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Sign conjugacy classes in symmetric groups

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ABSTRACT

A special type of conjugacy classes in symmetric groups is studied and used to answer a question about odd-degree irreducible characters.

This work was initiated by a question about associating suitable signs to odd-degree irreducible characters in the symmetric groups S_n, posed by I.M. Isaacs and G. Navarro. The question is related to their work [3].

The positive answer to the question is given below. It is in a sense the best possible and it involves a special conjugacy class in S_n. The has led the author to a general definition of sign classes in finite groups. This general definition is discussed briefly in Section 1. In Section 2 we consider special types of sign classes in S_n and apply this to the Isaacs–Navarro question in Section 3. The final section contains a general result on sign classes in S_n and some thoughts about a possible classification of them.

1. Sign classes in finite groups

A *sign class* in a finite group G is a conjugacy class on which all irreducible characters of G take one of the values 0, 1 or -1. Elements in sign classes are called *sign elements*.

Sign elements of prime order p may occur when you have a self-centralizing p-Sylow subgroup of order p in G. This occurs for example for $p = 7$ in the simple group M_{11} which also has sign elements of order 6. In $\mathrm{SL}(2, 2^n)$ there is an involution on which all irreducible characters except the Steinberg character take the values 1 or -1. Thus this is a sign element. Non-central involutions in dihedral groups are also examples of sign elements.

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Column orthogonality for the irreducible characters of G shows that a sign element s gives rise to two disjoint multiplicity-free characters Θ_s^+ and Θ_s^- which coincide on all conjugacy classes except the class of s. They are defined as follows

$$\Theta_s^+ = \sum_{\{\chi \in \mathrm{Irr}(G) \mid \chi(s)=1\}} \chi \quad \text{and} \quad \Theta_s^- = \sum_{\{\chi \in \mathrm{Irr}(G) \mid \chi(s)=-1\}} \chi.$$

An example for symmetric groups is given below.

Block orthogonality shows that if p is a prime number dividing the order of the sign element s and if you split Θ_s^+ and Θ_s^- into components according to the p-blocks of characters of G, then the values of these components for a given p-block still coincide on all p-regular elements in G. This has consequences for the decomposition numbers of G at the prime p.

2. Sign partitions

In this paper we are concerned with sign classes in the symmetric groups S_n. The irreducible characters of S_n are all integer valued. Let $\mathcal{P}(n)$ be the set of partitions of n. We write the entries of the character table $X(n)$ of S_n as $[\lambda](\mu)$, for $\lambda, \mu \in \mathcal{P}(n)$. This is the value of the irreducible character of S_n labelled by λ on the conjugacy class labelled by μ.

We call $\mu \in \mathcal{P}(n)$ a *sign partition* if the corresponding conjugacy class is a sign class, i.e. if $[\lambda](\mu) \in \{0, 1, -1\}$ for all $\lambda \in \mathcal{P}(n)$. The *support* of a sign partition μ is defined as

$$\mathrm{supp}(\mu) = \{\lambda \in \mathcal{P}(n) \mid [\lambda](\mu) \neq 0\}.$$

For example the Murnaghan–Nakayama formula ([5, 2.4.7] or [4, 21.1]) shows that (n) is always a sign partition. Indeed $[\lambda](n) \neq 0$ if and only if $\lambda = (n-k, 1^k)$ is a hook partition and then $[\lambda](n) = (-1)^k$. Using column orthogonality for irreducible characters this has as a consequence that the generalized character

$$\Theta_{(n)} = \sum_{k=0}^{n-1} (-1)^k [n-k, 1^k]$$

takes the value 0 everywhere except on the class (n) where is has value n.

For an arbitrary sign partition μ

$$\Theta_\mu = \sum_{\lambda \in \mathrm{supp}(\mu)} [\lambda](\mu)[\lambda]$$

is a generalized character vanishing outside the conjugacy class of μ and it is the difference between disjoint multiplicity-free characters Θ_μ^+ and Θ_μ^-. (See Section 1.)

