Stabilizers of Classes of Representable Matroids*

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Let $\mathcal{M}$ be a class of matroids representable over a field $F$. A matroid $N \in \mathcal{M}$ stabilizes $\mathcal{M}$ if, for any 3-connected matroid $M \in \mathcal{M}$, an $F$-representation of $M$ is uniquely determined by a representation of any one of its $N$-minors. One of the main theorems of this paper proves that if $\mathcal{M}$ is minor-closed and closed under duals, and $N$ is 3-connected, then to show that $N$ is a stabilizer it suffices to check 3-connected matroids in $\mathcal{M}$ that are single-element extensions or coextensions of $N$, or are obtained by a single-element extension followed by a single-element coextension. This result is used to prove that a 3-connected quaternary matroid with no $U_{3,4}$-minor has at most $(q-2)(q-3)$ inequivalent representations over the finite field $GF(q)$. New proofs of theorems bounding the number of inequivalent representations of certain classes of matroids are given. The theorem on stabilizers is a consequence of results on 3-connected matroids. It is shown that if $N$ is a 3-connected minor of the 3-connected matroid $M$, and $|E(M) - E(N)| \geq 3$, then either there is a pair of elements $x, y \in E(M)$ such that the simplifications of $M/x$, $M/y$, and $M \setminus x, y$ are all 3-connected with $N$-minors or the cosimplifications of $M/x$, $M/y$, and $M \setminus x, y$ are all 3-connected with $N$-minors, or it is possible to perform a $2\times 2$ exchange to obtain a matroid with one of the above properties. © 1999 Academic Press

1. INTRODUCTION

Possibly the major obstacle to progress in matroid representation theory is the fact that a matroid typically has inequivalent representations over a field. For some classes this problem does not arise; for example, binary matroids are uniquely representable over any field, ternary matroids are uniquely representable over $GF(3)$, and 3-connected quaternary matroids are uniquely representable over $GF(4)$. As has been often noted, all known proofs of the excluded-minor characterizations of binary, ternary, quaternary and regular matroids use these unique representability properties in essential ways [2, 5–7, 9, 15, 18].

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One case where the problem caused by inequivalent representations has been overcome is in [20, 22], where matrix characterizations of the matroids representable over $GF(3)$ and other fields are given. While ternary matroids are uniquely representable over $GF(3)$, they are not, in general, uniquely representable over other fields. The reason why inequivalent representations of ternary matroids do not cause too much difficulty is that their behaviour is predictable. It follows from results in [21] that if $M$ is a 3-connected ternary matroid with a $U_{2,4}$-minor, then a representation of $M$ over a field is uniquely determined by a representation of that minor. (Note that not all representations of the $U_{2,4}$-minor need extend to a representation of $M$, but one that extends, does so uniquely.) This means that the behaviour of inequivalent representations of $M$ is, in essence, no more complex than the behaviour of inequivalent representations of $U_{2,4}$.

Also, by extending a generic representation of $U_{2,4}$, it is possible to produce inductive arguments for the class of matroids representable over $GF(3)$ and some other field.

A primary motivation for this paper is the desire to generalise these ideas to other classes. Say that $\mathcal{M}$ is a class of matroids representable over a field $F$, and let $N$ be a matroid in $\mathcal{M}$. Then, broadly speaking, $N$ “stabilizes” $\mathcal{M}$ if, for any 3-connected matroid $M \in \mathcal{M}$ with an $N$-minor, a representation of $M$ is uniquely determined by a representation of any one of its $N$-minors. Evidently $U_{2,4}$ stabilizes the class of ternary matroids representable over $F$.

How hard is it to check whether $N$ is a stabilizer? One of the main results of this paper, Theorem 5.8, shows that to check that $N$ stabilizes $\mathcal{M}$, it suffices to check all 3-connected matroids in $\mathcal{M}$ that are either single-element extensions of $N$, single-element coextensions of $N$, or obtained by a single-element extension of $N$ followed by a single-element coextension. Thus a task that, at first glance, may seem to be potentially infinite, turns out to be reducible to an elementary finite check. Theorem 5.8 is worth comparing with some theorems of Seymour. It is a potentially infinite task to show that a matroid is a splitter for a class, or is 1- or 2-rounded. However, Seymour [14, 16, 17] has shown that these tasks all reduce to finite checks very similar to those needed to determine whether $N$ is a stabilizer.

