for all \( u, v \in E \), we have

(a) \( \|u + v\|^2 \leq \|u\|^2 + 2\langle v, J(u + v) \rangle \);

(b) \( \langle u - v, Ju - Jv \rangle \leq 2d^2 \rho_E(4\|u - v\|/d), \) where \( d = \sqrt{(\|u\|^2 + \|v\|^2)/2}, \) \( \rho_E(t) = \sup\{\frac{\|u\| + \|v\|}{2} - 1 : \|u\| = 1, \|v\| = t\} \) is called the modulus of smoothness of \( E \).

Lemma 2.2. (See [20].) Let \( X \) be a complete metric space and let \( T_1, T_2 : X \to C(X) \) be \( \theta \)-\( H \)-contraction mappings, then

\[
H(F(T_1), F(T_2)) \leq (1 - \theta)^{-1} \sup_{x \in X} H(T_1(x), T_2(x)),
\]

where \( F(T_1) \) and \( F(T_2) \) are the sets of fixed points of \( T_1 \) and \( T_2 \), respectively.

Let \( \Omega \) be a nonempty open subset of \( E \) in which the parameter \( \lambda \) takes values; let \( P : E \to E; N, M : E \times E \times \Omega \to E; g, m : E \times \Omega \to E \) be single-valued mappings such that \( g \neq 0 \) and let \( A, B, C, D, F : E \times \Omega \to C(E) \) be set-valued mappings. Suppose that \( W : E \times E \times \Omega \to 2^E \) is a set-valued mapping such that for each \( (y, \lambda) \in E \times \Omega \), \( W(., y, \lambda) : E \to 2^E \) is \( P, \eta \)-accretive and \( \text{Range}(g - m)(E \times \{\lambda\}) \cap \text{domain} W(., y, \lambda) \neq \emptyset \), where \( (g - m)(x, \lambda) = \)
g(x, λ) - m(x, λ), for any (x, λ) ∈ E × Ω. For each (f, λ) ∈ E × Ω, we consider the parametric generalized implicit quasi-variational-like inclusion problem (PGIQVLIP, for short):

Find x = x(λ) ∈ E, u = u(x, λ) ∈ A(x, λ), v = v(x, λ) ∈ B(x, λ), w = w(x, λ) ∈ C(x, λ), y = y(x, λ) ∈ D(x, λ) and z = z(x, λ) ∈ F(x, λ) such that (g - m)(x, λ) ∈ domain W(., z, λ) and

\[ f \in N(u, v, λ) - M(w, y, λ) + W((g - m)(x, λ), z, λ). \]  

(2.2)

Some special cases of PGIQVLIP (2.2):

I. If \( E \equiv H, \) a Hilbert space; \( P \equiv I, \) an identity mapping, and \( η(x, t) \equiv x - t, \) \( ∀x, t \in H, \) then PGIQVLIP (2.2) reduces to the problem of finding \( x = x(λ) ∈ H, u = u(x, λ) ∈ A(x, λ), v = v(x, λ) ∈ B(x, λ), w = w(x, λ) ∈ C(x, λ), \) \( y = y(x, λ) ∈ D(x, λ) \) and \( z = z(x, λ) ∈ F(x, λ) \) such that \( (g - m)(x, λ) ∈ \text{domain} \ W(., z, λ) \) and

\[ f \in N(u, v, λ) - M(w, y, λ) + W((g - m)(x, λ), z, λ), \]

which has been studied by Liu et al. [21].

II. If \( E \equiv H; P \equiv I; η(x, t) \equiv x - t, \) \( ∀x, t \in H; M \equiv 0, \) a zero mapping, and \( f \equiv 0, \) then PGIQVLIP (2.2) reduces to the problem of finding \( x = x(λ) ∈ H, u = u(x, λ) ∈ A(x, λ), v = v(x, λ) ∈ B(x, λ) \) and \( z = z(x, λ) ∈ F(x, λ) \) such that \( (g - m)(x, λ) ∈ \text{domain} \ W(., z, λ) \) and

\[ 0 \in N(u, v, λ) + W((g - m)(x, λ), z, λ), \]

which has been studied by Ding [8].

III. If \( E \equiv H; P \equiv I; η(x, t) \equiv x - t, \) \( ∀x, t \in H; f \equiv 0; M \equiv C \equiv D \equiv m \equiv 0; \) \( g \equiv I \) and \( A(λ, λ) \equiv B(x, λ) \equiv F(x, λ) \equiv x, \) \( ∀x \in H \times Ω. \) Then PGIQVLIP (2.2) reduces to the problem finding \( x = x(λ) ∈ H \) such that

\[ 0 \in N(x, x, λ) + M(x, x, λ), \]

which has been studied by Adly [1].

IV. If \( E \equiv H; P \equiv I; f \equiv 0; M \equiv B = C = D = E \equiv m \equiv 0; \) \( A(λ, λ) \equiv x, \) \( ∀(x, λ) \in H \times Ω; N(x, y, λ) \equiv N₁(x, λ); W(x, y, λ) \equiv W₁(x, λ), \) \( ∀(x, y, λ) \in H \times H × Ω, \) where \( N₁ : H \times Ω → H; W₁ : H \times Ω → 2^H, \) then PGIQVLIP (2.2) reduces to the problem of finding \( x = x(λ) ∈ H \) such that \( g(x, λ) ∈ \text{domain} \ W(., λ) \) and

\[ 0 \in N₁(x, λ) + W₁(g(x, λ), λ), \]

which has been studied by Adly [1].

For a suitable choices of the mappings \( A, B, C, D, F, N, M, W, g, P, m, η, \) it is easy to see that PGIQVLIP (2.2) includes a number of known classes of parametric quasi-variational inclusions, parametric generalized quasi-variational inclusions, parametric quasi-variational inequalities, studied by many authors as special cases, for example [1–3,5,7,8,10,21,22,24,26] and the references therein.