Below is a list of all sign partitions for $n = 2, \ldots, 10$:

$n=2$: (2), (1^2);
$n=3$: (3), $(2,1)$;
$n=4$: (4), $(3,1)$, $(2,1^2)$;
$n=5$: (5), $(4,1)$, $(3,2)$, $(3,1^2)$;
$n=6$: (6), $(5,1)$, $(4,2)$, $(4,1^2)$, $(3,2,1)$;
$n=7$: (7), $(6,1)$, $(5,2)$, $(5,1^2)$, $(4,3)$, $(4,2,1)$, $(3,2,1^2)$;
$n=8$: (8), $(7,1)$, $(6,2)$, $(6,1^2)$, $(5,3)$, $(5,2,1)$, $(4,3,1)$;
$n=9$: (9), $(8,1)$, $(7,2)$, $(7,1^2)$, $(6,3)$, $(6,2,1)$, $(5,4)$, $(5,3,1)$, $(5,2,1^2)$;
$n=10$: (10), $(9,1)$, $(8,2)$, $(8,1^2)$, $(7,3)$, $(7,2,1)$, $(6,4)$, $(6,3,1)$, $(6,2,1^2)$, $(5,4,1)$, $(4,3,2,1)$.

The sign partition $(4,2)$ of 6 yields two characters of degree 20

$$\Theta^{+}_{(4,2)} = [6] + [4,2] + \left[2^2, 1^2\right] + \left[1^6\right],$$
$$\Theta^{-}_{(4,2)} = [5,1] + \left[3^2\right] + \left[2^3\right] + \left[2, 1^4\right]$$

coinciding everywhere except on the class $(4,2)$ where they differ by a sign.

An important class of sign partitions are the *unique path*-partitions (for short *up*-partitions). They are described as follows. If $\mu = (a_1, a_2, \dots, a_k)$ and λ are partitions of n, then a μ-path in λ is a sequence $\lambda = \lambda_0, \lambda_1, \dots, \lambda_k = (0)$, of partitions, where for $i = 1, \dots, k$ λ_i is obtained by removing an a_i-hook in λ_{i-1}. Then we call μ is an *up*-partition for λ if the number of μ-paths in λ is at most 1. We call μ is an *up*-partition if it is an *up*-partition for all partitions λ of n.

Proposition 1. *An up-partition is also a sign partition.*

Proof. This follows immediately by repeated use of the Murnaghan–Nakayama formula. If there is no μ-path for λ, then $[\lambda](\mu) = 0$. Otherwise $[\lambda](\mu) = (-1)^k$, where k is the sum of the leg lengths of the hooks involved in the unique μ-path for λ. □

Remarks. 1. If $\mu = (a_1, a_2, \dots, a_k)$ is an *up*-partition with $a_k = 2$, then also $\mu' = (a_1, a_2, \dots, a_{k-1}, 1^2)$ is an *up*-partition.

2. If $\mu = (a_1, a_2, \dots, a_k)$ is an *up*-partition with $k \geqslant 2$, then also $\mu^* = (a_2, \dots, a_k)$ is an *up*-partition. Indeed, if a partition λ^* of $n - a_1$ has two or more μ^*-paths then a partition of n obtained be adding an a_1-hook to λ^* has two or more μ-paths.

3. The partition $(3,2,1)$ is a sign partition, but not an *up*-partition, since there are two $(3,2,1)$-paths in the partition $(3,2,1)$. Also $(4,3,2,1)$ is a sign partition, but not an *up*-partition, since there are two $(4,3,2,1)$-paths in the partition $(7,2,1)$.

Proposition 2. *Let $m > n$. If $\mu^* = (a_1, a_2, \dots, a_k)$ is a partition of n, and $\mu = (m, a_1, a_2, \dots, a_k)$ then μ^* is a sign partition* (*respectively an up-partition*) *of n if and only if μ is a sign partition* (*respectively an up-partition*) *of $m+n$.*

Proof. Let λ be a partition of $m+n$. Since $2m > m+n$, λ cannot contain more than at most one hook of length m, e.g. by 2.7.40 in [5]. This clearly implies that μ^* is an *up*-partition if and only if μ is an *up*-partition. If λ has no hook of length m, then $[\lambda](\mu) = 0$. If λ has a hook of length m, then remove the unique hook of that length to get the partition λ_1. Then $[\lambda](\mu) = \pm[\lambda_1](\mu^*)$. If μ^* is a sign partition we get that $[\lambda_1](\mu^*) \in \{0, 1, -1\}$ and thus $[\lambda](\mu) \in \{0, 1, -1\}$. This shows that if μ^* is a sign partition then μ is a sign partition. If μ is a sign partition and if $\lambda_1 \in \mathcal{P}(n)$, then add a hook of length m to λ_1 to get a partition λ. Since by assumption $[\lambda](\mu) \in \{0, 1, -1\}$, the same is true for $[\lambda_1](\mu^*)$. □

It is an interesting question whether it is possible to recognize from the parts of μ, whether or not μ is an *up*-partition or a sign partition. The final section of this paper contains results related to this question.