Section 6 gives some applications of stabilizers. It is shown that if $M$ is a 3-connected quaternary matroid with no $U_{3,6}$-minor that is representable over the finite field $GF(q)$ where $q > 3$, then $M$ has at most $(q - 2)(q - 3)$ inequivalent representations over $GF(q)$. Also new proofs are given of the following theorems: a ternary matroid is uniquely representable over $GF(3)$ [4]; a 3-connected quaternary matroid is uniquely representable over $GF(4)$ [8]; a 3-connected ternary matroid has at most $(q - 2)$ inequivalent representations over $GF(q)$, $q > 2$ [21]; and a 3-connected $GF(5)$-representable matroid has at most six inequivalent $GF(5)$-representations [11].
I do not believe that aesthetically the proofs given here are an improvement on the original ones; our proofs constitute case checking that certain matroids stabilize certain classes, while the original ones typically use a roundedness result together with some geometric insight. The point of the exercise is to develop a case that the theory of stabilizers provides a useful and systematic technique in matroid representation theory.

Sections 3 and 4 are devoted to proving 3-connectivity theorems. To some extent the results in this section can be regarded as lemmas for Theorem 5.8, but they are also motivated by independent considerations; in particular they are motivated by the desire to extend the characterizations of ternary matroids representable over other fields to characterizations of matroids representable over $GF(q)$ and other fields for values of $q > 3$. The results of [20, 22] rely crucially on a technical lemma for non-binary matroids [20, Theorem 3.1]. But the application to representation theory of this lemma is limited to classes for which $U_{2,4}$ is a stabilizer. The lemma is no help, for example, in characterizing the matroids representable over $GF(4)$ and other fields. The main 3-connectivity result of this paper, Theorem 4.3, can be regarded as a broad generalization of [20, Theorem 3.2] but, strictly speaking, Theorem 4.3 does not specialize to [20, Theorem 3.1]. However, it is not hard to see that, when appropriately specialized, Theorem 4.3 does give a workable alternative to [20, Theorem 3.1]. Of course there are many hurdles left to overcome to obtain techniques that would enable one to give characterizations of the matroids representable over $GF(q)$ and other fields in general. But it is clear that a result like Theorem 4.3 is a necessary step towards the first hurdle.

Other theorems in Sections 3 and 4 can be regarded as lemmas for Theorem 4.3 but they have some interest in their own right. Perhaps the reader does not belong to that small group of masochists that take an interest in details of proofs of technical 3-connectivity theorems. For such a reader the only information needed from Sections 3 and 4 is the statement of Theorem 4.3. For a reader that is interested in such details, the following comments may assist them in a reading.

Since writing the initial version of this paper Jim Geelen has pointed out to me that there is a more direct route to Theorem 4.3 than that given in this paper. This is obtained by generalizing the techniques of the connectivity theorems of [5] as follows. Let $N$ be a 3-connected minor of the 3-connected matroid $M$ with $|E(M) - E(N)| \geq 5$. Then, apart from some special cases that are easily dealt with, one can use the splitter theorem to obtain a sequence of elements, marked for deletion or contraction, that reduces $M$ to a minor isomorphic to $N$ while keeping 3-connectivity. It can be shown that amongst the first five elements of such a sequence is a pair of elements with the properties of Theorem 4.3. This gives a proof of Theorem 4.3 that does not rely on earlier results in this paper; although
note that the bulk of the proof of Theorem 4.3 is devoted to cases where $M$ and $N$ are “close” and therefore not covered by the above technique. I have kept the present proof for the reason that the earlier results of Sections 3 and 4 are potentially of some interest and, given that we have them in the paper, we may as well use them.

