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On Azumaya Algebras and Finite Dimensional Representations of Rings*

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This paper grew out of an attempt to understand some beautiful results of *Kaplansky* and others on associative rings with polynomial identities. It follows from their theory that these rings tend to have large centers. Now the study of rings which are finite modules over their centers can be to a large extent, reduced to the study of the centers themselves, i.e., to commutative algebra. It is therefore natural to ask for conditions on a ring A satisfying a polynomial identity which assure that A is in fact finite over its center. *Kaplansky*'s theorem [*9*] asserts that primitivity is such a condition. On the other hand, examples of *Procesi* [*13*] show that a ring with a polynomial identity is in general not finite over its center, so that some conditions are needed. Our main result (8.3) is a characterization of those rings A which are Azumaya algebras of rank n^2 over their center in intrinsic terms, i.e., without a priori reference to the center. (In the theorem, we assume that A is an algebra over a field k. The problem in the general case is still open.) This result may be viewed as a generalization of *Kaplansky*'s theorem. It is also related to an analogous result for C^* -algebras due to *Takesaki* and *Tomiyama* [*15*], [*16*].

As a by product of our methods, we get a precise description of the locally closed subset of the spectrum of a ring corresponding to irreducible n-dimensional representations. It is homeomorphic, with its "Zariski topology", to a certain scheme (Section 11). In this connection, we want to raise the following question, which seems very interesting to us: Let A be a finitely generated k-algebra. Denote by d_n the Krull dimension of the space of irreducible representations of A of dimension n (cf. Section 9). Is there a reasonable asymptotic behavior of d_n? For instance, is the generating function $\sum_n d_n t^n$ a rational function?

Our techniques are a combination of techniques of *Amitsur*, *Herstein*, and

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Procesi ([*1*], [*6*], [*13*]) with familiar methods of commutative algebra. The translation of these methods for our purposes may be of some independent interest; it is done in Part I. The main results are in Part II. Some propositions which we consider sufficiently trivial are stated without proof.

We make fundamental use of the concept of *extension* of *Procesi*.

PART I: GENERALITIES.

1. Bimodules

By *ring*, we mean associative ring with unit element. A ring homomorphism is assumed to preserve the unit.

Let A be a ring, and M a two-sided A-module. Its *center* is defined to be the set

$$Z(M) = Z_A(M) = \{m \in M \mid am = ma, \quad all \quad a \in A\}. \tag{1.1}$$

We call M a *bimodule* if M is generated by its subset $Z(M)$, so that any $m \in M$ has the form

$$m = \sum_i a_i\mu_i = \sum_i \mu_i a_i$$

for some $\mu_i \in Z(M)$, and $a_i \in A$.

If M is a bimodule, then the operations of A on the left and right on M commute, i.e., the associative law

$$(bm)\, c = b(mc)$$

holds for all $b, c \in A$, and $m \in M$. For,

$$\left(b\left(\sum a_i\mu_i\right)\right) c = \left(\sum \mu_i b a_i\right) c = \sum \mu_i(ba_ic)$$

$$= \sum (ba_ic)\, \mu_i = b\left(\left(\sum a_i\mu_i\right) c\right).$$

If A is a commutative ring, then a bimodule is just a module in the usual sense.

By *homomorphism* $\phi : M \to N$ of bimodules, we mean a homomorphism of 2-sided modules. It necessarily carries $Z(M)$ to $Z(N)$. One sees easily that, conversely, a map ϕ which carries $Z(M)$ to $Z(N)$ is a homomorphism iff. it is left linear, or right linear. The set $\mathrm{Hom}_A(M, N)$ of homomorphisms is a module over the center $Z = Z(A)$ of A.

Clearly the ring A itself is an A-bimodule. Its center Z is the center in the usual sense. Homomorphisms from A to any two-sided module M correspond in a 1–1 way to elements μ of $Z(M)$, the homomorphism sending

$$a \mapsto a\mu, \quad a \in A.$$

A free bimodule F is one isomorphic to a direct sum of copies of A. Such an F has a *basis* $\{\phi_i\} \subset Z(F)$. Any bimodule M is a *quotient* of some free module F, so that there is an exact sequence

$$0 \to R \to F \to M \to 0 \tag{1.2}$$

where $R \subset F$ is a *two-sided* submodule. Note that the "module of relations" R is in general *not* a bimodule. We say that M is of *finite type* if it is finitely generated as a left (or right, or two-sided) module, which implies that an exact sequence (1.2) can be found in which F has a finite basis.

Let $Z = Z(A)$. It is immediately seen that the functor $A \otimes_Z .$ carries Z-modules N to A-bimodules $A \otimes_Z N$ with the definition

$$(a \otimes n)\, b = ab \otimes n \quad \text{for} \quad a, b \in A \quad \text{and} \quad n \in N.$$

In the other direction, we have the functor $Z(.)$ from A-bimodules to Z-modules.

PROPOSITION (1.3). *With the above notation, the functors* $A \otimes_Z .$ *and* $Z(.)$ *are adjoint*, i.e.,

$$\mathrm{Hom}_A(A \otimes_Z N, M) \approx \mathrm{Hom}_Z(N, Z(M)).$$

2. Some Special Cases

In this section, we give some examples in which the functor $A \otimes_Z .$ introduced in the last section is an equivalence of categories.

PROPOSITION (2.1). *The functor* $A \otimes_Z .$ *from* free Z*-modules to* free A*-bimodules is an equivalence of categories.*

Proof. Every free A-bimodule is isomorphic to $A \otimes_Z F_0$ for some free Z-module F_0. Let F_0 have the basis $\{\phi_i\}$. Then $\{1 \otimes \phi_i\}$ is a basis for $A \otimes_Z F_0$, and it is clear that an element $m = \sum a_i \otimes \phi_i$ is in $Z(A \otimes_Z F_0)$ iff. each a_i is in Z, i.e., iff. m is of the form $m = 1 \otimes m_0$. Since a map f from a free bimodule $A \otimes_Z G_0$ to $A \otimes_Z F_0$ must carry elements of a basis (in the center of $A \otimes_Z G_0$) to $Z(A \otimes_Z F_0)$, it follows that f is induced from a unique map $f_0 : G_0 \to F_0$.

Remark (2.2). The *rank* of a free bimodule is therefore uniquely determined. Note also that a change of basis $\{\phi_i\} \subset Z(F)$ of a free bimodule F comes from a change of basis of the corresponding free Z-module, i.e., is given (say in the finite case) by an invertible matrix with entries in Z.

PROPOSITION (2.3). *Suppose A has the property that every two-sided submodule R of a free A-bimodule F is also a bimodule. Then the functor $A \otimes_Z \cdot$ from Z-modules to A-bimodules is an equivalence of categories.*

Proof. Let M be any A-bimodule, and write M as a quotient of a free bimodule F, with relations R (cf. (1.2)). Then R is a bimodule, hence is a quotient of a free bimodule G. Thus there is an exact sequence

$$G \xrightarrow{f} F \to M \to 0.$$

By (2.1), the map f is induced up to isomorphism by tensoring a map $f_0 : G_0 \to F_0$ of free Z-modules with A, where $F \approx A \otimes_Z F_0$ and $G \approx A \otimes_Z G_0$. Let M_0 be the cokernel of f_0. Then since $A \otimes_Z \cdot$ is a right exact functor, $M \approx A \otimes_Z M_0$.

Given a map $g : M \to M'$ of A-bimodules, it is easily seen as above that the map may be embedded in a row-exact diagram

$$\begin{array}{ccccccc} G & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \\ G' & \longrightarrow & F' & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

where G, F, G', F' are free. Then by (2.1), the entire left hand square of this diagram is obtained by tensoring from a diagram of Z-modules, whence g is obtained by tensoring the induced map of cokernels.

Recall that a *simple* ring A is a nonzero ring having no proper 2-sided ideals. The center of a simple ring is a field, as is easily seen.

PROPOSITION (2.4). *If A is simple, then every twosided submodule of an A-bimodule is a bimodule. Hence the functor $A \otimes_Z \cdot$ from Z-vector spaces to A-bimodules is an equivalence of categories, and every A-bimodule is free.*

Proof. The last assertions follow from (2.3) if we prove that every submodule of a free A-bimodule is free. Thus it suffices to prove the first assertion for a free bimodule M. Let $R \subset M$ be a twosided submodule, and $m \in R$. Write

$$m = \sum_{i=1}^{n} a_i \mu_i$$

with $\mu_i \in Z(M)$ part of a basis, and $a_i \in A$. We want to write m as a linear combination of elements of the center of R, and we proceed by induction on n. If $a_1 = 0$, we are through. If not, then since A is simple, we can write

$$1 = \sum_\nu b_\nu a_1 c_\nu \qquad b_\nu, c_\nu \in A.$$

Then $\sum b_\nu m c_\nu = m' \in R$ is of the form

$$m' = \mu_1 + \sum_2^n a_i' \mu_i .$$

If $a_i' \in Z$ for all i, then $m' \in Z(M)$, and $m - a_1 m'$ is a linear combination of fewer μ_i, whence we are through by induction. If say $a_2' \notin Z$, then for some $b \in A$

$$a_2'' = ba_2' - a_2'b \neq 0.$$

Hence

$$m'' = bm' - m'b \in R$$

is nonzero, and is a linear combination $m'' = \sum_2^n a_i'' \mu_i$ of fewer terms. Thus m'' is a linear combination of elements of the center of R. Since $a_2'' \neq 0$, we can write

$$1 = \sum_\nu b_\nu'' a_2'' c_\nu'' ,$$

whence $m''' = \sum b_\nu'' m'' c_\nu'' \in R$ is of the form

$$m''' = \mu_2 + a_3''' \mu_3 + \cdots + a_n''' \mu_n .$$

m''' is also a linear combination of elements of $Z(R)$, and $m - a_2 m'''$ involves fewer than n terms. Thus we are again done by induction.

Proposition (2.5). *Let A be an azumaya algebra of rank n^2 over its center Z.*

(i) *Let M be a two-sided A-module such that the operations of A on the left and right commute. Then M is an A-bimodule if and only if it is a Z-(bi)module by restriction of scalars.*

(ii) *The categories of Z-modules and A-bimodules are equivalent, via $A \otimes_Z \cdot$.*

Proof. It is immediately seen that an A-bimodule is a Z-module by restriction of scalars. Moreover, a two-sided submodule of an A-bimodule satisfies the assumptions of (i) and is a Z-module by restriction. Therefore assertion (ii) follows from (i) and (2.3).