However the above proposition suggests the following definition of a class of sign partitions, given in terms of its parts.

If $\mu = (a_1, a_2, \dots, a_k)$ is a partition we call it *strongly decreasing* (for short an *sd*-partition) if we have $a_i > a_{i+1} + \dots + a_k$ for $i = 1, \dots, k-1$.

Remarks. 1. Obviously, if $\mu = (a_1, a_2, \dots, a_k)$ is an *sd*-partition with $k \geqslant 2$ then $\mu^* = (a_2, \dots, a_k)$ is also an *sd*-partition.

2. The partition $(3, 1^2)$ is an *up*-partition, but not an *sd*-partition.

Proposition 3. *An sd-partition is an up-partition and thus also sign partition.*

Proof. That an *sd*-partition is an *up*-partition is proved by repeated use of Proposition 2. □

Remark. The *sd*-partitions are closely related to the so-called “non-squashing” partitions. A partition $\mu = (a_1, a_2, \dots, a_k)$ is called non-squashing if $a_i \geqslant a_{i+1} + \cdots + a_k$ for all $i = 1, \dots, k-1$. It is known that the number non-quashing partitions of n equals the binary partitions of n, i.e. the number of partitions of n into parts which are powers of 2 [2,9]. Let $s(n)$ denote the number of *sd*-partitions of n. Put $s(0) = 1$. Ordering the set of *sd*-partitions according to their largest part shows that

$$s(n) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} s(i).$$

Thus for all $k \geqslant 1$ we have $s(2k-1) = s(2k)$. Putting $t(k) = 2s(2k) = s(2k-1) + s(2k)$ it can be shown that $t(k)$ is then equal to the number of binary partitions of $2k$.

Proposition 4. *If $\mu = (a_1, a_2, \dots, a_k)$ is a sign partition of n then the number of irreducible characters λ with $[\lambda](\mu) \neq 0$ is z_μ, the order of the centralizer of an element of type μ in S_n. In particular, for an sd-partition $z_\mu = a_1 a_2 \cdots a_k$.*

Proof. Since the non-zero values of irreducible characters on μ are 1 or -1 this follows from column orthogonality. □

3. The Isaacs–Navarro question

Some background for this may be found in [3].

Question *(Isaacs–Navarro).* Let P be 2-Sylow subgroup of S_n and $\mathrm{Irr}_{2'}(S_n)$ be the set of odd-degree irreducible characters of S_n. Does there exist signs e_χ for $\chi \in \mathrm{Irr}_{2'}(S_n)$ such that the character

$$\Theta = \sum_{\chi \in \mathrm{Irr}_{2'}(S_n)} e_\chi \chi$$

satisfies that

$$\text{(i)} \quad \Theta(x) \text{ is divisible by } |P/P'| \text{ for all } x \in S_n$$

and

$$\text{(ii)} \quad \Theta(x) = 0 \text{ for all } x \in S_n \text{ of odd order?}$$

This is answered positively by

Theorem 5. *Write $n = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_t}$, where $r_1 > r_2 > \cdots > r_t \geqslant 0$. Then $\mu = (2^{r_1}, 2^{r_2}, \dots, 2^{r_t})$ is a sd-partition with support $\mathrm{supp}(\mu) = \mathrm{Irr}_{2'}(S_n)$. Moreover Θ_μ satisfies the conditions* (i) *and* (ii) *above. Indeed Θ_μ vanishes everywhere except on μ where it takes the value $|P/P'|$.*

Proof. Clearly μ is an *sd*-partition and thus a sign partition, which implies that Θ_μ vanishes everywhere except on μ where it takes the value $z_\mu = 2^{r_1+r_2+\cdots+r_t}$. This is the cardinality of $\mathrm{supp}(\mu)$ (Proposition 4). If C_i is the iterated wreath product of i copies of the cyclic group of order 2 then C_i/C_i' is an elementary abelian group of order 2^i. Since $P \simeq C_{r_1} \times C_{r_2} \times \cdots \times C_{r_t}$ we get $|P/P'| = 2^{r_1+r_2+\cdots+r_t}$.