2. PRELIMINARIES

Familiarity is assumed with the elements of matroid theory. In particular it is assumed that the reader is familiar with the theory of matroid representations as set forth in Oxley [10, Chap. 6] and the theory of matroid connectivity as set forth in [10, Chap. 8]. Notation and terminology follows [10] apart from two exceptions that we now discuss.

Recall that the simplification of a matroid $M$ is obtained by deleting all loops of $M$ and all but one member of each parallel class of $M$. Dually, the cosimplification is defined by contracting all coloops and all but one member of each series class of $M$. We differ from [10] in that we denote the simplification and cosimplification of $M$ by $\text{si}(M)$ and $\text{co}(M)$, respectively.

A minor irritant for us is the fact that if $N$ is a 3-connected minor of the matroid $M$, then it is possible that the simplifications or cosimplifications of $M$ may not have an $N$-minor. This is because there exist 3-connected matroids that are not simple or not cosimple. It is easily checked that these are the matroids $U_{0,1}$. $U_{1,1}$, $U_{1,2}$, $U_{1,3}$ and $U_{2,3}$. All other 3-connected matroids are both simple and cosimple and, apart from the trivial matroid $U_{0,0}$, have both rank and corank at least two. Let $N$ be a 3-connected matroid. The problem for us is this: in almost all cases, the statement “$\text{si}(M)$ is 3-connected with an $N$-minor” is equivalent to “$M$ has an $N$-minor, and $\text{si}(M)$ is 3-connected.” The only exceptions occurs when $r(M) = 1$, and $N$ is one of the abovementioned matroids. I have worried about this point much more than I should have, but it seems excessively pedantic to have to use the latter expression to deal with an essentially trivial exception. Thus, in this paper, the statement “$\text{si}(M)$ is 3-connected with an $N$-minor” will mean that “$M$ has an $N$-minor, and $\text{si}(M)$ is 3-connected”, and dually, the statement $\text{co}(M)$ is 3-connected with an $N$-minor” will mean that “$M$ has an $N$-minor an $\text{co}(M)$ is 3-connected.”

For the other exception consider representations of a rank-2 matroid over a field $F$. Since, regarded as a matroid, the automorphism group of $PG(1, F)$ is the symmetric group, all $F$-representations of a rank-2 matroid are regarded in [10] as being equivalent. However, from the perspective of projective geometry one wants automorphisms to preserve cross-ratios, and not all permutations of $PG(1, F)$ do this. In this paper we adopt the later viewpoint, and a rank-2 matroid may well have inequivalent representations over a field. This is discussed further in Section 5.
On 3-Connectivity and Related Matters. We are most interested in the case where a matroid $M$ may not be 3-connected, but either $si(M)$ or $co(M)$ is 3-connected. For such matroids the following hold.

(2.1) If $si(M)$ is 3-connected and $M'$ is an extension of $M$ with $r(M') = r(M)$, then $si(M')$ is 3-connected.

(2.2) If $M$ is connected and $si(M)$ (respectively $co(M)$) is 3-connected, then any 2-separation $\{X, Y\}$ of $M$ has the property that either $X$ or $Y$ is contained in a parallel (respectively series) class.

We use (2.1) and (2.2) freely without citation in proofs. Further straightforward facts that are also used freely are

(2.3) $si(si(M/x,y)) = si(M/x,y)$ and $co(co(M/x,y)) = co(M/x,y)$.

(2.4) If $M$ is 3-connected and $x$ is on a line of $M$ having at least four points, then $M/x$ is 3-connected.

Condition (2.4) generalizes immediately to

(2.5) Let $M$ be a matroid such that $si(M)$ is 3-connected. If $l$ is a rank-2 flat of $M$ containing at least four distinct rank-1 flats, and $x \not\in l$, then $si(M/x)$ is 3-connected.

We use the next two results frequently. The first is a theorem of Bixby [3] (see also [10, Proposition 8.4.6]).