It remains to prove the "if" part of (i). Assume first that A is free as a Z-module. Now a structure of 2-sided A-module on a Z-module M (such that the operations commute) is equivalent with a structure of left $A \otimes_Z A^\circ = E$-module on M. The ring E is well known to be ring of endomorphisms of A as Z-module ([3], p. 180, exc. 13), hence it is isomorphic to a full ring of matrices over Z, since A was assumed free. Therefore E is isomorphic to a direct sum of copies of A, as left E-module. Thus every left E-module M is isomorphic to a quotient of a sum of copies of A, which shows that the corresponding 2-sided A-module structure makes M into an A-bimodule.

In the general case, A is anyhow a projective Z-module, hence there exists a set of elements $S \subset Z$ which generates the unit ideal, such that A_s is a free Z_s-module for each $s \in S$ (where the subscript s denotes localization with respect to that element). The hypotheses of (i) on M are clearly preserved by localization. Hence each M_s is an A_s-bimodule by the above reasoning. Thus we are reduced to the following:

LEMMA (2.6). *Let A, M satisfy the assumptions of* (2.5)(i). *The property of M of being an A-bimodule is local for the Zariski topology on* Spec Z, i.e., *if $S \subset Z$ is a set of elements which generate the unit ideal, then M_s is an A_s-bimodule for each $s \in S$* iff. *M is an A-bimodule.*

3. ALGEBRAS

Let $f : A \to B$ be a homomorphism of rings. Following *Procesi* [13], we say that f is an *extension*, or that B is an A-*algebra* (associative with 1), if f makes B into an A-bimodule, i.e., if B is generated as an A-module by its centralizer = its center as two-sided A-module

$$Z_A(B) = \{\beta \in B \mid f(a)\,\beta = \beta f(a) \quad \text{for all} \quad a \in A\}.$$

This is equivalent with the assertion that B is generated as a ring by $f(A)$ and $Z_A(B)$. (This is immediate.)

The universal example of an A-algebra is the algebra of noncommutative polynomials in a set of variables $\{x_i\}$, with coefficients in A. We will denote this algebra by $A\{x_i\}$. Its elements are finite sums

$$\sum_\nu a_\nu m_\nu$$

where $0 \neq a_\nu \in A$, and the m_ν are distinct noncommutative monomials in the x_i's. Multiplication and addition are as usual. Thus the x_i's commute with A, i.e., are in $Z_A(A\{x_i\})$. In fact, $Z_A(A\{x_i\})$ is just the ring

$$Z_A(A\{x_i\}) = Z\{x_i\}, \qquad Z = Z(A).$$

It is easily checked that any A-algebra B is isomorphic to a quotient of some $A\{x_i\}$ by a (two-sided) ideal.

An A-algebra B will be called a *central A-algebra* or a *central extension* if it is generated as A-algebra (or equivalently, as A-bimodule) by its center $Z(B) = Z_B(B)$. The universal example is the (commutative) polynomial ring $A[x_i]$ on a set of variables $\{x_i\}$ with coefficients in A. In this ring the elements x_i commute with elements of A and with each other. Any central A-algebra is isomorphic to a quotient of some $A[x_i]$ by an ideal.

As with A-bimodules, the structure of A-algebras and central A-algebras is closely related to the center Z of A. For instance, we obtain immediately from (2.4), (2.5):

COROLLARY (3.1). *If A is a simple ring or an azumaya algebra over its center, then the categories of A-algebras (resp. central A-algebras) and of Z-algebras (resp. commutative Z-algebras) are equivalent via $A \otimes_Z \cdot$.*

This corollary for simple rings is well known. It is just a restatement of ([*8*] Theorem 1, p. 109).

COROLLARY (3.2). *A central extension of an azumaya algebra is again an azumaya algebra.*

4. TENSOR PRODUCTS

Let M, N be A-bimodules. Then the left multiplication on M and right multiplication on N make $M \otimes_A N$ into an *A-bimodule*. For, the set $\{\mu \otimes \nu \mid \mu \in Z(M), \nu \in Z(N)\}$ generates $M \otimes_A N$, and it is clearly contained in $Z(M \otimes_A N)$.

PROPOSITION (4.1). *Let M, N be A-bimodules. There is a unique isomorphism $M \otimes_A N \approx N \otimes_A M$ sending $\mu \otimes \nu$ to $\nu \otimes \mu$ for $\mu \in Z(M)$, $\nu \in Z(N)$. The isomorphism sends $m \otimes n$ to $n \otimes m$ if either $m \in Z(M)$ or $n \in Z(N)$.*

PROPOSITION (4.2). *Let $A \to B$ be an extension, and M an A-bimodule. Then $B \otimes_A M$ is a left B-module. Via the isomorphism $B \otimes_A M \approx M \otimes_A B$, of* (4.1), *it is also a right B-module. It is a B-bimodule with these operations.*

COROLLARY (4.3). *If $I \subset A$ is an ideal, and M is an A-bimodule, then $IM = MI = IMI$, and M/IM is an A/I-bimodule.*

PROPOSITION (4.4). *Let $A \to B$ be an extension, and N a B-bimodule.*

(i) *N is an A-bimodule by restriction of scalars.*

(ii) *Let $A \cdot Z_B(N)$ denote the A-submodule of N generated over A by its center $Z_B(N)$. It is an A-bimodule, and*

$$Z_B(N) = Z_A(A \cdot Z_B(N)).$$

(iii) *If M is an A-bimodule, then*

$$\mathrm{Hom}_B(B \otimes_A M, N) \approx \mathrm{Hom}_A(M, A \cdot Z_B(N)).$$

PROPOSITION (4.5). *Let $A \to B$, $A \to C$ be extensions. There is a unique ring structure on $B \otimes_A C$ having the property that*

$$(\beta \otimes 1)(1 \otimes \gamma) = (1 \otimes \gamma)(\beta \otimes 1) = \beta \otimes \gamma$$

for all $\beta \in Z_A(B)$ and $\gamma \in Z_A(C)$, and such that the arrows in the commutative diagram

$$\begin{tikzcd} & C \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & B \otimes_A C \\ & B \arrow[ur] & \end{tikzcd}$$

are extensions. They are central extensions if $A \to B$, $A \to C$ are. This ring structure is compatible with the symmetry (4.1). *Moreover, we have*

$$(b \otimes c)(b' \otimes c') = bb' \otimes cc'$$

if either $b' \in Z_A(B)$ or $c \in Z_A(C)$.

Note: (i) It is not enough for the existence of this ring structure that one or the other of the ring homomorphisms $A \to B$, $B \to C$ be an extension. In general, we need that both are extensions.

(ii) I don't know if the ring $B \otimes_A C$ of the proposition is actually the coproduct of B and C in the category of A-algebras and extensions.

Example. Let $A = M_a(k)$ be an $a \times a$ matrix algebra over a field. Let $B = M_{ab}(k)$, and view A as a subring of B by repetition of an $a \times a$ matrix b times along the diagonal. Then B is an extension of A by (2.5)(i). The ring $B \otimes_A B$ is isomorphic to $M_{abb}(k)$.

5. Nakayama Lemma

Let (a_{ij}) be an $n \times n$ matrix with values in a ring A. By its *determinant* we mean the element of A computed by the usual rule

$$\det(a_{ij}) = \sum_{\sigma} (-1)^{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)} ,$$

where σ runs over all permutations of the set $\{1,..., n\}$. If M is a two-sided A-module such that the operations of A on the left and right commute, and if one row or one column of the matrix has its values in M, the rest being in A, then the determinant is again defined by the same formula, and it takes its value in M.

The following is a weak version of the usual Cramer's rule:

Proposition (5.1). *Let M be as above, and let*

$$a_{ij} \in A; \quad x_i \in Z(M); \quad m_i \in M \qquad i, j = 1,..., n.$$

Suppose that the equations

$$\begin{array}{ccc} a_{11}x_1 + \cdots + a_{1n}x_n & = & m_1 \\ \vdots \qquad\qquad \vdots & & \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n & = & m_n \end{array}$$

hold in M. Then the equations

$$\det(a_{ij})\, x_k = D_k \qquad k = 1,..., n$$

also hold, where D_k is the determinant of the matrix obtained by substituting m_j for a_{kj} $(j = 1,..., n)$ in (a_{ij}).

We leave the proof as an exercise. Note that we do *not* make any assertion of *existence* of solutions $\{x_i\}$ to the equations.

For an A-bimodule M, it is clear that its left and right annihilators coincide. We refer to this (two-sided) ideal $\mathfrak{a}$ of A as the *annihilator* of M.

Proposition (5.2). (*Nakayama lemma*). *Let M be an A-bimodule of finite type with annihilator $\mathfrak{a}$, and let $\mathfrak{p}$ be a prime ideal of A. Put $\bar{A} = A/\mathfrak{p}$, $\bar{M} = M/\mathfrak{p}M$. Then the annihilator of $\bar{M}$ as $\bar{A}$-bimodule is zero if and only if $\mathfrak{p} \supset \mathfrak{a}$. In particular, suppose p is a maximal ideal, so that A is simple. Then $\bar{M} \neq 0$ if and only if $\mathfrak{a} \subset \mathfrak{p}$.*

Proof. The last assertion follows immediately from the first and (2.4). Now it is clear that the residues of elements of $\mathfrak{a}$ in $\bar{A}$ annihilate $\bar{M}$. Therefore the annihilator of $\overline{M}$ is non-zero if $\mathfrak{p} \not\supset \mathfrak{a}$. Suppose conversely that $\bar{a} \in \bar{A}$ is a non-zero annihilator of $\overline{M}$, but that $\mathfrak{a} \subset \mathfrak{p}$. Let a represent $\bar{a}$ in A, and let $x_1, ..., x_n$ be central generators of M. Since $\bar{a}$ annihilates $\overline{M}$, we can find $b_{ij} \in \mathfrak{p}$ so that

$$(*) \qquad ax_i = \sum_j b_{ij} x_j$$

for each i. Therefore by the above proposition, the determinant $\det(aI - (b_{ij}))$ annihilates each x_i, and hence M, where I denotes the identity matrix. Thus the determinant is in $\mathfrak{p}$. The expansion of this determinant yields

$$\det(aI - (b_{ij})) \equiv a^n \pmod{\mathfrak{p}}.$$

Therefore $a^n \in \mathfrak{p}$. Now we can multiply each of the equations (*) by an arbitrary element $c_i \in A$, say on the left. In the same way, we obtain

$$c_1 a c_2 a \cdots c_n a \in \mathfrak{p} \quad \text{for all} \quad c_1, ..., c_n \in A.$$

Since $\mathfrak{p}$ is a prime ideal, this implies $a \in \mathfrak{p}$, which is a contradiction.