Now, for each fixed \( λ ∈ Ω, \) the solution set \( S(λ) \) of PGIQVLIP (2.2) is denoted as

\[ S(λ) := \{ x = x(λ) ∈ E; u = u(x, λ) ∈ A(x, λ), v = v(x, λ) ∈ B(x, λ), w = w(x, λ) ∈ C(x, λ), \] \( y = y(x, λ) ∈ D(x, λ), z = z(x, λ) ∈ F(x, λ), \) \( f \in N(u, v, λ) - M(w, y, λ) + W((g - m)(x, λ), z, λ) \}. \]

(2.3)

The aim of this paper is to study the behavior and sensitivity analysis of the solution set \( S(λ) \), and the conditions on mappings \( A, B, C, D, F, N, M, W, g, P, m, η, \) under which the solution set \( S(λ) \) of PGIQVLIP (2.2) is nonempty and Lipschitz continuous with respect to the parameter \( λ ∈ Ω. \)

3. Sensitivity analysis of the solution set \( S(λ) \)

First, we define the following concepts.

**Definition 3.1.** A mapping \( g : E × Ω → E \) is said to be

(i) \((L_g, l_g)-\text{mixed Lipschitz continuous}, \) if there exist constants \( L_g, l_g > 0 \) such that
there exist constants $\lambda > 0$ such that

$$\|g(x_1, \lambda_1) - g(x_2, \lambda_2)\| \leq L_g \|x_1 - x_2\| + l_g \|\lambda_1 - \lambda_2\|, \quad \forall (x_1, \lambda_1), (x_2, \lambda_2) \in E \times \Omega;$$

(ii) $s$-strongly monotone, if there exists a constant $s > 0$ such that

$$\langle g(x_1, \lambda) - g(x_2, \lambda), x_1 - x_2 \rangle \geq s \|x_1 - x_2\|^2, \quad \forall (x_1, x_2, \lambda) \in E \times E \times \Omega.$$

Remark 3.1. When $\lambda$ is fixed, then mixed-Lipschitz continuity of $g$ implies Lipschitz continuity in the first argument.

Definition 3.2. A set-valued mapping $A : E \times \Omega \to C(E)$ is said to be $(L_A, l_A)$-$H$-mixed Lipschitz continuous, if there exist constants $L_A, l_A > 0$ such that

$$H(A(x_1, \lambda_1), A(x_2, \lambda_2)) \leq L_A \|x_1 - x_2\| + l_A \|\lambda_1 - \lambda_2\|, \quad \forall (x_1, \lambda_1), (x_2, \lambda_2) \in E \times \Omega.$$

Definition 3.3. Let $P : E \to E, g, m : E \times \Omega \to E$ be mappings and let $A, B : E \times \Omega \to C(E)$ be set-valued mappings. A mapping $N : E \times E \times \Omega \to E$ is said to be

(i) $(L_{(N,1)}, L_{(N,2)}, l_N)$-mixed Lipschitz continuous, if there exist constants $L_{(N,1)}, L_{(N,2)}, l_N > 0$ such that

$$\|N(x_1, y_1, \lambda_1) - N(x_2, y_2, \lambda_2)\| \leq L_{(N,1)} \|x_1 - x_2\| + L_{(N,2)} \|y_1 - y_2\| + l_N \|\lambda_1 - \lambda_2\|, \quad \forall (x_1, y_1, \lambda_1), (x_2, y_2, \lambda_2) \in E \times E \times \Omega;$$

(ii) $\xi$-strongly mixed $P \circ (g - m)$-accretive with respect to $A$ and $B$, if there exists a constant $\xi > 0$ such that

$$\|N(u_1, v_1, \lambda) - N(u_2, v_2, \lambda)\| \leq \xi \|x - y\|^2,$$

where $P \circ (g - m)$ denotes $P$ composition $(g - m)$;

(ii) $\sigma$-generalized mixed $P \circ (g - m)$-pseudocontractive with respect to $A$ and $B$, if there exists a constant $\sigma > 0$ such that

$$\|N(u_1, v_1, \lambda) - N(u_2, v_2, \lambda)\| \leq \sigma \|x - y\|^2,$$

where $P \circ (g - m)$ denotes $P$ composition $(g - m)$;

(iv) $\nu$-relaxed mixed $P \circ (g - m)$-Lipschitz with respect to $A$ and $B$, if there exists a constant $\nu > 0$ such that

$$\|N(u_1, v_1, \lambda) - N(u_2, v_2, \lambda)\| \leq \nu \|x - y\|^2,$$

We now transfer the PGIQVLIP (2.2) into a fixed point problem.

Lemma 3.1. For each $(f, \lambda) \in E \times \Omega, (x, u, v, w, y, z)$ with $x = x(\lambda) \in E, u = u(x, \lambda) \in A(x, \lambda), v = v(x, \lambda) \in B(x, \lambda), w = w(x, \lambda) \in C(x, \lambda), y = y(x, \lambda) \in D(x, \lambda)$ and $z = z(x, \lambda) \in F(x, \lambda)$ such that $(g - m)(x, \lambda) \in \text{domain } W(., z, \lambda)$ is a solution of PGIQVLIP (2.2) if and only if the set-valued mapping $G : E \times \Omega \to 2^E$ defined by

$$G(t, \lambda) = \bigcup_{u \in A(t, \lambda), v \in B(t, \lambda), w \in C(t, \lambda), y \in D(t, \lambda), z \in F(t, \lambda)} \left[t - (g - m)(t, \lambda) + J_{\rho}^W(z, \lambda)[P \circ (g - m)(t, \lambda) - \rho N(u, v, \lambda) + \rho M(w, y, \lambda) + \rho f]ight], \quad t \in E,$$

has a fixed point, where $P : E \to E$ and $P \circ (g - m)$ denotes $P$ composition $(g - m); J_{\rho}^W(z, \lambda) = (P + \rho W(., z, \lambda))^{-1},$ and $\rho > 0$ is a constant.