(2.6) Let $M$ be a 3-connected matroid and let $e$ be an element of $M$. Then either $si(M/e)$ or $co(M\setminus e)$ is 3-connected.

The second is a theorem of Tutte [19] (see also [10, lemma 8.4.9]). Recall that a triangle of a matroid is a 3-element circuit while a triad is a 3-element cocircuit.

(2.7) (Tutte’s Triangle Lemma) Let $M$ be a 3-connected matroid having at least four elements and suppose that $\{e, f, g\}$ is a triad of $M$ such that neither $M/e$ nor $M/f$ is 3-connected. Then $M$ has a triangle that contains $e$ and exactly one of $f$ and $g$.

$\Delta - Y$ and $Y - \Delta$ Exchanges. Say that $\{a, b, c\}$ is a triangle of a matroid $M$. The operation of performing a $\Delta - Y$ exchange on $\{a, b, c\}$ replaces this triangle by a triad. It may be that $\{a, b, c\}$ is already a triad. In this case a $\Delta - Y$ exchange leaves $M$ unchanged. Otherwise $\{a, b, c\}$ is coindependent. Then formally, a $\Delta - Y$ exchange is the 3-sum of $M$ and $M(K_4)$ across the triangle $\{a, b, c\}$. For details, see [1]. Informally one adds a triad $\{a', b', c'\}$ in such a way that $\{a', b', c\}$, $\{a', c', b\}$, and $\{b', c', a\}$ are all triangles. The triangle $\{a, b, c\}$ is then removed; see Fig. 1. A $Y - \Delta$ exchange is the dual of a $\Delta - Y$ exchange.
Throughout this paper we adopt the convention that ground sets are preserved under \( A - Y \) and \( Y - A \) exchanges, that is, after removing the triangle or triad \( \{a, b, c\} \) we then relabel \( a' \) by \( a \), \( b' \) by \( b \), and \( c' \) by \( c \).

**Partial Fields.** A partial field \( F \) is a structure that behaves very much like a field except that addition may be a partial operation in that, for \( a, b \in F \), \( a + b \) may not be defined. Partial fields are introduced in [12]. The point of partial fields is that one can develop a theory of matroid representation for them. Many of the properties of matroids representable over fields hold in the more general setting of partial fields; for example, the class of matroids representable over a partial field is minor-closed and closed under direct sums, 2-sums and duality. Moreover a number of natural classes of matroids such as regular matroids and the matroids studied in [20, 22] can be characterized as classes of matroids representable over partial fields. The results of Section 5 are stated in the generality of partial fields. Readers not familiar with partial fields should simply treat these as results for fields.

## 3. Contracting A Pair of Elements

Let \( M \) be a 3-connected matroid with a 3-connected matroid \( N \) as a minor. In this section and Section 4 we prove theorems that tell us when we can find a pair of elements \( x \) and \( y \) such that minors obtained by removing \( x, y \) or both \( x \) and \( y \) have a prescribed connectivity and an \( N \)-minor. In applications of such results there are times when one can freely dualise so that it does not matter whether the elements are deleted or contracted. At other times, for example with certain classes of graphic matroids, the situation is quite different, and one may distinctly prefer...
deletion to contraction or conversely. The next three results tell us what we can do if we insist on contracting.

**Theorem 3.1.** Let \( N \) be a 3-connected minor of the 3-connected matroid \( M \). If \( r(M) \geq r(N) + 2 \), then there exist distinct elements \( x, y \in E(M) \) such that \( \text{si}(M/x) \) and \( \text{si}(M/y) \) are both 3-connected with an \( N \)-minor.

**Theorem 3.2.** Let \( N \) be a 3-connected minor of the 3-connected matroid \( M \), where \( r(M) \geq r(N) + 3 \). If \( x \in E(M) \) has the property that \( \text{si}(M/x) \) is 3-connected with an \( N \)-minor, then there exists an element \( y \neq x \) such that \( \text{si}(M/y) \) and \( \text{si}(M/x, y) \) are both 3-connected with \( N \)-minors.