6. Flatness

Throughout this section a tensor product, when not otherwise indicated, is assumed to be with respect to a fixed ring A as scalars.

Let $f : A \to B$ be an extension. We recall that B is *left flat* over A if the functor $B \otimes_A \cdot$ from left A-modules to left B-modules is exact. Right flatness is defined similarly (cf. [3] for a treatment). We say that f is *flat for bimodules* if for every exact sequence

$$M' \to M \to M''$$

of *A-bimodules*, the sequence of B-bimodules

$$B \otimes M' \to B \otimes M \to B \otimes M''$$

is exact. Thus left flatness (resp. right flatness) implies flatness for bimodules.

Proposition (6.1). *The following assertions on an extension $f : A \to B$ are equivalent. When f has these properties, B will be said to be* faithfully flat *over A* for bimodules.

(i) *Let $M' \to M \to M''$ be a sequence of A-bimodules. It is an exact sequence if and only if the sequence $B \otimes M' \to B \otimes M \to B \otimes M''$ is exact.*

(ii) *B is flat over A for bimodules, and for every A-bimodule the canonical map $M \to B \otimes M$ (sending $m \mapsto 1 \otimes m$) is injective.*

Proof. Suppose (i) holds, and let M be an A-bimodule. The kernel R of the map $M \to B \otimes M$ is a twosided submodule. Let $\overline{M} = M/R$. Since the kernel of $B \otimes M \to B \otimes \overline{M}$ is the image of $B \otimes R$ in $B \otimes M$, which is zero, it follows that the sequence $0 \to B \otimes M \to B \otimes \overline{M}$ is exact. By (i), we have $0 \to \overline{M} \to M$, whence $R = 0$, as was to be shown.

Conversely, suppose (ii) holds, and let (*): $M' \to M \to M''$ be a sequence of A-bimodules which becomes exact when tensored with B. Let $N \subset M$ be the image of M', and $\overline{M} = M/N$. Then since B is flat over A, $B \otimes N$ is the image of $B \otimes M'$ in $B \otimes M$, whence $B \otimes \overline{M} \approx B \otimes M/B \otimes N$. Since (*) becomes exact when tensored by B, it follows that $B \otimes \overline{M} \to B \otimes M''$ is injective. Now since $\overline{M} \to B \otimes \overline{M}$ is injective by (ii), this implies that $\overline{M} \to M''$ is injective, i.e., that $N = \ker(M \to M'')$, which is what was to be proved.

For the moment, let $f : A \to B$ be any ring homomorphism. The *Amitsur complex* for f is the cosimplicial object $S = (S^n, d^i, s^i)$ in the category of two-sided A-modules

$$B \rightrightarrows B \otimes B \begin{smallmatrix}\rightarrow\\ \rightarrow\\ \rightarrow\end{smallmatrix} B \otimes B \otimes B \begin{smallmatrix}\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow\end{smallmatrix} \cdots$$

so that

$$S^n = B \otimes_A \cdots \otimes_A B \quad (n\text{-fold tensor product}),$$

and where

$$d^i(b_0 \otimes \cdots \otimes b_n) = b_0 \otimes \cdots \otimes 1 \otimes \cdots \otimes b_n \in S^{n+1} \ (1 \text{ in the } i\text{th position})$$

$$s^i(b_0 \otimes \cdots \otimes b_n) = b_0 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n \in S^{n-1}.$$

The complex is *augmented* by the map $f : A \to B$.

If B is an A-algebra, then the objects $B \otimes_A \cdots \otimes_A B$ are also A-algebras (4.5), and the face maps are A-extensions. The degeneracies are however not usually ring homomorphisms.

PROPOSITION (6.2). *Let $f : A \to B$ be an extension which is faithfully flat for bimodules. Then the Amitsur complex for f is a resolution of A*, i.e., *the sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{\delta^1} B \otimes B \xrightarrow{\delta^2} \cdots$$

is exact, where $\delta^n = \sum_{i=0}^{n} (-1)^i d^i$. In particular, f maps A onto the subring of B consisting of elements b such that $b \otimes 1 = 1 \otimes b$ in $B \otimes B$. More generally, if M is an A-bimodule (resp. if M is a left A-module and if f is left faithfully flat [3]), then the sequence

$$0 \to M \to B \otimes M \to B \otimes B \otimes M \to \cdots$$

obtained from the above sequence by tensoring with M, is exact.

Proof. This is standard (cf. [3], [5] exp. VIII). It suffices to show that the sequence in question becomes exact after tensoring on the left by B, and the resulting complex is homotopically trivial, the homotopy being multiplication of the first two entries in a tensor, i.e., $h : B^{\otimes n+2} \to B^{\otimes n+1}$ by

$$b \otimes b_0 \otimes \cdots \otimes b_n \mapsto bb_0 \otimes \cdots \otimes b_n .$$

The case of a module is the same.

Proposition (6.2) is the basis of *Grothendieck's* theory of flat descent for modules, and his theory ([5], exp. VIII) extends without difficulty to the case of left modules with respect to a left flat extension $f : A \to B$. Since we do not need these results, we will omit their proofs. On the other hand, we do not know reasonable conditions on an extension which allow one to descend a bimodule. Such conditions might be very useful.

7. The Spectrum

Recall that the *spectrum* Spec A of a ring A is its set of prime ideals [8]. It has a Zariski topology whose closed sets are of the form $V(I) = \{\mathfrak{p} \mid \mathfrak{p} \supset I\}$ for an ideal I. (This topology was actually introduced by *Jacobson* in this general situation.) We define the support of an A-bimodule M of finite type as the closed set $V(\mathfrak{a})$, where $\mathfrak{a}$ is the annihilator of M. It follows from the Nakayama lemma (5.2) that this is a reasonable notion.

The spectrum of a ring does not vary functorially. However, we have

Proposition (7.1). *(Procesi [13].) Let $f : A \to B$ be an extension. Then if $\mathfrak{p}$ is a prime ideal of B, $f^{-1}(\mathfrak{p})$ is a prime ideal of A. The map ϕ:* Spec $B \to$ Spec A *defined by*

$$\mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$$

is continuous.

Proposition (7.2). *With the notation of* (7.1), *let K be the kernel of f. Then $V(K) \subset$* Spec A *is the closure of the image Z of ϕ.*

Proof. Clearly $V(K) \supset Z$. Hence we may suppose $K = 0$, so that f is injective. We need then to show that the image Z is dense. Let I be an ideal such that $V(I) \supset Z$. Then $I \subset f^{-1}(\mathfrak{p})$ for all $\mathfrak{p} \in$ Spec B. Hence $IB \subset \mathfrak{p}$ for all such $\mathfrak{p}$. Thus $IB \subset N(B)$ where $N(B)$ is the nilradical of B [8]. What we need to show is that then I is contained in the nilradical $N(A)$ of A. Thus we are reduced to the following lemma:

Lemma (7.3). *Let $f : A \to B$ be an injective extension. Then $f^{-1}(N(B)) \subset N(A)$.*

Proof. Clearly any nilpotent ideal $J \subset B$ has nilpotent intersection with A. The lemma follows from this by transfinite induction.

PROPOSITION (7.4). *Let $f : A \to B$ be an injective extension, and suppose A is a prime ring. Then there is an ideal $\mathfrak{p} \in \operatorname{Spec} B$ such that $f^{-1}(\mathfrak{p}) = (0)$.*

Proof. Let $\mathfrak{p}$ be a maximal element among ideals J of B such that $J \cap A = (0)$. If I, J are ideals of B such that $IJ \subset \mathfrak{p}$ but neither I not J are in $\mathfrak{p}$, then we can find elements $a \in I$, $b \in J$, $p, p' \in \mathfrak{p}$ such that

$$0 \neq a + p \in A \qquad \text{and} \qquad 0 \neq b + p' \in A.$$

Since A is a prime ring, there is an element $x \in A$ with

$$0 \neq (a + p)\, x(b + p').$$

But this element is then both in A and in $\mathfrak{p}$, a contradiction.

COROLLARY (7.5). *Let $f : A \to B$ be an extension which is faithfully flat for bimodules. Then the map $\phi : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.*

Proof. Let $\mathfrak{p}$ be a prime ideal of A, and set $\bar{A} = A/\mathfrak{p}$, $\bar{B} = \bar{A} \otimes_A B$. Then by (6.1)(ii), we have $\bar{A} \subset \bar{B}$. Now apply (7.4).

The following propositions show that if $f : A \to B$ is faithfully flat for bimodules, then the topology on Spec A is a quotient of the topology on Spec B. They are routine generalizations of ([5], exp. VIII).

PROPOSITION (7.6). *Let $g : A \to A'$ be a flat extension for bimodules, and let $f : A \to B$ be arbitrary. Put $B' = B \otimes_A A'$, and let $f' : A' \to B'$ be the map. If $K = \ker(f)$, $K' = \ker(f')$, then $K' = KA'$.*

Proof. Let $\bar{A} = A/K$, and $\bar{A}' = A'/KA'$, so that $A' = A' \otimes_A \bar{A}$ (4.3). Since $0 \to \bar{A} \to B$ is exact, so is $0 \to \bar{A}' \to B'$, which proves the proposition.

Applying (7.6) to (7.2), we obtain the following:

COROLLARY (7.7). *With the notation of* (7.6), *let*

$$\begin{array}{ccc} X & \longleftarrow & X' \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi'} \\ Y & \xleftarrow{\psi} & Y' \end{array}$$

be the associated diagram of spectra, where $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, etc. Let $Z = \operatorname{im}\phi$, $Z' = \operatorname{im}\phi'$. Then their closures satisfy $\bar{Z}' = \psi^{-1}(\bar{Z})$.