Proof. For each $(f, \lambda) \in E \times \Omega$, PGIQVLIP (2.2) has a solution $(x, u, v, w, y, z)$ with $x = x(\lambda) \in E, u = u(x, \lambda) \in A(x, \lambda), v = v(x, \lambda) \in B(x, \lambda), w = w(x, \lambda) \in C(x, \lambda), y = y(x, \lambda) \in D(x, \lambda)$ and $z = z(x, \lambda) \in F(x, \lambda)$ such that $(g - m)(x, \lambda) \in \text{domain } W(., z, \lambda)$ if and only if
mapping with respect to $\lambda$ of $W(.,z,\lambda)$, and $N$ is $P$-mixed Lipschitz continuous in first two arguments. Suppose that the set-valued mapping $PGIQVLIP(2.2)$ is nonempty and closed. Since for each $z, \lambda \in E \times \Omega$, $W(.,z,\lambda)$ is $P$-$\eta$-accretive, by definition of $P$-$\eta$-proximal-point mapping $J^W_{\rho}(z,\lambda)$ of $W(.,z,\lambda)$, preceding inclusion holds if and only if
\[
(g - m)(x, \lambda) = J^W_{\rho}(z,\lambda)[P \circ (g - m)(x, \lambda) - \rho N(u, v, \lambda) + \rho M(w, y, \lambda) + \rho f],
\]
i.e., $x \in G(x, \lambda)$. This completes the proof. □

**Theorem 3.1.** Let $E$ be a real uniformly smooth Banach space with $\rho_E(t) = ct^2$ for some $c > 0$. Let the set-valued mappings $A, B, C, D, F : E \times \Omega \rightarrow C(E)$ be $H$-Lipschitz continuous in the first argument with constants $L_A$, $L_B$, $L_C$, $L_D$, $L_F$, respectively; let the mapping $\eta : E \times E \rightarrow E$ be $\tau$-Lipschitz continuous and $P : E \rightarrow E$ be $\delta$-strongly monotone. Let the mappings $g, m : E \times \Omega \rightarrow E$ such that $(g - m)$ is $s$-strongly accretive and $L_{(g - m)}$-Lipschitz continuous in the first argument and let the mapping $P \circ (g - m)$ be $L_{P \circ (g - m)}$-Lipschitz continuous in the first argument; let the mapping $N : E \times E \times \Omega \rightarrow E$ be $\xi$-strongly mixed $P \circ (g - m)$-accretive with respect to $A$ and $B$, and $(L_{(N,1)}, L_{(N,2)})$-mixed Lipschitz continuous in first two arguments; let the mapping $M : E \times E \times \Omega \rightarrow E$ be $\sigma$-generalized mixed $P \circ (g - m)$-pseudocontractive with respect to $C$ and $D$, and $(L_{(M,1)}, L_{(M,2)})$-mixed Lipschitz continuous in first two arguments. Suppose that the set-valued mapping $W : E \times E \times \Omega \rightarrow 2^E$ is such that for each $(y, \lambda) \in E \times \Omega$, $W(., y, \lambda) : E \rightarrow 2^E$ is $P$-$\eta$-accretive with range $(g - m)(E \times \{\lambda\}) \cap \text{domain } W(., y, \lambda) \neq \emptyset$. Suppose that there exist constants $\mu_1, \mu_2 > 0$ such that
\[
\|J^W_{\rho}(x,\lambda)(z) - J^W_{\rho}(y,\lambda)(z)\| \leq \mu_1 \|x - y\| + \mu_2 \|\lambda - \tilde{\lambda}\|, \quad \forall x, y, z \in E; \quad \lambda, \tilde{\lambda} \in \Omega, \quad (3.2)
\]
and suppose that there exists a constant $\rho > 0$ such that
\[
\theta = q + \epsilon(\rho); \quad q := \mu_1 L_F + \sqrt{1 - 2s + 64cL^2_{(g - m)}},
\]
\[
\epsilon(\rho) := r^{-1}\sqrt{L^2_{P \circ (g - m)} - 2\rho(\xi - \sigma) + 132\rho^2cL^2_N + L^2_M}; \quad r := \frac{\delta}{\tau},
\]
\[
L_N := (L AL_{(N,1)} + L BL_{(N,2)}); \quad L_M := (L CL_{(M,1)} + L DL_{(M,2)});
\]
\[
|\rho - \frac{(\xi - \sigma)^2}{132c(L^2_N + L^2_M)}| < \frac{\sqrt{(\xi - \sigma)^2 - 132c(L^2_N + L^2_M)(L^2_{P \circ (g - m)} - (1-q^2)r^2)}}{132c(L^2_N + L^2_M)}. \quad (3.4)
\]
Then, for each fixed $f \in E$, the set-valued mapping $G$ defined by (3.1) is a compact-valued uniform $\theta$-$H$-contraction mapping with respect to $\lambda \in \Omega$, where $\theta$ is given by (3.3). Moreover, for each $\lambda \in \Omega$, the solution set $S(\lambda)$ of $PGIQVLIP(2.2)$ is nonempty and compact-valued, and the mixed Lipschitz continuity of $N$ and $M$, we estimate
\[
\|J^W_{\rho}(z,\lambda)(\lambda) - J^W_{\rho}(y,\lambda)(\lambda)\| \leq \|J^W_{\rho}(z,\lambda)(\lambda) - J^W_{\rho}(y,\lambda)(\lambda)\| + \|J^W_{\rho}(z,\lambda)(\lambda) - J^W_{\rho}(y,\lambda)(\lambda)\| + \|J^W_{\rho}(z,\lambda)(\lambda) - J^W_{\rho}(y,\lambda)(\lambda)\| \leq \|J^W_{\rho}(z,\lambda)(\lambda) - J^W_{\rho}(y,\lambda)(\lambda)\| + \|J^W_{\rho}(z,\lambda)(\lambda) - J^W_{\rho}(y,\lambda)(\lambda)\| + \|J^W_{\rho}(z,\lambda)(\lambda) - J^W_{\rho}(y,\lambda)(\lambda)\|.
\]