As an immediate consequence of Theorems 3.1 and 3.2 we have

**Corollary 3.3.** Let \( N \) be a 3-connected minor of the 3-connected matroid \( M \). If \( r(M) \geq r(N) + 3 \), then there exist distinct elements \( x, y \in E(M) \) such that \( \text{si}(M/x), \text{si}(M/y) \) and \( \text{si}(M/x, y) \) are all 3-connected with \( N \)-minors.

Note that in Corollary 3.3, both \( M/x \) and \( M/y \) are guaranteed to be connected, but \( M/x, y \) is not, since it may have loops. If \( M \) is a finite projective geometry, then every pair of elements of \( M \) is in a triangle, so that the matroid obtained by contracting any pair is not connected, so in a sense Corollary 3.3 is best possible. However, for certain classes, for example graphic matroids, it may be possible to strengthen the result to guarantee that \( M/x, y \) is connected.

The proof technique for Theorem 3.1 and 3.2 is as follows. By an easy corollary of the splitter theorem one has no difficulty finding an element \( x \) such that \( \text{si}(M/x) \) is 3-connected with an \( N \)-minor. Moreover, one can find an element \( p \in E(\text{si}(M/x)) \) such that \( \text{si}(M/x, p) \) is 3-connected with an \( N \)-minor. The problem is that \( \text{si}(M/p) \) may not be 3-connected with an \( N \)-minor. In this case, the structure described in Lemma 3.6 arises, namely \( x \) is on a rank-3 cocircuit \( C^* \) of \( M \) and \( p \in (\text{cl}(C^*) − C^*) \). The proofs of Lemma 3.9 and Theorems 3.1 and 3.2 amount to a close analysis of the situations that can arise in such structures. The details are at times technical, but I see no way that the technicalities can be avoided.

As noted above, a straightforward consequence of Seymour’s Splitter Theorem [16] is

**Lemma 3.4.** Let \( N \) be a 3-connected minor of the 3-connected matroid \( M \). If \( r(M) > r(N) \), then there is an element \( e \) of \( E(M) \) with the property that \( \text{si}(M/e) \) is 3-connected with an \( N \)-minor.

The next lemma is essentially well known, but it is vital so we provide a proof.
Since \( M \) is 3-connected. We deduce that it follows that \( Y \). Hence \( \leq \).

If \( \leq \). Therefore \( \leq \). Thus we may assume without loss of generality that \( Y \). If \( \leq \), then it is readily checked that \( X \). Since \( \leq \), \( \leq \). Thus \( \leq \) is exact. Hence \( \leq \). If \( \leq \), then \( \leq \). We know that \( X \). Also, since \( \leq \), \( \leq \) spans \( X \), and hence \( \leq \). Therefore \( \leq \). This establishes the lemma.

Much of the argument in the proofs of the theorems of this paper focuses on rank-3 cocircuits. The next two lemmas establish some properties related to these structures.

**Lemma 3.5.** Let \( M \) be a 3-connected matroid. An element \( e \) of \( E(M) \) has the property that \( \leq(e) \) is not 3-connected if and only if \( M \) has a 3-separation \( \{ X, Y \} \) with the properties that \( r(X) \geq 3, r(Y) \geq 3, \) and \( e \in \leq(X) \cap \leq(Y) \).

Proof. Assume that \( \leq(e) \) is not 3-connected. Then \( M/e \) has a 2-separation \( \{ X', Y' \} \), where \( r_M(X') \geq 2 \) and \( r_M(Y') \geq 2 \). Say \( e \in \leq_a(X') \); then \( r_M(X') = r_M_a(Y') \). By the definition of contraction, \( r_M(Y' \cup e) = r_M(Y') + 1 \). It follows that \( X', Y' \cup e \) is a 2-separation of \( M \), contradicting the fact that \( M \) is 3-connected. We deduce that \( e \in \leq_a(Y') \) and similarly that \( e \in \leq(Y') \). Hence \( r_M(X') = r_M(Y') + 1 \), and similarly \( r_M(Y') = r_M(Y') + 1 \). It now follows easily that \( \{ X' \cup e, Y' \} \) (and, indeed, \( \{ X', Y' \cup e \} \)) is a 3-separation of \( M \) with the required properties. The proof of the converse is even more straightforward.