PROPOSITION (7.8). *Let $A \to A'$ be a faithfully flat extension for bimodules, and let*

$$Y \xleftarrow{\psi} Y' \underset{p_2}{\overset{p_1}{\leftleftarrows}} Y''$$

be the diagram of spectra corresponding to the diagram

$$A \to A' \rightrightarrows A' \otimes_A A'.$$

Then a closed set $C' \subset Y'$ is of the form $C' = \psi^{-1}(C)$ for some (unique) closed set $C \subset Y$ if and only if

$$C' = p_1 p_2^{-1}(C').$$

Proof. The only if part is trivial since the two composed maps $Y'' \to Y$ are the same. Suppose that $C' = p_1 p^{-1} C'$, where $C' = V(I')$. Put $B = A'/I'$, $X = \operatorname{Spec} B \approx C'$, $X' = \operatorname{Spec} B' = \operatorname{Spec} A' \otimes_A B \approx p_2^{-}(C')$, and consider the diagram

$$\begin{array}{ccc} X & \longleftarrow & X' \\ \phi \big\downarrow & & \big\downarrow \phi'. \\ Y & \xleftarrow{\psi} & Y' \end{array}$$

Since $X = \operatorname{Spec} B \approx C'$, we have $\phi(X) = \psi(C')$. Since $X' = \operatorname{Spec} B'$, we have $\phi'(X') = p_1(p_2^{-} C')$. Therefore by (7.7)

$$\psi^{-1}\overline{(\psi(C'))} = \psi^{-1}\overline{(\phi(X))} = \overline{\phi'(X')} = \overline{p_1(p_2^{-1}(C'))} = C'.$$

Thus we can take $C = \overline{\psi(C')}$.

PART II: A CHARACTERIZATION OF AZUMAYA ALGEBRAS.

8. STATEMENT OF THE THEOREM

Let k_0 be the prime field of a given characteristic p, and let $f(x_1, ..., x_n) \in k_0\{x_1, ..., x_n\}$ be a noncommutative polynomial with coefficients in k_0. We may substitute $n \times n$ matrices $M_1, ..., M_n \in M_n(R)$ with entries in some commutative k_0-algebra R into f, and the result is again an $n \times n$ matrix $f(M_1, ..., M_n) \in M_n(R)$. We call $f(x)$ a *k_0-identity*, or usually just an *identity* among $n \times n$ matrices in characteristic p, if $f(M_1, ..., M_n) = 0$ for R and all $M_1, ..., M_n$. This is clearly equivalent with the assertion that $f(X_1, ..., X_n) = 0$ when the X_i are *generic* matrices

$$X_i = (x^i_{\alpha\beta}) \qquad \alpha, \beta = 1, ..., n$$

with the "variable" entries $x^i_{\alpha\beta}$.

We leave the following proposition as an exercise:

PROPOSITION (8.1). *Let R be a commutative k_0-algebra and $F(x_1, ..., x_n) \in R\{x_1, ..., x_n\}$ a polynomial such that $F(X_1, ..., X_n) = 0$ for generic matrices X_i. Then F is a linear combination of k_0-identities for $n \times n$ matrices, with coefficients in R.*

Little seems to be known about the nature of these identities, other than the results of *Amitsur* and *Levitski* [1] which assert that the *standard identity*

$$[x_1, ..., x_{2n}] = 0 \tag{8.2}$$

holds, where

$$[x_1, ..., x_{2n}] = \sum_{\sigma} (-1)^{\sigma} x_{\sigma(1)} \cdots x_{\sigma(2n)},$$

and that no other identity of degree $\leqslant 2n$ holds. An example of an identity which is not a consequence of (8.2) for the case $n = 2$, is the following: Given 2×2 matrices x, y, z, w, the element

$$(xy - yx)(zw - wz) + (zw - wz)(xy - yx)$$

is identically in the center. Therefore there is an identity of degree 5 which asserts that this element commutes with another matrix t. It can be shown that this is not a consequence of the standard identity.

Our theorem is the following:

THEOREM (8.3). *Let A be a ring which is a k_0-algebra. Then A is an azumaya algebra of rank n^2 over its center $Z = Z(A)$ if and only if the following two conditions hold*:

(i) *A has no representation in a vector space of rank $< n$ over a field $K \supset k_0$*, i.e., *if $\phi : A \to M_r(K)$ is a homomorphism, then $r \geqslant n$.*

(ii) *The identities which hold among $n \times n$ matrices in characteristic p also hold identically in A.*

Note that the necessity of these conditions is clear. In fact, condition (8.3)(i) follows immediately for an azumaya algebra from (3.2). To see (8.3)(ii), note that if A is an azumaya algebra, then there is a faithfully flat

Z-algebra Z' such that $A' = Z' \otimes_Z A$ is isomorphic to the full ring of matrices $M_n(Z')$. Thus $A \subset M_n(Z')$, which shows that (ii) holds.

Let $\phi : A \to M_r(K)$ be a representation, and let $I = \ker \phi$. Then the standard identity $[x_1, ..., x_{2r}] = 0$ holds in the ring A/I. Thus if $\mathfrak{m}$ is a maximal ideal containing I, the same identity holds in $A/\mathfrak{m}$, whence by *Kaplansky*'s theorem [9] and the above result of *Amitsur* and *Levitski*, the ring $A/\mathfrak{m}$ is an azumaya algebra of rank $\leqslant r$ over a field. Thus we obtain

COROLLARY (8.4). *Condition* (i) *of* (8.3) *can be replaced by the following*:

(i') *For every maximal ideal* $\mathfrak{m}$ *of* A, *the ring* $A/\mathfrak{m}$ *is an azumaya algebra of rank* n^2 *over a field.*

If A is a prime ring satisfying the standard identity (8.2), then it follows from the theorem of *Posner* [11] that A is contained in an $n \times n$ matrix algebra over a commutative field K. Hence all identities of $n \times n$ matrices hold in A.

COROLLARY (8.5). *With the notation of* (8.3), *suppose in addition that* A *is a prime ring. Then condition* (8.3)(ii) *can be replaced by the following condition*: *The standard identity* $[x_1, ..., x_{2n}] = 0$ *holds in* A.

The assertion of (8.3) seems of some interest however, even in the case of a ring with minimum condition.

Let us introduce the ring $k_0\{X_i\}$ generated by generic $n \times n$ matrices $X_i = (x^i_{\alpha\beta})$ (possibly infinitely many) over k. This is the subring of the ring of all $n \times n$ matrices whose entries are polynomials in the variables $x^i_{\alpha\beta}$, which is generated by the matrices X_i and the scalars. We identify the scalars $c \in k_0$ with the $n \times n$ matrices cI, where I is the $n \times n$ identity matrix. Clearly, $k_0\{X_i\}$ is isomorphic to the quotient of the ring $k_0\{x_i\}$ of non-commutative polynomials in the variables x_i by the ideal consisting of all polynomials which are identities among $n \times n$ matrices. Moreover if $f(y_1, ..., y_n)$ is an identity, then it vanishes for all substitutions in $k_0\{X_i\}$.

It follows immediately from (8.1) that

COROLLARY (8.6). *If* R *is a commutative* k_0*-algebra, then the ring* $R \otimes_{k_0} k_0\{X_i\}$ *is isomorphic to the ring generated by the generic matrices* X_i *over* R.

Moreover, we obtain

COROLLARY (8.7). *With the notation of* (8.3), *suppose that* A *is an algebra over a commutative ring* R. *Then* (8.3)(ii) *is equivalent with the assertion that* A *is isomorphic to a quotient of a ring* $R\{X_i\}$ *generated over* R *by generic* $n \times n$ *matrices, for a suitably large index set* I.

9. The Space of Irreducible n-Dimensional Representations

Let A be a ring. By *n-dimensional representation* of A we mean a homomorphism

$$\phi : A \to M_n(K) \tag{9.1}$$

of A into the ring $M_n(K)$ of $n \times n$ matrices over a field K. We call a representation ϕ *irreducible* if the image of A generates $M_n(K)$ as K-algebra. It is clear that this is equivalent with the assertion that ϕ is a central extension (Section 3).

Two representations

$$\phi_i : A \to M_{n_i}(K_i) \qquad i = 1, 2$$

are said to be *isomorphic* if $K = K_1 = K_2$, $n = n_1 = n_2$ and if ϕ_1, ϕ_2 differ by an inner automorphism of $M_n(K)$. We call ϕ_i *equivalent* if there is a diagram of fields

$$\begin{tikzcd} K_1 \arrow[dr, "\epsilon_2"] & \\ & L \\ K_2 \arrow[ur, "\epsilon_1"'] & \end{tikzcd}$$

such that the representations $M_{n_i}(\epsilon_i)\, \phi_i$ are isomorphic, where $M_{n_i}(\epsilon_i) : M_{n_i}(K_i) \to M_{n_i}(L)$ is the induced map. It is easily seen that this is an equivalence relation.

The following result is a consequence of *Posner*'s theorem [*12*]. I don't know if there is an elementary proof.

Theorem (9.2). *Two finite dimensional irreducible representations ϕ_1, ϕ_2 of A are equivalent if and only if* $\ker \phi_1 = \ker \phi_2$.

Proof. We may clearly replace A by $A/(\ker \phi_i)$, i.e., suppose that ϕ_i are injective. Then since $M_{n_i}(K_i)$ is a prime ring and ϕ_i is a central extension, A is itself a prime ring (7.1). It satisfies the identies holding in $M_{n_i}(K_i)$.

Lemma (9.3). *Let A be a prime ring satisfying a polynomial identity, and let Q be its left ring of fractions, which is an azymaya algebra over a field k* (cf. *Posner* [*11*]). *Then Q is a central extension of A.*

Proof. Consider the central extension $A[k] \subset Q$ of A generated by k. The ring $A[k]$ is a k-subalgebra of Q. If $a \in A$ is not a left zero divisor, so that it has a two-sided inverse $a^{-1} \in Q$, then it follows that a is a regular element in Q, whence in $A[k]$. Since $A[k]$ is a finite k-algebra, we have $a^{-1} \in A[k]$. Thus $A[k] = Q$.