**Proof.** Let $(x, \lambda)$ be an arbitrary element in $E \times \Omega$. Since $A, B, C, D, F$ are compact-valued, then, for any sequences $\{u_n\} \subset A(x, \lambda), \{v_n\} \subset B(x, \lambda), \{u_n\} \subset C(x, \lambda), \{v_n\} \subset D(x, \lambda), \{z_n\} \subset F(x, \lambda)$, there exist subsequences $\{u_{n_i}\} \subset \{u_n\}, \{v_{n_i}\} \subset \{v_n\}, \{u_{n_i}\} \subset \{u_n\}, \{v_{n_i}\} \subset \{v_n\}, \{z_{n_i}\} \subset \{z_n\}$ and elements $u \in A(x, \lambda), v \in B(x, \lambda), w \in C(x, \lambda), y \in D(x, \lambda), z \in F(x, \lambda)$ such that $u_{n_i} \rightarrow u, v_{n_i} \rightarrow v, w_{n_i} \rightarrow w, y_{n_i} \rightarrow y, z_{n_i} \rightarrow z$ as $i \rightarrow \infty$. By using Theorem 2.4 and (3.2) and the mixed Lipschitz continuity of $N$ and $M$, we estimate
\[
\|J^W_{\rho}(z,\lambda)(\lambda) - J^W_{\rho}(y,\lambda)(\lambda)\| \leq \|J^W_{\rho}(z,\lambda)(\lambda) - J^W_{\rho}(y,\lambda)(\lambda)\| + \|J^W_{\rho}(z,\lambda)(\lambda) - J^W_{\rho}(y,\lambda)(\lambda)\| + \|J^W_{\rho}(z,\lambda)(\lambda) - J^W_{\rho}(y,\lambda)(\lambda)\|.
\]
\[
\mu_1 \|z_{n_i} - z\| + \rho \frac{T}{\delta} \left[ \|N(u_{n_i}, v_{n_i}, \lambda) - N(u, v, \lambda)\| + \|M(w_{n_i}, y_{n_i}, \lambda) - M(w, y, \lambda)\| \right] \\
\leq \mu_1 \|z_{n_i} - z\| + \rho \frac{T}{\delta} \left[ L(N,1)\|u_{n_i} - u\| + L(N,2)\|v_{n_i} - v\| + L(M,1)\|w_{n_i} - w\| + L(M,2)\|y_{n_i} - y\| \right] \\
\to 0, \quad \text{as } i \to \infty. \tag{3.5}
\]

Thus (3.1) and (3.5) yield that \(G(x, \lambda) \in C(E)\).

Now, for each fixed \(\lambda \in \Omega\), we prove that \(G(x, \lambda)\) is a uniform \(\theta\)-\(H\)-contraction mapping. Let \((x_1, \lambda), (x_2, \lambda)\) be arbitrary elements in \(E \times \Omega\) and any \(t_1 \in G(x_1, \lambda)\), there exist \(u_1 = u_1(x_1, \lambda) \in A(x_1, \lambda)\), \(v_1 = v_1(x_1, \lambda) \in B(x_1, \lambda)\), \(w_1 = w_1(x_1, \lambda) \in C(x_1, \lambda), y_1 = y_1(x_1, \lambda) \in D(x_1, \lambda)\) and \(z_1 = z_1(x_1, \lambda) \in F(x_1, \lambda)\) such that

\[
t_1 = x_1 - (g - m)(x_1, \lambda) + J^W_{\rho} \left[ P \circ (g - m)(x_1, \lambda) - \rho N(u_1, v_1, \lambda) + \rho M(w_1, y_1, \lambda) + \rho f \right]. \tag{3.6}
\]

It follows from the compactness of \(A(x_2, \lambda), B(x_2, \lambda), C(x_2, \lambda), D(x_2, \lambda), \) and \(F(x_2, \lambda)\) and \(H\)-Lipschitz continuity of \(A, B, C, D, F\) that there exist \(u_2 = u_2(x_2, \lambda) \in A(x_2, \lambda)\), \(v_2 = v_2(x_2, \lambda) \in B(x_2, \lambda)\), \(w_2 = w_2(x_2, \lambda) \in C(x_2, \lambda)\), \(y_2 = y_2(x_2, \lambda) \in D(x_2, \lambda)\) and \(z_2 = z_2(x_2, \lambda) \in F(x_2, \lambda)\) such that

\[
\|u_1 - u_2\| \leq H(A(x_1, \lambda), A(x_2, \lambda)) \leq L_A \|x_1 - x_2\|, \\
\|v_1 - v_2\| \leq H(B(x_1, \lambda), B(x_2, \lambda)) \leq L_B \|x_1 - x_2\|, \\
\|w_1 - w_2\| \leq H(C(x_1, \lambda), C(x_2, \lambda)) \leq L_C \|x_1 - x_2\|, \\
\|y_1 - y_2\| \leq H(D(x_1, \lambda), D(x_2, \lambda)) \leq L_D \|x_1 - x_2\|, \\
\|z_1 - z_2\| \leq H(F(x_1, \lambda), F(x_2, \lambda)) \leq L_F \|x_1 - x_2\|. \tag{3.7}
\]

Let

\[
t_2 = x_2 - (g - m)(x_2, \lambda) + J^W_{\rho} \left[ P \circ (g - m)(x_2, \lambda) - \rho N(u_2, v_2, \lambda) + \rho M(w_2, y_2, \lambda) + \rho f \right], \tag{3.8}
\]

then we have \(t_2 \in G(x_2, \lambda)\).