Note that it follows from Lemma 3.5 that if \( M \) is 3-connected and \( r(M) \leq 3 \), then \( \leq(M/z) \) is 3-connected for all \( z \in E(M) \). The next lemma describes a situation that arises frequently.

**Lemma 3.6.** Let \( M \) be a 3-connected matroid, and let \( x \) and \( p \) be elements of \( E(M) \) with the properties that \( \leq(M/x) \) and \( \leq(M/x, p) \) are 3-connected, but \( \leq(M/p) \) is not 3-connected. Then \( r(M) \geq 4 \), and \( M \) has a rank-3 cocircuit \( C^* \) containing \( x \) and complementary hyperplane \( H \) such that \( p \in \leq(C^*) \cap H \).

Proof. If \( r(M) \leq 3 \), then, as noted above, \( \leq(M/z) \) is 3-connected for all \( z \in E(M) \). Hence \( r(M) \geq 4 \). By Lemma 3.5, \( M \) has a 3-separation \( \{ X, Y \} \) with \( r(X) \geq 3, r(Y) \geq 3, \) and \( p \in \leq(X) \cap \leq(Y) \). Since \( \leq(M/x) \) is 3-connected, \( x \notin \leq(X) \cap \leq(Y) \). Thus we may assume without loss of generality that \( x \in X - \leq(Y) \). If \( r(X) \geq 3 \), then it is readily checked that \( \{ X - \{ x, p \}, Y - \{ p \} \} \) is a 2-separation of \( M/x, p \) and that \( r_{M/x, p}(X - \{ x, p \}) \geq 2 \), and \( r_{M/x, p}(Y - \{ p \}) \geq 2 \), so that \( \leq(M/x, p) \) is not 3-connected. Thus \( r(X) = 3 \). Since \( M \) is 3-connected, the 3-separation \( \{ X, Y \} \) is exact. Hence \( r(Y) = r(M) - r(X) + 2 = r(M) - 1 \), and we deduce that \( C^* = X - \leq(Y) \) is a cocircuit of \( M \). If \( r(C^*) < 3 \), then \( M \) is not 3-connected. Therefore \( r(C^*) = 3 \). We know that \( x \in C^* \). Also, since \( r(C^*) = r(X) \), \( C^* \) spans \( X \), and hence \( \leq(X) \). Therefore \( p \in \leq(C^*) \). This establishes the lemma.

Much of the argument in the proofs of the theorems of this paper focuses on rank-3 cocircuits. The next two lemmas establish some properties related to these structures.

**Lemma 3.7.** Let \( C^* \) be a rank-3 cocircuit of a matroid \( M \) with complementary hyperplane \( H \) and let \( p \) be an element of \( H \cap \leq(C^*) \).

(i) If \( z_1, z_2 \in C^* \), then \( \leq(M/p, z_1) \cong \leq(M/p, z_2) \).
(ii) If, for some element $c \in C^*$, $\text{si}(M/p, c)$ is 3-connected with an $N$-minor, then, for all $z \in C^*$, $\text{si}(M/p, z)$ is 3-connected with an $N$-minor.

Proof. Since $p \in H$, $r_{M/p}(H - p) = r_M(H) - 1$, and since $p \in \text{cl}(C^*)$, $r_{M/p}(C^*) = r_M(C^*) - 1 = 2$. Thus $\{H - p, C^*\}$ is a 2-separation of $M/p$ corresponding to a 2-sum decomposition $M_{H - p} \oplus_2 M_{C^*}$ of $M/p$. It is now routinely checked that, for all $z \in C^*$, $\text{si}(M/p, z) \cong \text{si}(M_{H - p})$. This establishes part (i). Part (ii) is an immediate corollary of part (i).