Returning to the proof of the theorem, let Q be as in the lemma, and choose a central extension $Q \subset M_d(K')$ which splits the azumaya algebra Q, so that

$$\phi' : A \xrightarrow{}_0 M_d(K')$$

is a *flat* ([3] p. 162, exc. 22 ζ) central extension. We claim that the representation ϕ' is equivalent to any $\phi : A \to M_n(K)$ having kernel zero. This will clearly complete the proof. In fact, it follows from the flatness of ϕ' and the injectivity of ϕ that the ring

$$M_d(K') \otimes_A M_n(K)$$

is nonzero, hence has a simple quotient B which is a central extension of A. Let $L = Z(B)$. Since B is a central extension of $M_d(K')$, we have by (3.1) an L-isomorphism

$$B \approx M_d(K') \otimes_{K'} L = M_d(L).$$

Similarly,

$$B \approx M_n(K) \otimes_K L = M_n(L)$$

whence $d = n$. Since the representations ϕ', ϕ induce the same map $A \to B$, the equivalence follows.

The above theorem asserts that the set of equivalence classes of n-dimensional irreducible representations of a is in natural 1–1 correspondence with a certain subset of Spec A, which we will denote by

$$\operatorname{Spec}_n A.$$

One sees easily that $\operatorname{Spec}_n A$ is a locally closed subset of Spec A in the Zariski topology. In fact, if $I_n \subset A$ is the ideal generated by the standard identity $[x_1, ..., x_{2n}] = 0$ in A, then it is clear from *Posner's* theorem that $\operatorname{Spec}_n A$ is the subset

$$\operatorname{Spec}_n A = V(I_n) - V(I_{n-1}).$$

We call this set, with its induced topology from Spec A, the *space of irreducible n-dimensional representations* of A.

PROPOSITION (9.4). *Let $f: A \rightarrow B$ be a central extension. Then* $\operatorname{Spec}_n B$ *is the inverse image of* $\operatorname{Spec}_n A$, *under the map* $\operatorname{Spec} B \rightarrow \operatorname{Spec} A$.

10. Beginning of the Proof

We are going to work with the ring generated by generic matrices introduced in Section 7. However, it will be convenient to replace the prime field k_0 by an infinite extension field k. We will replace A by $k \otimes_{k_0} A$, and we will see later that A is an azumaya algebra if and only if $k \otimes_{k_0} A$ is (cf. (11.1)). It is clear that condition (7.3)(i) is equivalent for these two rings, and so is (7.3)(ii), because of (7.5), (7.6). We therefore consider the ring $k\{X_i\}$ generated by the generic matrices $\{X_i\}$ over the infinite field k. Note that the theorem is trivial for the case $n = 1$, since the standard identity in the case $n = 1$ is just the commutative law. We therefore assume that $n > 1$. Then the ring A is not commutative, and hence the index set I will consist of *at least two elements*.

The following lemmas are easy and well-known:

LEMMA (10.1). *The only matrices with entries in a field which commute with two matrices which are generic over the prime field are those of the form aI, where I is the identity matrix and a is a scalar.*

LEMMA (10.2). *For any field k, the generic matrices $\{X_i\}$* (card$(I) > 1$) *generate the full matrix algebra $M_n(k(x^i_{\alpha\beta}))$ over the field $k(x^i_{\alpha\beta})$. Equivalently, the inclusion $k\{X_i\} \hookrightarrow M_n(k(x^i_{\alpha\beta}))$ is a central extension. In particular, $k\{X_i\}$ is a prime ring.*

We will also need the following:

LEMMA (10.3). *For any field k, let Q be the left ring of fractions of $k\{X_i\}$ [11]. The ring Q has a unique embedding into $M_n(k(x^i_{\alpha\beta}))$ compatible with the inclusion of $k\{X_i\}$. An element of $k\{X_i\}$ is left regular if and only if it is an invertible matrix in $M_n(k(x^i_{\alpha\beta}))$.*

Proof. The last assertion follows immediately from the first. By (9.3), Q is a central extension and an azumaya algebra over a field, clearly of rank n^2. Let $Q \hookrightarrow M_n(K')$ be a central extension which splits this azumaya algebra. Then by (9.2) the two irreducible representations

$$k\{X_i\} \hookrightarrow M_n(K')$$

$$k\{X_i\} \hookrightarrow M_n(k(x^i_{\alpha\beta}))$$

are equivalent. Hence there is a field extension $L \supset k(x_{\alpha 0})$ and a commutative diagram of injective central extensions

$$\begin{tikzcd} & Q \arrow[dr] & \\ k\{X_i\} \arrow[ur] \arrow[dr] & & M_n(L) \\ & M_n(k(x^i_{\alpha\beta})) \arrow[ur] & \end{tikzcd}$$

It follows that an element a of $k\{X_i\}$ which is invertible in Q is not a zero divisor in $M_n(L)$, hence not in $M_n(k(x^i_{\alpha\beta}))$. Therefore a is an invertible matrix in $M_n(k(x^i_{\alpha\beta}))$. Since Q is a ring of fractions of $k\{X_i\}$, the result follows from ([*3*], p. 163 exc. 22(c)).

LEMMA (10.4). *There are polynomials* $f(u, v)$, $g(u, v) \in k_0\{u, v\}$ *such that for* $n \times n$ *matrices* U, V *the equation*

$$f(U, V) \operatorname{tr} U = g(U, V)$$

holds identically, where tr U *is the trace of the matrix, and where* $\det(f(U, V))$ *is not identically zero.*

Proof. Replace $\{X_i\}$ by $\{U, V\}$ and k by k_0 in (10.3). The existence and uniqueness of the reduced trace function on azumaya algebras shows that $\operatorname{tr} U \in Q$, from which the result follows.

Let us denote by B the ring of matrices generated over the infinite field k by the generic matrices X_i and by the traces tr z of all elements $z \in k\{X_i\}$, where we identify tr z with the matrix tr $z.I$. The ring B is again a subring of $M_n(k(x^i_{\alpha\beta}))$. Since tr z is in the center of B, the inclusion $k\{X_i\} \hookrightarrow B$ is a central extension. Trace is a linear function, and so it follows immediately that tr $b \in B$ for every $b \in B$. Note that the identities of $n \times n$ matrices hold in B, and that since it has the central extension $M_n(k(x^i_{\alpha\beta}))$, B is a prime ring.

PROPOSITION (10.5). *The map* $\operatorname{Spec} B \to \operatorname{Spec} k\{X_i\}$ *induces a homeomorphism of* $U' = \operatorname{Spec}_n B$ *onto* $U = \operatorname{Spec}_n k\{X_i\}$.

Note that U', U are open sets of Spec B, Spec $k\{X_i\}$ respectively.

Proof. First of all, U' is the inverse image of U in Spec B, by (9.4). Let $\mathfrak{p} \in U$. Then $k\{X_i\}/\mathfrak{p}$ can be embedded in a matrix algebra $M_n(K)$ over a field K containing k. The map $k\{X_i\} \to M_n(K)$ is given by assigning the images of the generic matrices X_i, and it obviously extends to a map on the traces, i.e., to a map $B \to M_n(K)$. Hence U' maps *onto* U.

Now let $\mathfrak{p}' \in U'$, so that $B/\mathfrak{p}'$ embeds into some $M_n(K)$. The image of $k\{X_i\}$ generates $M_n(K)$ as a K-module. Since k is an infinite field, it is clear that we can find "sufficiently general" $u, v \in k\{X_i\}$ such that $\det(f(u, v)) \neq 0$ and that also $\det(f(\bar{u}, \bar{v})) \neq 0$, where f is the polynomial of (10.4) and where the bar denotes the image in $M_n(K)$. It follows that for any $z \in k\{X_i\}$,

$$\det(f(u + cz, v)) \qquad \text{and} \qquad \det(f(\bar{u} + \overline{cz}, \bar{v}))$$

are nonzero for "sufficiently general" $c \in k$.

Let $b \in B$, and write

$$b = \sum_j a_j T_j \qquad a_j \in k\{X_i\}$$

where each T_j is a product of traces of elements of $k\{X_i\}$. Since trace is a linear function we can rewrite $\operatorname{tr} z$ as $1/c(\operatorname{tr} u - \operatorname{tr}(u + cz))$. Then if we substitute for the traces appearing in the T_i the formula of Lemma (10.4), we can multiply b on the left by enough elements $f(u, v), f(u + cz, v)$ as above to clear the denominator and obtain an element of $k\{X_i\}$ (this works because the traces are central elements). The elements $f(u, v)$, $f(u + cz, v)$ have images in $M_n(K)$ which have nonzero determinants, and therefore they are not zero divisors in $k\{X_i\}/\mathfrak{p}$ ($\mathfrak{p} = \mathfrak{p}' \cap k\{X_i\}$. Hence we have expressed the element $b \in B$ in the form

$$b = a^{-1}c \tag{10.6}$$

where a, c $k\{X_i\}$, and a is congruent to a regular element (modulo $\mathfrak{p}$).

It follows that we can characterize the ideal $\mathfrak{p}'$ in terms of $\mathfrak{p} = \mathfrak{p}' \cap k\{X_i\}$ as follows: $\mathfrak{p}'$ is the set of elements $b \in B$ such that $ab \in \mathfrak{p}$ for some $a \in k\{X_i\}$ which is congruent to a regular element (modulo $\mathfrak{p}$). This shows that the map $U' \to U$ is one to one, hence bijective.

To show the map is a homeomorphism, we must show that its inverse is continuous, which amounts to showing that for every $b \in B$, its locus of zeros $C' = V(b) \cap U'$ in U' is the inverse image of a closed set $C \subset U$. But it is clear from the above discussion that we can take for C the common zeros of all elements of $k\{X_i\}$ of the form ab, where $a \in k\{X_i\}$. This completes the proof.

LEMMA (10.7). *The open set* $\operatorname{Spec}_n B = U' \subset \operatorname{Spec} B$ *is a union of spectra of the form* $\operatorname{Spec} B_s$, *where* B_s *is obtained from* B *by inverting the element* $s \in Z(B)$, *and where* B_s *is an azumaya algebra of rank* n^2 *over its center.*

Proof. Recall that for a commutative ring R, the function $\operatorname{tr}(uv) : M_n(R) \times M_n(R) \to M_n(R)$ is a non-degenerate inner product. Therefore, if $\{z_{ij}\}(i, j = 1,\dots, n)$ is a basis for $M_n(R)$ as an R-module, then the $n^2 \times n^2$ matrix $(\mathrm{tr}(z_{ij}z_{kl}))$ is invertible. Thus if we are given a matrix m, then the n^2 equations

$$c_{11}\,\mathrm{tr}(z_{11}z_{ij}) + \cdots + c_{nn}\,\mathrm{tr}(z_{nn}z_{ij}) = \mathrm{tr}(mz_{ij}) \tag{10.8}$$

have a unique solution for $c_{ij} = c_{ij}(m)$ which is an integral expression in the elements $\mathrm{tr}(mz_{ij})$, $\mathrm{tr}(z_{ij}z_{kl})$, $1/d$ (where $d = \det(\mathrm{tr}(z_{ij}z_{kl}))$. It follows immediately that the expression

$$m = \sum_{i,j} c_{ij}(m)\, z_{ij} \tag{10.9}$$

holds identically among $n \times n$ matrices $\{m, z_{ij}\}$ in any ring R, provided d is invertible.