Next, using Theorem 2.4 and (3.2), we estimate

\[
\|t_1 - t_2\| \leq \|x_1 - x_2 - ((g - m)(x_1, \lambda) - (g - m)(x_2, \lambda))\| \\
+ \|J^W_{\rho} \left[ P \circ (g - m)(x_1, \lambda) - \rho N(u_1, v_1, \lambda) + \rho M(w_1, y_1, \lambda) + \rho f \right]\| \\
- J^W_{\rho} \left[ P \circ (g - m)(x_2, \lambda) - \rho N(u_2, v_2, \lambda) + \rho M(w_2, y_2, \lambda) + \rho f \right]\| \\
+ \|J^W_{\rho} \left[ P \circ (g - m)(x_1, \lambda) - \rho N(u_1, v_1, \lambda) + \rho M(w_1, y_1, \lambda) + \rho f \right]\| \\
- J^W_{\rho} \left[ P \circ (g - m)(x_2, \lambda) - \rho N(u_2, v_2, \lambda) + \rho M(w_2, y_2, \lambda) + \rho f \right]\| \\
\leq \|x_1 - x_2 - ((g - m)(x_1, \lambda) - (g - m)(x_2, \lambda))\| + \mu_1 \|z_1 - z_2\| \\
+ \frac{T}{\delta} \|P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda)\| \\
- \rho \left( N(u_1, v_1, \lambda) - N(u_2, v_2, \lambda) - M(w_1, y_1, \lambda) + M(w_2, y_2, \lambda) \right). \tag{3.9}
\]

Since \((g - m)\) is \(s\)-strongly accretive and \(L_{(g - m)}\)-Lipschitz continuous, we have

\[
\|x_1 - x_2 - ((g - m)(x_1, \lambda) - (g - m)(x_2, \lambda))\|^2 \\
\leq \|x_1 - x_2\|^2 - 2\|(g - m)(x_1, \lambda) - (g - m)(x_2, \lambda), J(x_1 - x_2 - ((g - m)(x_1, \lambda) - (g - m)(x_2, \lambda))\| \\
\leq \|x_1 - x_2\|^2 - 2\|(g - m)(x_1, \lambda) - (g - m)(x_2, \lambda), J(x_1 - x_2)\| \\
- 2\|(g - m)(x_1, \lambda) - (g - m)(x_2, \lambda), J(x_1 - x_2 - ((g - m)(x_1, \lambda) - (g - m)(x_2, \lambda)) - J(x_1 - x_2)\| \\
\leq \|x_1 - x_2\|^2 - 2s \|x_1 - x_2\|^2 + 64cL^2_{(g - m)} \|x_1 - x_2\|^2 \\
\leq (1 - 2s + 64cL^2_{(g - m)}) \|x_1 - x_2\|^2. \tag{3.10}
\]

Since \(N\) is \((L_{(N,1)}, L_{(N,2)})\)-mixed Lipschitz continuous and \(M\) is \((L_{(M,1)}, L_{(M,2)})\)-mixed Lipschitz continuous and \(H\)-Lipschitz continuity of set-valued mappings \(A, B, C, D, F\) we have
\[
\| N(u_1, v_1, \lambda) - N(u_2, v_2, \lambda) \| \leq L_{(N,1)} \| u_1 - u_2 \| + L_{(N,2)} \| v_1 - v_2 \|
\]
\[
\leq L_{(N,1)} H(A(x_1, \lambda), A(x_2, \lambda)) + L_{(N,2)} H(B(x_1, \lambda), B(x_2, \lambda))
\]
\[
\leq (L_A L_{(N,1)} + L_B L_{(N,2)}) \| x_1 - x_2 \|,
\]
(3.11)

and

\[
\| M(w_1, y_1, \lambda) - N(w_2, y_2, \lambda) \| \leq L_{(M,1)} \| w_1 - w_2 \| + L_{(M,2)} \| y_1 - y_2 \|
\]
\[
\leq (L_C L_{(M,1)} + L_D L_{(M,2)}) \| x_1 - x_2 \|.
\]
(3.12)

Further, since \( N \) is \( \xi \)-strongly mixed \( P \circ (g - m) \)-accretive with respect to \( A \) and \( B \), \( M \) is \( \sigma \)-generalized mixed \( P \circ (g - m) \)-pseudocontractive with respect to \( C \) and \( D \), and \( P \circ (g - m) \) is \( L_{P \circ (g - m)} \)-Lipschitz continuous, then, using \( \| x + y \|^2 \leq 2 \| x \|^2 + \| y \|^2 \), we have

\[
\| P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda) - \rho(N(u_1, v_1, \lambda) - N(u_2, v_2, \lambda) - M(w_1, y_1, \lambda) + M(w_2, y_2, \lambda)) \|^2
\]
\[
\leq \| P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda) \|^2
\]
\[
- 2 \rho \| N(u_1, v_1, \lambda) - N(u_2, v_2, \lambda) - M(w_1, y_1, \lambda) + M(w_2, y_2, \lambda),
\]
\[
J(P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda)
\]
\[
- \rho(N(u_1, v_1, \lambda) - N(u_2, v_2, \lambda) - M(w_1, y_1, \lambda) + M(w_2, y_2, \lambda)) \|
\]
\[
\leq L^2_{P \circ (g - m)} \| x - y \|^2 - 2 \rho \| N(u_1, v_1, \lambda) - N(u_2, v_2, \lambda), J(P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda))
\]
\[
+ 2 \rho \| M(w_1, y_1, \lambda) - M(w_2, y_2, \lambda), J(P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda))
\]
\[
- 2 \rho \| (N(u_1, v_1, \lambda) - N(u_2, v_2, \lambda)) - (M(w_1, y_1, \lambda) - M(w_2, y_2, \lambda)),
\]
\[
J(P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda)
\]
\[
- \rho(N(u_1, v_1, \lambda) - N(u_2, v_2, \lambda) - M(w_1, y_1, \lambda) + M(w_2, y_2, \lambda))
\]
\[
- J(P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda))
\]
\[
\leq L^2_{P \circ (g - m)} \| x_1 - x_2 \|^2 - 2 \rho \xi \| x_1 - x_2 \|^2 + 2 \rho \sigma \| x_1 - x_2 \|^2
\]
\[
+ 64 \rho^2 c \| (N(u_1, v_1, \lambda) - N(u_2, v_2, \lambda)) - (M(w_1, y_1, \lambda) - M(w_2, y_2, \lambda)) \|^2
\]
\[
\leq L^2_{P \circ (g - m)} \| x_1 - x_2 \|^2 - 2 \rho (\xi - \sigma) \| x_1 - x_2 \|^2
\]
\[
+ 132 \rho^2 c [L_{(N,1)} L_A + L_{(N,2)} L_B]^2 + (L_{(M,1)} L_C + L_{(M,2)} L_D)^2 \| x_1 - x_2 \|^2
\]
\[
\leq (L^2_{P \circ (g - m)} - 2 \rho (\xi - \sigma) + 132 \rho^2 c [L_{(N,1)} L_A + L_{(N,2)} L_B]^2 + (L_{(M,1)} L_C + L_{(M,2)} L_D)^2) \| x_1 - x_2 \|^2.
\]
(3.13)