We need then only the fact that $\operatorname{supp}(\mu) = \operatorname{Irr}_{2'}(S_n)$. By [8, Theorem 4.1], $\operatorname{supp}(\mu) \subseteq \operatorname{Irr}_{2'}(S_n)$. (Since we here know that non-zero values on μ are ± 1, this also follows from a general character theoretic result [1, (6.4)].) On the other hand $|\operatorname{Irr}_{2'}(S_n)| = 2^{r_1+r_2+\cdots+r_t}$ by [6, Corollary (1.3)], so that the $\operatorname{supp}(\mu)$ cannot be properly contained in $\operatorname{Irr}_{2'}(S_n)$. □

Remark. The results from [6,8] quoted in the above proof are formulated for arbitrary primes. However Theorem 5 does not have an analogue for odd primes. J. McKay has pointed out, that in [7] it was shown that the partition μ of the theorem is a sign partition.

Example. In $\mathrm{SL}(2, 2^n)$ the 2-Sylow subgroup is self-centralizing. It has a unique conjugacy class of involutions and $2^n + 1$ irreducible characters, all of which (with the exception of the Steinberg character) have odd degrees. The involutions are sign elements, so that Θ_t, t involution, vanishes on all elements of odd order. The value on t is 2^n. Thus this is another example of the existence of signs for odd-degree irreducible characters such that the signed sum satisfy the conditions mentioned above.

4. Repeated parts in a sign partition

We want to show that repeated parts are very rare in sign partitions. Indeed only the part 1 may be repeated.

Lemma 6. *A sign partition μ cannot have its smallest part repeated except for the part* 1*, which may be repeated once.*

Proof. If 1 is repeated $m \geqslant 2$ times in the sign partition μ then $[n-1, 1](\mu) = [m-1, 1](1^m) = m-1$. Thus $m = 2$. If $b > 1$ is the smallest part, repeated $m \geqslant 2$ times in μ, then $[n-b, b](\mu) = m$. □

Theorem 7. *A sign partition cannot have repeated parts except for the part* 1*, which may be repeated once.*

Proof. We are going to assume that a is the smallest repeated part > 1 in the partition μ and that the multiplicity of a in μ is $m \geqslant 2$. We want to determine a partition λ satisfying that *all hook lengths outside the first row are* $\leqslant a$ and in addition $|[\lambda](\mu)| \geqslant m$.

Divide the parts of μ into

$a_1 \geqslant \cdots \geqslant a_{i-1}$ (all greater than a) (sum s, say);
$a_i, \ldots, a_{i+m-1}$ (m parts all equal to a);
$a_{i+m} > \cdots > a_k$ (all parts smaller than a) (sum t, say). (However we allow $a_{k-1} = a_k = 1$.)

We let $\mu_0 = (a_{i+m}, \ldots, a_k)$, a partition of t.

By Lemma 6 we may assume that $t > 0$. An easy analysis shows that we may assume $a \geqslant 4$. (To do this we just have to show that partitions on the form

$$\left(2^m, 1\right), \quad \left(2^m, 1^2\right), \quad \left(3^m, 2, 1\right), \quad \left(3^m, 2, 1^2\right), \quad \left(3^m, 1\right), \quad \left(3^m, 1^2\right), \quad m \geqslant 2,$$

are not sign partitions. For example $[2m-1, 1^2](2^m, 1) = [2m, 1^2](2^m, 1^2) = -m$.)

First we notice that we need only consider the case that $s = 0$. Indeed, if λ' is a partition of $n - s$ satisfying that all hook lengths outside the first row are $\leqslant a$ and that $|[\lambda'](\mu')| \geqslant m$, where $\mu' = (a_i, \ldots, a_k)$ and λ is obtained by adding s to the largest part of λ' then MN shows that $[\lambda](\mu) = [\lambda'](\mu')$ and we are done. (Here and in the following MN refers to the Murnaghan–Nakayama formula.) Thus we may assume that $a = a_1$ is the only repeated part of μ, apart (possibly) from 1.

We have then $n = ma + t$. Let for $0 \leqslant \ell \leqslant m$, μ_ℓ be μ_0 with ℓ parts equal to a added. Thus $\mu_m = \mu$.

Now $(n-a, 1^a)$ has only two hooks of length a so MN shows $[n-a, 1^a](\mu) = (-1)^{a-1}[n-a](\mu_{m-1}) + [n-2a, 1^a](\mu_{m-1}) = (-1)^{a-1} + [n-2a, 1^a](\mu_{m-1})$.

Inductively we get $[n-a, 1^a](\mu) = (m-1)(-1)^{a-1} + [t, 1^a](\mu_1)$.

If $t \leqslant a$ then $[t, 1^a]$ has only one hook of length a and we get $[t, 1^a](\mu_1) = (-1)^{a-1}[t](\mu_1) = (-1)^{a-1}$ and consequently $[n-a, 1^a](\mu) = m(-1)^{a-1}$. Thus $[n-a, 1^a]$ may be chosen as the desired λ.