Returning to our ring B, let $\mathfrak{p}' \in U'$, so that $B/\mathfrak{p}'$ is an order in an azumaya algebra of rank n^2 over a field. Choose elements $z_{ij} \in k\{X_i\}$ which generate this azumaya algebra over its center, and such that $d = \det(\mathrm{tr}(z_{ij}z_{kl})) \neq 0$ (as element of $k[x^i_{\alpha\beta}]$). Then the localized ring B_d may be viewed as a subring of the ring $M_n(k(x^i_{\alpha\beta}))$. We claim that B_d is a free module with basis $\{z_{ij}\}$ over its center $Z(B_d)$. Since d is invertible, the elements z_{ij} are linearly independent in $M_n(k(x^i_{\alpha\beta}))$ over $k(x^i_{\alpha\beta})$, hence a fortiori over $Z(B_d)$ (cf. (10.1)!). To show that the z_{ij} generate, it suffices to write any $b \in B$ as a linear combination with coefficients in $Z(B_d)$, since $d \in Z(B_d)$. But in the identity (10.9), the coefficients $c_{ij}(b)$ are integral expressions in $\mathrm{tr}(bz_{ij})$, $\mathrm{tr}(z_{ij}z_{kl})$, $1/d$ and these are elements of $Z(B_d)$.

Conceivably the ring B_d is already an azumaya algebra. In any case, the condition that a free algebra $\mathscr{A}$ over a commutative ring Z be an azumaya algebra is that $\mathscr{A} \otimes_Z \mathscr{A}^0 \xrightarrow{\sim} \mathrm{End}_A(\mathscr{A})$, which is an open condition on Z, hence on Spec $\mathscr{A}$. Moreover, for $\mathscr{A} = B_d$ this open set is exactly the intersection of Spec B_d with U' (this follows from ([*3*], p. 180 exc. 14 α). Thus Spec $B_d \cap U'$ is covered by spectra of rings of the form $(B_d)_\sigma$, where $\sigma \in Z(B_d)$, which are azumaya algebras. Write $\sigma = d^{-r}s$, $s \in B$. Then since $\sigma \in Z(B_d)$ and d is not a zero divisor, $s \in Z(B)$. We have $(B_d)_\sigma = B_{ds}$. The point $\mathfrak{p}'$ is in one of the spectra Spec B_{ds}, which completes the proof.

LEMMA (10.10). *The azumaya algebras B_s of* (10.7) *are left flat* (*by symmetry also right flat*) *central extensions of $k\{X_i\}$.*

Proof. The ring B_s is a composition of central extensions, hence is a central extension of $k\{X_i\}$. Let $Z = Z(B_s)$ be its center. To show that a sequence of left B_s-modules is exact, we can view them as Z-modules, and then it suffices to show that for each $\mathfrak{p} \in$ Spec Z the sequence of $Z_\mathfrak{p}$-modules obtained by localization is exact. Let $(B_s)_\mathfrak{p} = Z_\mathfrak{p} \otimes_Z B_s$. Then if M is a B_s-module, we have $M_\mathfrak{p} \approx (B_s)_\mathfrak{p} \otimes_B M$. Thus to show that B_s is left flat over $k\{X_i\}$, it clearly suffices to show that for each $\mathfrak{p}$ the ring $(B_s)_\mathfrak{p}$ is left flat over $k\{X_i\}$. To prove this, we will show that $(B_s)_{\mathfrak{p}}$ is a *left ring of fractions* of $k\{X_i\}$. The flatness follows (cf. [3], p. 162 exc. 22 ζ). To begin with, it is easily seen that $k\{X_i\}$ maps injectively to $(B_s)_{\mathfrak{p}}$. The ideals $\mathfrak{p}B_s$, $\mathfrak{p}B_s \cap B$, $\mathfrak{p}B_s \cap k\{X_i\}$ are prime ideals in the rings B_s, B, $k\{X_i\}$ respectively. Let us denote all of these ideals by $\mathfrak{p}$.

Now $(B_s)_{\mathfrak{p}}$ is an azumaya algebra over the local ring $Z_{\mathfrak{p}}$. It follows that an element of $(B_s)_{\mathfrak{p}}$ is invertible if and only if its residue (modulo $\mathfrak{p}$) is a regular element of $(B_s)_{\mathfrak{p}}/\mathfrak{p}$. This is easy to see. Thus an element $a \in k\{X_i\}$ is invertible in $(B_s)_{\mathfrak{p}}$ iff. its residue (modulo $\mathfrak{p}$) is a regular element in $(B_s)_{\mathfrak{p}}/\mathfrak{p}$ (since $k\{X_i\}/\mathfrak{p}$ is an order in an azumaya algebra, it suffices that the residue of a be regular in $k\{X_i\}/\mathfrak{p}$). Let $S = \{a \in k\{X_i\} \mid a \text{ is congruent to a regular element } (\text{mod } \mathfrak{p})\}$. To show that $(B_s)_{\mathfrak{p}}$ is isomorphic to the ring of fractions $S^{-1}k\{X_i\}$, it remains to show that every element $z \in (B_s)_{\mathfrak{p}}$ if of the form $z = \alpha^{-1}\beta$ with $\alpha \in S$ and $\beta \in k\{X_i\}$ (cf. [3] p. 162 exc. 22 (b)).

Since $(B_s)_{\mathfrak{p}}$ is obtained by localization from B, we know that we can write z in the form

$$z = g^{-1}h$$

with $g, h \in B$ and g regular (mod $\mathfrak{p}$) (in fact, g can be taken to be in the center). Moreover, we saw in (10.6) that every $b \in B$ can be written in the form $b = a^{-1}c$ with $a, c \in k\{X_i\}$ and a regular (mod $\mathfrak{p}$). Write g accordingly:

$$g = a^{-1}c \qquad a, c \in k\{X_i\}.$$

Then g, a are regular (mod $\mathfrak{p}$), hence so is c. Hence

$$cg^{-1}h = c(c^{-1}a)\,h = ah \in B.$$

Now write ah as in (10.6):

$$ah = \gamma^{-1}\beta \qquad \gamma, \beta \in k\{X_i\}$$

with γ regular (mod $\mathfrak{p}$). Then setting $\alpha = (\gamma c)^{-1}$, we have

$$g^{-1}h = c^{-1}ah = c^{-1}\gamma^{-1}\beta = (\gamma c)^{-1}\,\beta = \alpha^{-1}\beta$$

as required.

11. Completion of the Proof

Lemma (11.1). *Let $f: A \to B$ be a central extension which is faithfully flat for bimodules. Then B is an azumaya algebra of rank n^2 over its center iff. A is.*

Proof. We already remarked the "if" part (3.2). Suppose B is an azumaya algebra. Since B has a faithfully flat central extension which is a matrix algebra, and since the composition of faithfully flat extensions is faithfully flat, we may as well suppose that $B = M_n(R)$ is a full matrix algebra over its center R. Let $a \in A$, and let $t \in R$ be the trace of the matrix $f(a)$. Now $B \otimes_A B$ is a central extension of B, hence is a full matrix algebra over its center (cf. (3.1)). We have $1 \otimes f(a) = f(a) \otimes 1$. Since trace is compatible with extension of scalars, it follows that also $1 \otimes t = t \otimes 1$. Therefore by (6.2), t is the image of an element of A, necessarily in its center. Thus if we view A as a subring of B, then the trace of any element of A is in the center of A.

Let $\mathfrak{p} \in \operatorname{Spec} A$ be a closed point. Since f is faithfully flat, there is a closed point $\mathfrak{p}' \in \operatorname{Spec} B$ lying over A, and $B/\mathfrak{p}'$ is a matrix algebra over its center $R/\mathfrak{p}' \cap R = K$, which is a field. Since $A \to B/\mathfrak{p}'$ is central, we can find elements $z_{ij} \in A$ whose images in $B/\mathfrak{p}'$ form a basis for that algebra over K. Let $d \in Z(A)$ be the element $d = \det(\operatorname{tr}(z_{ij} z_{kl}))$. Then as in the proof of (10.7), it follows that A_d is a free module with basis $\{z_{ij}\}$ over the localized ring $(Z(A))_d$. Moreover, $d \notin \mathfrak{p}$. Therefore a set of such d's, say $\{d_1, \ldots, d_N\} \subset Z(A)$, can be found which generates the unit ideal in A. Hence we can write

$$1 = \sum a_i d_i \qquad a_i \in A.$$

However, we want to show that $\{d_1, \ldots, d_N\}$ generates the unit ideal in $Z(A)$. Now if the rank n^2 of the matrix algebra is not divisible by the characteristic, it suffices to consider the expression

$$n = \operatorname{tr}(1) = \sum d_i \operatorname{tr}(a_i).$$

Since n is invertible, and $\operatorname{tr}(a_i) \in Z(A)$, we are done in that case.

In general, consider for instance the determinant function from $M_n(R)$ to R. This function is not linear. However, if we write the expansion of the expression $\det(t_1 X_1 + \cdots + t_N X_N)$ (where the X_i are matrices and the t_i are variables) in terms of the various monomials in $\{t_i\}$,

$$\det(t_1 X_1 + \cdots + t_N X_N) = \sum_{(i)} t^{(i)} D_{(i)}$$

where $(i) = (i_1, \ldots, i_N)$ with $i_1 + \cdots + i_N = n$, then the expressions $D_{(i)} = D_{(i)}(X)$ are obtained by polarization from the determinant ([*17*], p. 5). The important thing is that each $D_{(i)}(X)$ is invariant under simultaneous conjugation of the matrices $X_1, \ldots, X_N$ by a given matrix, hence is a uniquely determined function $D_{(i)}(X)$ from N-tuples of matrices in $M_n(R)$ to R, independent of automorphism and compatible with extension of scalars. Therefore it follows that when $a_1, \ldots, a_N \in A$, also $D_{(i)}(a_1, \ldots, a_N) \in Z(A)$. Hence

$$1 = \det(1) = \sum_{(i)} d^{(i)} D_{(i)}(a)$$

yields the required expression.