Now, from (3.9)–(3.13), it follows that

\[
\| t_1 - t_2 \| \leq \theta \| x_1 - x_2 \|,
\]
(3.14)

where

\[
\theta = q + \epsilon(\rho); \quad q := \mu_1 L_F + \sqrt{(1 - 2s + 64c L^2_{(g - m)})};
\]
\[
r := \frac{\delta}{\tau}; \quad \epsilon(\rho) := r^{-1} \sqrt{L^2_{P \circ (g - m)} - 2 \rho (\xi - \sigma) + 132 \rho^2 c [L^2_{N} + L^2_{M}]};
\]
\[
L_N := (L_{(N,1)} L_A + L_{(N,2)} L_B); \quad L_M := (L_{(M,1)} L_C + L_{(M,2)} L_D).
\]

Hence, we have

\[
d(t_1, G(x_2, \lambda)) = \inf_{t_2 \in G(x_2, \lambda)} \| t_1 - t_2 \| \leq \theta \| x_1 - x_2 \|.
\]
(3.15)
Since \( t_1 \in G(x_1, \lambda) \) is arbitrary, we obtain
\[
\sup_{t_1 \in G(x_1, \lambda)} d(t_1, G(x_2, \lambda)) \leq \theta \|x_1 - x_2\|.  \tag{3.16}
\]
By using same argument, we can prove
\[
\sup_{t_2 \in G(x_2, \lambda)} d(t_2, G(x_1, \lambda)) \leq \theta \|x_1 - x_2\|.  \tag{3.17}
\]
By the definition of the Hausdorff metric \( H \) on \( C(E) \), we obtain that for all \((x_1, \lambda), (x_2, \lambda) \in E \times \Omega\),
\[
H(G(x_1, \lambda), G(x_2, \lambda)) \leq \theta \|x_1 - x_2\|,  \tag{3.18}
\]
that is, \( G(x, \lambda) \) is a uniform \( \theta \)-\( H \)-contraction mapping with respect to \( \lambda \in \Omega \). Also, it follows from condition (3.4) that \( \theta < 1 \) and hence \( G(x, \lambda) \) is a set-valued contraction mapping which is uniform with respect to \( \lambda \in \Omega \). By a fixed point theorem of Nadler [23], for each \( \lambda \in \Omega \), \( G(x, \lambda) \) has a fixed point \( x = x(\lambda) \in E \), i.e., \( x = x(\lambda) \in G(x, \lambda) \), and hence Lemma 3.1 ensures that \( S(\lambda) \neq \emptyset \). Further, for any sequence \( \{x_n\} \subset S(\lambda) \) with \( \lim_{n \to \infty} x_n = x_0 \), we have \( x_n \in G(x_n, \lambda) \) for all \( n \geq 1 \). By virtue of (3.18), we have that
\[
d(x_0, G(x_0, \lambda)) \leq \|x_0 - x_n\| + H(G(x_n, \lambda), G(x_0, \lambda)) \leq (1 + \theta) \|x_n - x_0\| \to 0 \quad \text{as} \quad n \to \infty,
\]
that is, \( x_0 \in G(x_0, \lambda) \) and hence \( x_0 \in S(\lambda) \). Thus \( S(\lambda) \) is closed in \( E \). This completes the proof. \( \square \)

**Theorem 3.2.** Let \( E \) be a real uniformly smooth Banach space with \( \rho_E(t) \leq ct^2 \) for some \( c > 0 \). Let the mappings \( A, B, C, D, F, \eta, P, g, m, W \) be same as in Theorem 3.1. Let the mapping \( N \) be \((L(N,1), L(N,2))-mixed Lipshitz continuous in first two arguments, and let the mapping \( M \) be \( \nu \)-relaxed mixed \( P \circ (g - m) \)-Lipschitz with respect to \( C \) and \( D \), and \((L(M,1), L(M,2))\)-mixed Lipschitz continuous in first two arguments. If condition (3.3) is satisfied, then there exists a constant \( \rho > 0 \) such that
\[
\begin{align*}
\theta_1 &:= q + \epsilon(\rho); \quad q := \mu_1 LF + \sqrt{1 - 2s + 64\epsilon c L_2^2 \rho_{g-m}}; \quad r := \frac{\delta}{\tau}; \\
\epsilon(\rho) &:= r^{-1}[\rho L_N + \frac{\sqrt{L_2^2 \rho_{g-m}} - 2\rho v + 64\rho^2 c(L_2^2 + L_2^2)}{c L_2^2 \rho_{g-m}}]; \\
L_N &:= (L_{A}L(N,1) + L_{B}L(N,2)); \quad L_M := (L_{C}L(M,1) + L_{D}L(M,2)); \\
\rho &- \frac{v - r(1 - q)L_N}{64\epsilon c L_2^2 - L_2^2} < \frac{\sqrt{[\frac{v - r^2(1 - q)^2 L_N}{L_2^2 \rho_{g-m}}] - (\frac{\sqrt{L_2^2 \rho_{g-m}} - (1 - q^2)r^2)}{64\epsilon c L_2^2 - L_2^2}}.  \tag{3.19}
\end{align*}
\]

Then, for given \( f \in E \), the set-valued mapping \( G \) defined by (3.1) is a compact-valued uniform \( \theta_1 \)-\( H \)-contraction mapping with respect to \( \lambda \in \Omega \), where \( \theta_1 \) is given by (3.19). Moreover, for each \( \lambda \in \Omega \), the solution set \( S(\lambda) \) of PGIQVLIP (2.2) is nonempty and closed.