Lemma 3.8. Let $C^*$ be a rank-$3$ cocircuit of the 3-connected matroid $M$. If $x \in C^*$ has the property that $\text{cl}_M(C^*) - x$ contains a triangle of $M/x$, then $\text{si}(M/x)$ is 3-connected.

Proof. Assume that $x \in C^*$ has the property that $\text{cl}_M(C^*) - x$ contains a triangle $T$ of $M/x$. Assume that $\text{si}(M/x)$ is not 3-connected. Then $M/x$ has a 2-separation $\{X', Y'\}$ where $r_{M/x}(X') \geq 2$ and $r_{M/x}(Y') \geq 2$. Without loss of generality we may assume that $X'$ contains two points of $T$. Consider another point $z$ of $\text{cl}_M(C^*) - x$. Assume that $z \in Y'$. Certainly $z \notin \text{cl}_{M/x}(X')$. Evidently, if $z$ is a cocircuit of $(M/x)/Y'$, then $\{X' \cup z, Y' - z\}$ is a separation of $M/x$, so that $M/x$ is not connected. But this implies that $M$ is not connected. This contradiction shows that $z$ is not a cocircuit of $(M/x)/Y'$. Therefore $\{Y' \cup z, Y' - z\}$ is a 2-separation of $M/x$ with $r_{M/x}(X' \cup z) \geq 2$ and $r_{M/x}(Y' - z) \geq 2$. It follows that we may assume without loss of generality that $(\text{cl}(C^*) - x) \subseteq X'$. In this case $Y'$ is contained in the hyperplane $E(M) - C^*$. Hence $r_{M}(Y') = r_{M/x}(Y')$. It now follows that $\{X' \cup x, Y'\}$ is a 2-separation of $M$, contradicting the fact that $M$ is 3-connected. We conclude that $\text{si}(M/x)$ is indeed 3-connected.

Now say that $M$ is a 3-connected matroid with the 3-connected matroid $N$ as a minor where $r(M) - r(N) \geq 2$. Then, by Lemma 3.4, $M$ has an element $x$ such that $\text{si}(M/x)$ is 3-connected with an $N$-minor, and $\text{si}(M/x)$ has an element $p$ such that $\text{si}(M/x, p) = \text{si}(\text{si}(M/x)/p)$ is 3-connected with an $N$-minor. If it is the case that $\text{si}(M/p)$ is 3-connected then both Theorems 3.1 and 3.2 hold. Thus argument focuses on the case when $\text{si}(M/p)$ is not 3-connected. In this case, the situation dealt with by the following lemma frequently arises.

Lemma 3.9. Let $M$ be a 3-connected matroid with a triad $\{x, a, b\}$ and a triangle $\{a, b, p\}$. Let $N$ be a 3-connected matroid and assume that $\text{si}(M/x)$ and $\text{si}(M/x, p)$ are both 3-connected with $N$-minors and that $M \setminus p$ is not 3-connected. Then there is an element $y \in E(M) - x$ with the property that both $\text{si}(M/y)$ and $\text{si}(M/x, y)$ are 3-connected with $N$-minors.