It follows that A is a locally free module of rank n^2 over its center $Z(A)$, and by ([3], p. 180, exc. 14 (α)), it remains to prove that $A/\mathfrak{p}A$ is an azumaya algebra for each maximal ideal $\mathfrak{p}$ of $Z(A)$. Now since the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, each simple quotient $\bar{A}$ of $A/\mathfrak{p}A$ has a central extension which is a simple quotient of B, and therefore an azumaya algebra of rank n^2. Thus $\bar{A}$ must also be an azumaya algebra of rank n^2, whence also $\bar{A} = A/\mathfrak{p}A$.

We can now complete the proof of (8.3). Let A be a ring satisfying (8.3)(i), (ii), and let k be an infinite extension of the prime field k_0. The ring $k \otimes_{k_0} A$ if faithfully flat over A, hence by (11.1) it suffices to show that $k \otimes_{k_0} A$ is an azumaya algebra. Thus we may suppose that A is a k-algebra, hence that A is a quotient of the ring $k\{X_i\}$ generated by generic matrices X_i, as in section 10.

The space $\operatorname{Spec} A$ identifies with a closed subset of $\operatorname{Spec} k\{X_i\}$ which is contained in U, by assumption (i) of (8.3). Hence it follows from (10.7) that every point of $\operatorname{Spec} A$ is an image of a point of $\operatorname{Spec} B_s$ for some B_s as in (10.7). Since $\operatorname{Spec} A$ is quasi-compact, it is covered by the images of finitely many $\operatorname{Spec} B_{s_i}(i = 1,..., N)$, whose images are open sets in U by (10.5). Put $B_i = A \otimes_{k\{X_i\}} B_{s_i}$, and $B = B_1 \times \cdots \times B_N$. Then it follows from (10.10) that B_i is left flat over A, hence that B is. Applying (11.1) again, we see that it suffices to show that B is faithfully flat over A for left modules.

Since $\operatorname{Spec} A$ is covered by the $\operatorname{Spec} B_{s_i}$, it follows that $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective. To show $A \to B$ faithfully flat, it suffices to show that if M is a simple left A-module, then $B \otimes_A M \neq 0$. But since A satisfies the identities of $n \times n$ matrices, every primitive quotient of A is finite dimensional over its center, i.e., is an azumaya algebra, by Kaplansky's theorem [9]. Let $A_0 = A/\mathfrak{a}$, where $\mathfrak{a} =$ annihilator (M). Then A_0 is isomorphic to a sum of copies of M. Since $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, $B/\mathfrak{a}B = B \otimes_A A_0 \neq 0$. Hence $B \otimes_A M \neq 0$ as well, as was to be shown.

12. Structure of the Space of Irreducible n-Dimensional Representations

In this section, we work over an infinite field k. Let A be a k-algebra. Then using the considerations of section 10 we can give the space $\operatorname{Spec}_n A$ a canonical structure consisting of a scheme X_n whose underlying topological space is $\operatorname{Spec}_n A$, together with a sheaf of azumaya algebras $\mathcal{A}_n$ of rank n^2 over the structure sheaf $\mathcal{O}_n = \mathcal{O}_{X_n}$, and a homomorphism

$$A \to \Gamma(X_n, \mathcal{O}_n),$$

such that the n-dimensional irreducible representations of A are induced up to equivalence from the composed maps

$$A \to \mathcal{O}\mathfrak{l}_n \otimes_{\mathcal{O}_n} \overline{k(x)} \approx M_n(\overline{k(x)})$$

(where $\overline{k(x)}$ is the algebraic closure of the residue field $k(x)$) for the various points $x \in X_n$.

Consider first a ring $k\{X_i\}$ generated by generic matrices. Let B be the central extension of section 10, and let $Z = Z(B)$ be the center of B. Since B is a prime ring, the nonzero elements of Z are regular in B, and it follows that for any nonzero $s \in Z$, we have

$$B_s = Z_s \otimes_Z B,$$

and

$$Z(B_s) = Z_s .$$

Let B_s be as in lemma (10.7), and let $S \subset Z$ denote the set of elements considered there. Then since it is an azumaya algebra, we have (3.1)

$$\operatorname{Spec} B_s \approx \operatorname{Spec} Z_s .$$

It follows immediately that the open set $U' = \operatorname{Spec}_n B$ of $\operatorname{Spec} B$ is also homeomorphic to the open set

$$U'' \subset \operatorname{Spec} Z$$

where

$$U'' = \bigcup_{s \in S} \operatorname{Spec} Z_s = \operatorname{Spec} Z - V(S). \tag{12.1}$$

Moreover, it is clear that the Z_s-algebras B_s glue together to give a sheaf of azumaya algebras, say $\mathscr{B}$, on the scheme U'', which settles that case.

Now if A is an arbitrary k-algebra, replace it first by the ring A/I_n, where I_n is the ideal generated by the identities of $n \times n$ matrices. Since $\operatorname{Spec}_n A = \operatorname{Spec}_n A/I_n$, this is permissible. Write A as a quotient of a ring $k\{X_i\}$. Then we obtain a diagram of central extensions

$$\begin{array}{ccccc} k\{X_i\} & \longrightarrow & B & \longrightarrow & B_s \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A_B & \longrightarrow & (A_B)_s \end{array}$$

where $B_A = B \otimes_{k\{X_i\}} A$. The vertical maps are surjective, hence induce closed immersions on the spaces Spec_n. Since $\operatorname{Spec}_n B \approx \operatorname{Spec}_n k\{X_i\}$, it follows that also $\operatorname{Spec}_n A \approx \operatorname{Spec}_n A_B$. Thus we may replace A by A_B. Then the ring $(A_B)_s$ is a quotient of B_s, hence is an azumaya algebra over its center

$Z((A_B)_s)$. Taking into account Corollary (3.1), it is clear that the spectra Spec $Z((A_B)_s)$ glue together to give the required scheme X_n as closed subscheme of U''.

We propose now to describe the space of irreducible n-dimensional representations of the noncommutative polynomial ring $k\{X_1, \ldots, X_r\}$, or equivalently, of the ring generated by r generic $n \times n$ matrices. In order to apply classical invariant theory, *we assume that the characteristic of k is zero.*

Consider the affine space E of dimension n^2r with the coordinates $x^i_{\alpha\beta}(i = 1, \ldots, r; \alpha, \beta = 1, \ldots, n)$. We view E as the space of r-tuples of $n \times n$ matrices. Then obviously E parametrizes k-homomorphisms of $k\{X_1, \ldots, X_r\}$ into the $n \times n$ matrix algebra. In order to obtain equivalence classes of representations, we have to identify homomorphisms differing by an inner automorphism. The projective general linear group PGL_n operates on E, via simultaneous conjugation of the matrices, viz. if P is an invertible matrix representing an element of PGL, then P operates by

$$(X_1, \ldots, X_r) \mapsto (PX_1P^{-1}, \ldots, PX_rP^{-1}). \tag{12.2}$$

We denote by $R \subset k[x^i_{\alpha\beta}]$ the ring of polynomials invariant under the action induced by (12.2).

Let $f(X_1, \ldots, X_r) \in k\{X_i\}$. The matrix $f(X)$ has a characteristic polynomial whose coefficients are in $k[x^i_{\alpha\beta}]$, and these coefficients are obviously invariant by conjugation of $f(X)$, hence lie in R. Let $R' \subset R$ be the subring generated over k by these coefficients, for all noncommutative polynomials $f(X) \in k\{X_i\}$. Since we assume that the characteristic of k is zero, the characteristic polynomial of a matrix M can be determined rationally from the traces of $M, M^2, \ldots, M^n$. Hence R' can also be viewed as the ring generated by trace $f(X)$ for all $f(X) \in k\{X_i\}$, and it suffices to take homogeneous polynomials $f(X)$. Hence R' is generated by homogeneous elements of $k[x^i_{\alpha\beta}]$.

We conjecture[1] *that* $R' = R$, and we can show the following: (12.3) R' is a finitely generated ring, R is integral and birational over R', and the map $\rho : \operatorname{Spec} R \to \operatorname{Spec} R'$ is bijective on geometric points (hence is a homeomorphism).

In fact, it is an immediate consequence of *Hilbert*'s general theory [7] that in order to prove all of these assertions, it is enough to show merely that ρ is *injective* on geometric points. For then *Hilbert*'s theorem implies that R is integral over R', and that in fact is integral over a finitely generated subring of R' ([7], p. 299), whence the finiteness and the surjectivity of ρ. The injectivity can be shown by elementary arguments as follows:

It is clear that R' is compatible with field extensions. So is R ([11], p. 27,

[1] For $n = 2$, this is true, and in fact the ring of invariants is a polynomial ring generated by tr X_i, det X_i, $\det(X_i + X_j)$ (cf. [4], Chapter VIII, section 136).

Theorem. 1.1). Hence we need only consider rational points of Spec R and an algebraically closed field k.

Let

$$\phi : k\{X_i\} \to M_n(k)$$

be a k-representation, i.e., a rational point of E, and say we view the matrices as linear transformations on the vector space V. Then if we choose a new basis for V compatibly with a composition series with respect to ϕ, the representation will have the familiar form (in matrix notation)

$$\phi = \begin{pmatrix} \bar{\phi}_1 & & N \\ & \ddots & \\ 0 & & \bar{\phi}_s \end{pmatrix} \tag{12.4}$$

where $\bar{\phi}_\nu : k\{X_i\} \to M_{n_\nu}(k)$ are the irreducible quotients of ϕ, so that $\bar{\phi}_\nu$ are surjective maps and that $n = n_1 + \cdots + n_s$, and where N is the nilradical of ϕ. The representation

$$\bar{\phi} = \bar{\phi}_1 \oplus \cdots \oplus \bar{\phi}_s \tag{12.5}$$

is the graded, or semi-simple, representation associated to ϕ; it is uniquely determined up to isomorphism (9).