Therefore we may now assume that $a < t$.

Consider then the case $t < 2a$ so that $t - a < a$. There are exactly a partitions of t obtained by adding an a-hook to the partition $(t - a)$. Suppose that κ_i is obtained by adding a hook with leg length i to $(t - a)$. Here $0 \leqslant i \leqslant a - 1$.

Since $t < 2a$ each κ_i has only one hook of length a (e.g. by 2.7.40 in [5]). Removing it we get $(t-a)$. Note that $\kappa_0 = (t)$. By Theorem 21.7 in [4] the generalized character $\sum_{i=0}^{a-1}(-1)^i\kappa_i$ takes the value 0 on μ_0, since μ_0 has no part divisible by a. Choose a $j > 0$ such that $(-1)^j[\kappa_j](\mu_0) \geqslant 0$. (Clearly, the $(-1)^j[\kappa_j](\mu_0)$ cannot all be < 0, since the contribution from $[t]$ is equal to 1 and $a \geqslant 4$.)

If $1 \leqslant j \leqslant 2a - t - 1$ then the hook of length a in κ_j lies in the first row. In this case let λ_1 be the partition $(t, a - (j - 1), 1^{j-1})$ of $t + a$. Then λ_1 has two hooks of length a, one in row 1 with leg length 1 and one in row 2 with leg length $j - 1$. If you remove the one in row 1 you get the partition κ_j and if you remove the one in row 2 you get κ_0. MN shows that

$$[\lambda_1](\mu_1) = (-1)^{j-1}[\kappa_0](\mu_0) - [\kappa_j](\mu_0) = (-1)^{j-1}\bigl(1 + (-1)^j[\kappa_j](\mu_0)\bigr).$$

If $2a - t \leqslant j \leqslant a - 1$ then the hook of length a in κ_j lies in the second row. In this case let λ_1 be the partition $(t, a - j, 1^j)$ of $t + a$. Then λ_1 has two hooks of length a, one in row 1 with leg length 0 and one in row 2 with leg length j. If you remove the one in row 1 you get the partition κ_j and if you remove the one in row 2 you get κ_0. MN shows that

$$[\lambda_1](\mu_1) = (-1)^j[\kappa_0](\mu_0) + [\kappa_j](\mu_0) = (-1)^j\bigl(1 + (-1)^j[\kappa_j](\mu_0)\bigr).$$

Putting $\epsilon_j = (-1)^{j-1}$ for $1 \leqslant j \leqslant 2a - t - 1$ and $\epsilon_j = (-1)^j$ for $2a - t \leqslant j \leqslant a - 1$, we then have

$$[\lambda_1](\mu_1) = \epsilon_j\bigl(1 + (-1)^j[\kappa_j](\mu_0)\bigr).$$

Let for $\ell \geqslant 2$ λ_ℓ be obtained from λ_1 by adding $(\ell - 1)a$ to its largest part. Thus the largest part of $\lambda := \lambda_m$ is $n - a$ so that all hook lengths outside the first row are $\leqslant a$. We claim that $|[\lambda](\mu)| \geqslant m$.

By MN we have for $\ell \geqslant 2$

$$[\lambda_\ell](\mu_\ell) = [\lambda_{\ell-1}](\mu_{\ell-1}) + \epsilon_j.$$

Thus

$$\begin{aligned}[\lambda](\mu) = [\lambda_m](\mu_m) &= [\lambda_{m-1}](\mu_{m-1}) + \epsilon_j \\ &= [\lambda_{m-2}](\mu_{m-2}) + 2\epsilon_j\end{aligned}$$

and so on. This shows

$$[\lambda](\mu) = [\lambda_1](\mu_1) + (m - 1)\epsilon_j.$$

Thus

$$[\lambda](\mu) = [\kappa_j](\mu_0) + m(-1)^j = \epsilon_j\bigl(m + (-1)^j[\kappa_j](\mu_0)\bigr).$$

The choice of j guarantees that this has absolute value $\geqslant m$, so that μ is not a sign class.