**Proof.** As in the proof of Theorem 3.1, we see that \( G \) is compact-valued and (3.5)–(3.8) and (3.10)–(3.12) hold. Since \( N \) is \((L(N,1), L(N,2))-mixed Lipshitz continuous, M is \( \nu \)-relaxed mixed \( P \circ (g - m) \)-Lipschitz with respect to \( C \) and \( D \), and \((L(M,1), L(M,2))\)-mixed Lipshitz continuous. It follows that
\[
\begin{align*}
\| P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda) + \rho(M(w_1, y_1, \lambda) - M(w_2, y_2, \lambda)) &\|^2 \\
\leq & \| P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda) \|^2 \\
& + 2\rho[M(w_1, y_1, \lambda) - M(w_2, y_2, \lambda), J(P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda))] \\
& + 2\rho[M(w_1, y_1, \lambda) - M(w_2, y_2, \lambda), J(P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda))] \\
& + \rho(M(w_1, y_1, \lambda) - M(w_2, y_2, \lambda) - J(P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda))] \\
& \leq L_{P \circ (g - m)}^2 \|x_1 - x_2\|^2 - 2\rho v \|x_1 - x_2\|^2 + 64\rho^2 c \|M(w_1, y_1, \lambda) - M(w_2, y_2, \lambda)\|^2 \\
& \leq (L_{P \circ (g - m)}^2 - 2\rho v + 64\rho^2 c (L_{C}L(M,1) + L_{D}L(M,2))^2) \|x_1 - x_2\|^2. \tag{3.21}
\end{align*}
\]
By Theorem 3.2, $G(x, \lambda) \leq \bar{\Omega}$ and $\bar{\Omega}$ is given by (3.19). By Lemma 2.2, we obtain

$$\|i_1 - i_2\| \leq \|x_1 - x_2 - ((g - m)(x_1, \lambda) - (g - m)(x_2, \lambda))\| + \mu_1 \|z_1 - z_2\|
+ \frac{\tau}{\delta} \left[ \|P \circ (g - m)(x_1, \lambda) - P \circ (g - m)(x_2, \lambda) + \rho [M(w_1, y_1, \lambda) - M(w_2, y_2, \lambda)]\|
+ \rho \|N(u_1, v_1, \lambda) - N(u_2, v_2, \lambda)\| \right]
\leq (1 - 2s + 64cL^2_{(g - m)})^\frac{1}{2} \|x_1 - x_2\| + \mu_1 L_F \|x_1 - x_2\|
+ \frac{\tau}{\delta} \|x_1 - x_2\| \left[ (L^2_{P_0(g - m)} - 2\rho v + 64\rho^2 \delta [LC(L_{M, 1}) + LD(L_{M, 2})]^2]^\frac{1}{2}
+ \rho [L_A L_{(N, 1)} + L_B L_{(N, 1)}] \right]
\leq \theta_1 \|x_1 - x_2\|. \quad (3.22)

The rest of the proof follows precisely as in the proof of Theorem 3.1. This completes the proof. \hfill \Box

**Theorem 3.3.** Let $E$ be a real uniformly smooth Banach space with $\rho_E(t) \leq ct^2$ for some $c > 0$. Let the set-valued mappings $A, B, C, D, F$ be $H$-mixed Lipschitz continuous with pairs of constants $(L_A, l_A)$, $(L_B, l_B)$, $(L_C, l_C)$, $(L_D, l_D)$, $(L_F, l_F)$, respectively. Let $\eta : E \times E \to E$ be a $\tau$-Lipschitz continuous mapping and let $P : E \to E$ be a $\delta$-strongly $\eta$-accretive mapping. Let the mappings $(g - m)$, $P \circ (g - m)$ be mixed Lipschitz continuous with pairs of constants $(L_{(g - m)}, l_{(g - m)})$ and $(L_{P_0(g - m)}, l_{P_0(g - m)})$, respectively; let the mapping $N$ be $(L_{(N, 1)}, L_{(N, 2)}, l_N)$-mixed Lipschitz continuous and let the mapping $M$ be $\nu$-relaxed mixed $P \circ (g - m)$-Lipschitz with respect to $C$ and $D$, and $(L_{(M, 1)}, L_{(M, 2)}, l_M)$-mixed Lipschitz continuous. Suppose that the set-valued mapping $W$ is same as in Theorem 3.1 and conditions (3.2), (3.19), (3.20) hold, then for each $\lambda \in \Omega$, the solution set $S(\lambda)$ of PGIQVLIP (2.2) is a $H$-Lipschitz continuous mapping from $\Omega$ into $E$.

**Proof.** For each $\lambda, \tilde{\lambda} \in \Omega$, it follows from Theorem 3.2, $S(\lambda)$ and $S(\tilde{\lambda})$ are both nonempty and closed subsets of $E$. By Theorem 3.2, $G(x, \lambda)$ and $G(x, \tilde{\lambda})$ are both set-valued $\theta_1$-$H$-contraction mappings with same contractive constant $\theta_1 \in (0, 1)$. By Lemma 2.2, we obtain

$$H(S(\lambda), S(\tilde{\lambda})) \leq \left( \frac{1}{1 - \theta_1} \right) \sup_{x \in E} H(G(x, \lambda), G(x, \tilde{\lambda})), \quad (3.23)$$

where $\theta_1$ is given by (3.19).