In order to prove the injectivity of the map ρ, we will show

(12.6) Two k-linear representations ϕ, ϕ' with isomorphic associated semi-simple representations have the same image in Spec R. If ϕ, ϕ' are two non-isomorphic semi-simple representations, then there is an element $f \in k\{X_i\}$ such that the characteristic polynomials of $\phi(f)$ and of $\phi'(f)$ are distinct.

To prove the first assertion, it suffices to show that the associated semi-simple representation $\bar{\phi}$ is in the closure of the orbit of ϕ in E, which is pretty trivial. One mimics in higher dimension the following conjugation, and then lets t specialize to zero:

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{\phi}_1 & n \\ 0 & \bar{\phi}_2 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \bar{\phi}_1 & tn \\ 0 & \bar{\phi}_2 \end{pmatrix}.$$

Now let $\phi = \bar{\phi}$ be semi-simple. We can first of all write it in the form

$$\phi = \psi_1^{e_1} \oplus \cdots \oplus \psi_s^{e_s}$$

where the exponents e_i are distinct integers, where each ψ_i is semi-simple, and where $\psi_1 \oplus \cdots \oplus \psi_s$ has no repeated isomorphic factors. If $f \in k\{X_i\}$, the characteristic polynomial $\lambda_{\phi(f)}$ of $\phi(f)$ will be a product

$$\lambda_{\phi(f)} = \lambda^{e_1}_{\psi_1(f)} \cdots \lambda^{e_s}_{\psi_s(f)},$$

and if f is sufficiently general, the eigenvalues of the matrices $\psi_i(f)$ will be distinct elements of k. Thus for sufficiently general f, the characteristic polynomial of $\psi_i(f)$ is determined from $\lambda_{\phi(f)}$ as that factor having exponent e_i. It is easily seen that since the field k is infinite, the characteristic polynomials of all f are determined by those for which f is sufficiently general in the above sense. Therefore the integers e_i and the characteristic polynomials of $\psi_i(f)$ are determined by elements of R'. Thus to prove the second assertion of (12.6), we may replace ϕ by ψ_i, which reduces us to the case that the semi-simple representations have no repeated isomorphic factors.

Then there is a $z \in k\{X_i\}$ such that $\phi(z)$ and $\phi'(z)$ have distinct eigenvalues, whence they can be diagonalized. Thus if we replace ϕ, ϕ' by isomorphic representations, we may assume that $\phi(z) = \phi'(z)$ is a diagonal matrix with distinct entries (assuming of course that $\lambda_{\phi(f)} = \lambda_{\phi'(f)}$ for all $f \in k\{X_i\}$). By choosing the ordering suitably, this may be done compatibly with a decomposition of $\bar{\phi}$ as sum of simple representations. Suitable polynomials, say $z_{11}, z_{22}, \ldots, z_{nn}$, in z can then be found such that

$$\phi(z_{ii}) = \phi'(z_{ii}) = E_{ii},$$

where E_{ij} denotes the matrix with the single nonzero entry 1 in the (i, j) position.

Now if a is any element of $k\{X_i\}$, then the matrix $\phi(z_{ii}az_{jj})$ has as single nonzero entry the (i, j) entry $\phi_{ij}(a) = \phi_{ij}(z_{ii}az_{jj})$, and similarly for ϕ'. Since ϕ' is represented up to ordering of the basis as a sum of simple representations, it follows that $\phi'_{ij}(a) = 0$ for all a implies that $\phi'_{ji}(a) = 0$ for all a, too. Suppose $\phi'_{ij}(a) \neq 0$ for some a, whence $\phi'_{ji}(b) \neq 0$ for suitable b. Then the matrix

$$\phi'(z_{ii}az_{jj}z_{jj}bz_{ii})$$

has as only nonzero entry the entry $\phi'_{ij}(a)\,\phi'_{ji}(b)$ in the (i, i) position, and the same is true for ϕ. Thus the fact that $\phi'_{ij}(a)\,\phi'_{ji}(b)$ is nonzero can be deduced from the trace of that matrix, whence $\phi_{ij}(a) \neq 0$ as well. It follows that ϕ' decomposes compatibly with the decomposition of ϕ into simple representations. This reduces us to the case that ϕ is simple, and by the same reasoning ϕ' is then also simple.

We can then extend the set z_{ii} to elements $z_{ij} \in k\{X_i\}$ such that

$$\phi(z_{ij}) = E_{ij}.$$

Replace z_{ij} by $z_{ii}z_{ij}z_{jj}$. Then we know that $\phi'(z_{ij})$ has at most one nonzero entry, that in the (i, j) position. By reasoning as in the previous paragraph, it follows that this entry $\phi'_{ij}(z_{ij})$ is in fact nonzero. Since we can still change basis in the representation ϕ' by conjugation with a diagonal matrix, we can adjust ϕ' so that

$$\phi'(z_{1j}) = E_{1j} \qquad j = 2,..., n.$$

Routine calculation shows that then $\phi'(z_{ij}) = E_{ij}$ for all (i, j), and then that in fact $\phi = \phi'$, whence (12.6) follows.

Now consider the set E_s of points of E corresponding to simple representations ϕ, i.e., such that ϕ is surjective. It is clear that this is a Zariski open set of E. Clearly E_s is stable under the operation of PGL. Moreover, the induced operation on E_s is scheme-theoretically free (this is just a restatement of the fact that the only invertible matrices with values in the ring of dual numbers $k[\epsilon]$ ($\epsilon^2 = 0$) which commute with all matrices are the scalar matrices). By (12.6) and general facts ([*11*], p. 27, Theorem 1.1) about group actions, it follows that E_s is the full inverse image of an open subscheme V of Spec R, that V consists of smooth points of Spec R, and that the map $E \to$ Spec R is smooth at all points of E_s. Since ρ : Spec $R \to$ Spec R' is a homeomorphism (12.3), V corresponds to an open set $V' \subset$ Spec R' and we claim

(12.7) The map ρ induces an isomorphism of schemes $V \xrightarrow{\sim} V'$.

Since ρ is integral and one to one, it is enough to show that if $\mathfrak{m}' \subset R'$, and $\mathfrak{m} \subset R$ are maximal ideals corresponding to a simple representation ϕ, then

$$\mathfrak{m} = \mathfrak{m}' \cdot R.$$

To show this, it suffices since E_s is smooth over Spec R to show that the locus $V(\mathfrak{m}')$ of the ideal $\mathfrak{m}'$ in $E =$ Spec $k[x^i_{\alpha\beta}]$ is the orbit of ϕ, with its scheme-theoretic (i.e., reduced) structure. We already know that $V(\mathfrak{m}')$ is set-theoretically the orbit of ϕ', since ρ is one to one. Therefore it suffices to show the following : If

$$\phi' : k\{X_i\} \to M_n(k[\epsilon])$$

is a representation with values in the dual numbers such that

$$\phi' = \phi + \epsilon\pi$$

for some map $\pi : k\{X_i\} \to M_n(k)$, and such that the characteristic polynomials $\phi(f)$ and $\phi'(f)$ are equal for all $f \in k\{X_i\}$, then ϕ and ϕ' are isomorphic, i.e., there is a matrix $P \in PGL_n(k[\epsilon])$ such that $\phi' = P\phi P^{-1}$.

Note first that under the above hypothesis

$$\ker \phi = \ker \phi'.$$

For, certainly $\ker \phi' \subset \ker \phi$. Suppose $\phi(f) = 0$, and choose $z_{ij} \in k\{X_i\}$ such that $\phi(z_{ij}) = E_{ij}$. Then

$$\phi'(z_{ij} f z_{ki}) = E_{ij} \cdot \epsilon\pi(f) \cdot E_{ki}$$

is a matrix whose only nonzero entry is $\epsilon\pi_{jk}(f)$ in the (i, i) position. Hence the trace of $\phi'(z_{ij} f z_{ki})$ is $\epsilon\pi_{jk}(f)$, whence $\pi_{jk}(f) = 0$. Thus $\pi(f) = 0$ as was to be shown.

Thus if $\mathfrak{p} = \ker \phi$, then $k\{X_i\}/\mathfrak{p} \approx M_n(k)$, and this isomorphism induces via ϕ' a k-linear inclusion

$$M_n(k) \xhookrightarrow{i} M_n(k[\epsilon]).$$

The image generates $M_n(k[\epsilon])$ as $k[\epsilon]$-module, as follows from the Nakayama lemma. Hence this inclusion is a central extension. Thus (3.1) it induces an isomorphism

$$M_n(k[\epsilon]) = M_n(k) \otimes_k k[\epsilon] \xrightarrow{i \otimes k[\epsilon]} M_n(k[\epsilon]).$$

By the Skolem–Noether theorem, this is an inner automorphism of $M_n(k[\epsilon])$, which proves our assertion.

Now consider the ring $M_n(k[x^i_{\alpha\beta}])$. The group PGL operates in two ways by automorphisms on this ring: first of all by conjugation, which is a $k[x^i_{\alpha\beta}]$-linear action, and secondly by operation on the entries of a matrix via the operation on $k[x^i_{\alpha\beta}]$ itself considered above. By construction, these two actions agree on the (non-invariant) subring $k\{X_i\}$. They also agree trivially on the ring R (viewed as diagonal matrices). Hence they agree on the ring B generated over $k\{X_i\}$ by the traces of elements of $k\{X_i\}$. Now the elements of B invariant under conjugation, i.e., under the first operation, are just the elements of the center $Z(B) = Z$, since $M_n(k[x^i_{\alpha\beta}])$ is a central extension of B (10.3). Hence it follows that $Z \subset R$, whence we have

$$R' \subseteq Z \subseteq R. \tag{12.8}$$

It is immediately seen that these inclusions induce homeomorphisms

$$V' \approx U'' \approx V$$

where U'' is the open set (12.1). Thus the underlying scheme X_n of $\operatorname{Spec}_n k\{X_i\}$ is identified canonically with the open subscheme V of Spec R. In particular, X_n is a smooth variety of dimension $n^2 r - n^2 + 1$ (cf. [*10*] in this connection).

Note: It is intuitively rather clear from the start that the point sets of V and of $\operatorname{Spec}_n k\{X_i\}$ should correspond. What seems less obvious is that their

Zariski topologies are the same. For, one matrix equation $f(X) = 0$ should involve (in general) n^2 conditions on the space E, invariant by the action of PGL. Thus one might expect the topology of $\mathrm{Spec}_n\, k\{X_i\}$ to be much weaker than that of V.

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