A similar argument may be used in the case $t \geqslant 2a$. Then $t - a \geqslant a$ and it is possible to add an a-hook to the partition $(t-a)$ in $a+1$ ways. Putting an a-hook with leg length i below $t-a$ gives you a partitions κ_i, $i = 0, \ldots, a - 1$. In addition we have the partition (t). Using again Theorem 21.7 in [4] we see that the generalized character $\sum_{i=0}^{a-1}(-1)^i\kappa_i$ takes the value -1 on μ_0. It is possible to choose a $j \geqslant 0$ such that $(-1)^j[\kappa_j](\mu_0) \geqslant 0$. Otherwise we would have $-1 = \sum_{i=0}^{a-1}(-1)^i[\kappa_i](\mu_0) \leqslant -a$. We then proceed as in the previous case. □

Corollary 8. *If μ is a sign partition, then the centralizer of elements of cycle type μ is abelian. In short: Centralizers of sign elements in S_n are abelian.*

Remark. G. Navarro has kindly pointed out that there exists a group of order 32 containing a sign element with a non-abelian centralizer.

Corollary 9. *Suppose that $n = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_t}$, where $r_1 > r_2 > \cdots > r_t \geqslant 0$. The sign classes of 2-elements in S_n have for n odd (i.e. $r_t = 0$) cycle type $(2^{r_1}, 2^{r_2}, \ldots, 2^{r_t})$. If $n = 4k + 2$ (i.e. $r_t = 1$) we have in addition $(2^{r_1}, 2^{r_2}, \ldots, 2^{r_{t-1}}, 1^2)$. If $n = 8k + 4$ (i.e. $r_t = 2$) we have in addition $(2^{r_1}, 2^{r_2}, \ldots, 2^{r_{t-1}}, 2, 1^2)$.*

Proof. If a sign class of a 2-element in S_n does not have the type $(2^{r_1}, 2^{r_2}, \ldots, 2^{r_t})$, then by Theorem 7, the part 1 has to be repeated twice. We have seen that (1^2) and $(2, 1^2)$ are sign partitions. Therefore Proposition 2 shows that the other two cycle types listed in the corollary are indeed cycle types for sign classes. For these values of n there can be no more sign classes. If $n = 8k$ (i.e. $r_t \geqslant 3$), then the possibility that 2^{r_t} is replaced by $2^{r_t-1}, 2^{r_t-2}, \ldots, 2, 1^2$ is excluded by Proposition 2 and the fact that $(4, 2, 1^2)$ is not a sign partition. □

Finally we formulate a conjecture about which partitions are sign partitions. It seems that sign partitions are close to being *sd*-partitions.

We fix the following notation: $\mu^* = (a_1, a_2, \ldots, a_r)$ for some $r \geqslant 2$ is a partition of t and $\mu = (a, a_1, \ldots, a_r)$ where $a > a_1$.

Then μ is called *exceptional* if $a \leqslant t$ and both μ and μ^* are sign partitions and in addition the partitions $(a_i, a_{i+1}, \ldots, a_r)$ are all sign partitions.

If we can determine the exceptional partitions, then we also know all the sign partitions. However there exist infinite series of exceptional partitions. Indeed it can be shown that the following partitions are exceptional:

- $(a, a-1, 1)$ for $a \geqslant 2$.
- $(a, a-1, 2, 1)$ for $a \geqslant 4$.
- $(a, a-1, 3, 1)$ for $a \geqslant 5$.

The author suspects strongly that these are the only infinite series of exceptional partitions and would like to state the following conjecture.

Conjecture. *Let $\mu = (a_1, a_2, \ldots, a_k)$ be a partition. Then μ is a sign partition if and only if one of the following conditions hold:*

(1) *μ is an sd-partition, i.e. $a_i > a_{i+1} + \cdots + a_k$ for $i = 1, \ldots, k-1$.*
(2) *$a_i > a_{i+1} + \cdots + a_k$ for $i = 1, \ldots, k-2$ and in addition $a_{k-1} = a_k = 1$.*
(3) *$a_i > a_{i+1} + \cdots + a_k$ for $i = 1, \ldots, k-3$ and in addition $(a_{k-2}, a_{k-1}, a_k) = (a, a-1, 1)$ for some $a \geqslant 2$.*
(4) *$a_i > a_{i+1} + \cdots + a_k$ for $i = 1, \ldots, k-4$ and in addition $(a_{k-3}, a_{k-2}, a_{k-1}, a_k)$ is one of the following*
   - *$(a, a-1, 2, 1)$ for some $a \geqslant 4$;*
   - *$(a, a-1, 3, 1)$ for some $a \geqslant 5$;*
   - *$(3, 2, 1, 1)$;*
   - *$(5, 3, 2, 1)$.*

We hope to be able return to this conjecture later. Its verification would also easily imply a classification of *up*-partitions.

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