Now, for any $i_1 \in G(x, \lambda)$, there exist $u = u(x, \lambda) \in A(x, \lambda)$, $v = v(x, \lambda) \in B(x, \lambda)$, $w = w(x, \lambda) \in C(x, \lambda)$, $y = y(x, \lambda) \in D(x, \lambda)$ and $z = z(x, \lambda) \in F(x, \lambda)$ satisfying

$$i_1 = x - (g - m)(x, \lambda) + J^W_{\rho}(w, \lambda) [P \circ (g - m)(x, \lambda) - \rho N(u, v, \lambda) + \rho M(w, y, \lambda) + \rho f]. \quad (3.24)$$

It is easy to see that there exist $\bar{u} = u(x, \tilde{\lambda}) \in A(x, \tilde{\lambda})$, $\bar{v} = v(x, \tilde{\lambda}) \in B(x, \tilde{\lambda})$, $\bar{w} = w(x, \tilde{\lambda}) \in C(x, \tilde{\lambda})$, $\bar{y} = y(x, \tilde{\lambda}) \in D(x, \tilde{\lambda})$ and $\bar{z} = z(x, \tilde{\lambda}) \in F(x, \tilde{\lambda})$ such that

$$\|u - \bar{u}\| \leq H(A(x, \lambda), A(x, \tilde{\lambda})) \leq l_A \|\lambda - \tilde{\lambda}\|,$$

$$\|v - \bar{v}\| \leq H(B(x, \lambda), B(x, \tilde{\lambda})) \leq l_B \|\lambda - \tilde{\lambda}\|,$$

$$\|w - \bar{w}\| \leq H(C(x, \lambda), C(x, \tilde{\lambda})) \leq l_C \|\lambda - \tilde{\lambda}\|,$$

$$\|y - \bar{y}\| \leq H(D(x, \lambda), D(x, \tilde{\lambda})) \leq l_D \|\lambda - \tilde{\lambda}\|,$$

$$\|z - \bar{z}\| \leq H(F(x, \lambda), F(x, \tilde{\lambda})) \leq l_F \|\lambda - \tilde{\lambda}\|. \quad (3.25)$$

Let

$$i_2 = x - (g - m)(x, \tilde{\lambda}) + J^W_{\rho}(w, \tilde{\lambda}) [P \circ (g - m)(x, \tilde{\lambda}) - \rho N(\bar{u}, \bar{v}, \tilde{\lambda}) + \rho M(\bar{w}, \bar{y}, \tilde{\lambda}) + \rho f]. \quad (3.26)$$
Clearly, \( i_2 \in G(x, \bar{\lambda}) \).

Since \( N \) and \( M \) are mixed Lipschitz continuous and in view of (3.3) and (3.23)–(3.26) and with \( t = P \circ (g - m)(x, \bar{\lambda}) - \rho N(\bar{u}, \bar{v}, \bar{\lambda}) + \rho M(\bar{w}, \bar{y}, \bar{\lambda}) \), we have

\[
\| i_1 - i_2 \| \leq \left\| (g - m)(x, \lambda) - (g - m)(x, \bar{\lambda}) \right\| \\
+ \| J^W_{\rho}((g - m)(x, \lambda)) \| \left[ P \circ (g - m)(x, \lambda) - \rho N(\bar{u}, \bar{v}, \bar{\lambda}) + \rho M(\bar{w}, \bar{y}, \bar{\lambda}) \right] - J^W_{\rho}(g - m)(x, \bar{\lambda})(t) \| \\
+ \| J^W_{\rho}((g - m)(\lambda)) \| \left[ J^W_{\rho}(g - m)(x, \lambda) - J^W_{\rho}(g - m)(x, \bar{\lambda}) \right] \|
\]

\[
\leq \left\| (g - m)(x, \lambda) - (g - m)(x, \bar{\lambda}) \right\| + \frac{\tau}{\delta} \left\| P \circ (g - m)(x, \lambda) - P \circ (g - m)(x, \bar{\lambda}) \right\|
\]

\[
+ \frac{\tau}{\delta} \left\| N(u, v, \lambda) - N(\bar{u}, \bar{v}, \bar{\lambda}) \right\| + \left\| M(w, y, \lambda) - M(\bar{w}, \bar{y}, \bar{\lambda}) \right\| + \mu_1 \| z - \bar{z} \| + \mu_2 \| \lambda - \bar{\lambda} \|
\]

\[
\leq l_{(g - m)} \| \lambda - \bar{\lambda} \| + \frac{\tau}{\delta} \| P \circ (g - m) \| \| \lambda - \bar{\lambda} \|
\]

\[
+ \frac{\tau}{\delta} \left[ l_{A(L, N, 1)} + l_{B(L, N, 2)} + l_{N} + I_{C(L, M, 1)} + I_{D(L, M, 2)} + l_{M} \right] \| \lambda - \bar{\lambda} \| + \mu_1 \| z - \bar{z} \| + \mu_2 \| \lambda - \bar{\lambda} \|
\]

\[
\leq \left( l_{(g - m)} + \mu_2 + \mu_1 I_{F} \right)
\]

\[
+ \frac{\tau}{\delta} \left[ l_{P \circ (g - m)} + \rho(l_{A(L, N, 1)} + l_{B(L, N, 2)} + l_{N} + I_{C(L, M, 1)} + I_{D(L, M, 2)} + l_{M}) \right] \| \lambda - \bar{\lambda} \|
\]

\[
\leq \theta_2 \| \lambda - \bar{\lambda} \|
\]

(3.27)

where

\[
\theta_2 := l_{(g - m)} + \mu_2 + \mu_1 I_{F} + \frac{\tau}{\delta} \left[ l_{P \circ (g - m)} + \rho(l_{A(L, N, 1)} + l_{B(L, N, 2)} + l_{N} + I_{C(L, M, 1)} + I_{D(L, M, 2)} + l_{M}) \right].
\]

Hence, we obtain

\[
\sup_{i_2 \in G(x, \lambda)} d(i_1, G(x, \bar{\lambda})) \leq \theta_2 \| \lambda - \bar{\lambda} \|
\]

By using similar argument, we have

\[
\sup_{i_2 \in G(x, \lambda)} d(i_2, G(x, \lambda)) \leq \theta_2 \| \lambda - \bar{\lambda} \|
\]

Hence, it follows that

\[
H(G(x, \lambda), G(x, \bar{\lambda})) \leq \theta_2 \| \lambda - \bar{\lambda} \|, \quad \forall (x, \lambda), (x, \bar{\lambda}) \in E \times \Omega.
\]

By Lemma 2.2, we obtain

\[
H(S(\lambda), S(\bar{\lambda})) \leq \left( \frac{\theta_2}{1 - \theta_1} \right) \| \lambda - \bar{\lambda} \|
\]

(3.28)

This implies that \( S(\lambda) \) is \( H \)-Lipschitz continuous in \( \lambda \in \Omega \), and this completes the proof. \( \square \)

References