Proof. Set $C^* = \{a, b, x\}$, and $H = E(M) - C^*$. If $r(M) \leq 3$, then $\text{si}(M/p)$ is 3-connected and certainly has an $N$-minor. Thus, in this case,
the lemma follows by setting \( y = p \). Assume that \( r(M) \geq 4 \), so that \( \{ C^*, H \} \) is a 3-separation of \( M \) with the property that \( r(C^*) = 3 \) and \( r(H) \geq 3 \). Since \( \{ a, b, p \} \) is a triangle, \( p \in cl(C^*) \). It now follows by Lemma 3.5 that \( si(M/p) \) is not 3-connected. Hence, by (2.6), \( co(M \setminus p) \) is 3-connected. But \( M \setminus p \) is not 3-connected, so \( M \setminus p \) has at least one coloop or series pair. Since \( M \) is 3-connected, \( M \setminus p \) is connected, so that \( M \setminus p \) has no coloops. Thus \( M \setminus p \) has at least one series pair. This cannot be a series pair of \( M \). Hence \( M \) has a triad containing \( p \). It is easily seen that a triad containing one element of a triangle must contain another element of that triangle. Hence, for some \( y \in E(M) \), either \( \{ p, a, y \} \) or \( \{ p, b, y \} \) is a triad of \( M \). Assume without loss of generality that \( \{ p, a, y \} \) is a triad. We now show that \( y \notin cl(C^*) \). Assume otherwise, that is, assume that \( y \in cl(C^*) \). Then \( cl(C^*) \) contains two distinct cocircuits, so \( r(E(M) - cl(C^*)) \leq r - 2 \). But then \( r(cl(C^*)) + r(E(M) - cl(C^*)) \leq r + 1 \), so that \( cl(C^*), E(M) - cl(C^*) \) is a 2-separation of \( M \), contradicting the fact that \( M \) is 3-connected. Therefore \( y \in (E(M) - cl(C^*)) \).

Certainly \( \{ p, a, y \} \) is a triad of \( M/x \), so that \( \{ p, y \} \) is a series pair of \( M/x \setminus a \). Hence \( M/x \setminus a/y \cong M/x \setminus a/p \), that is, \( M/x, p \setminus a \cong M/x, y \setminus a \). But \( \{ a, b \} \) is a parallel pair of \( M/x, p \). Hence \( si(M/x, p \setminus a) \) is 3-connected with an \( N \)-minor, so that \( si(M/x, y \setminus a) \) is 3-connected with an \( N \)-minor. It is now easily seen that \( si(M/x, y) \) is 3-connected with an \( N \)-minor. It immediately follows that \( si(M/y) \) has an \( N \)-minor. We complete the proof by showing that \( si(M/y) \) is 3-connected.

Since \( \{ a, p, y \} \) is a triad of \( M \), \( \{ a, p \} \) is a series pair of \( M \setminus y \). But \( \{ a, b, p \} \) is a triangle of \( M \setminus y \). We deduce that \( co(M \setminus y) \) has a non-trivial parallel class, so \( co(M \setminus y) \) is not 3-connected. It now follows by (2.6) that \( si(M/y) \) is 3-connected.}

**Proof of Theorem 3.1.** Assume that \( M \) has no element \( q \) such that \( M \setminus q \) is 3-connected with an \( N \)-minor. (The case where there is such an element follows easily once the case where there is no such element has been established and is treated later.)

By Lemma 3.4 there are distinct elements \( x \) and \( p \) of \( E(M) \) such that both \( si(M/x) \) and \( si(M/p) \) are 3-connected with \( N \)-minors. If \( si(M/p) \) is 3-connected, then the theorem holds, so assume that \( si(M/p) \) is not 3-connected. By Lemma 3.6, \( r(M) \geq 4 \) and \( M \) has a rank-3 cocircuit \( C^* \) with complementary hyperplane \( H \) such that \( x \in C^* \) and \( p \in cl(C^*) \cap H \). It follows from Lemma 3.7 that

\[ (3.1.1) \quad \text{If } z \in C^* \setminus x, \text{ then } M/z \text{ has an } N \text{-minor}. \]

If we can find an element \( z \) of \( C^* \setminus x \) such that \( si(M/z) \) is 3-connected, then, by (3.1.1), we are done. Unfortunately this is not always possible. The remainder of the proof is devoted to showing that in the case where no
such element exists \( \cl(M^*) \) has a specific structure. From now on we assume that if \( z \in (C^* - x) \) then \( \si(M/z) \) is not 3-connected. In this case we have

\[
(3.1.2) \quad \cl(M^*) - x \text{ is a rank-2 flat of } M.
\]

Proof. We first show that \( p \) is the only element of \( \cl(C^*) \cap H \). Assume not; say \( u \notin p \) belongs to \( \cl(C^*) \cap H \). Choose a basis \( \{ x, u_1, u_2 \} \) of \( C^* \). Then \( M |\{u, u_1, u_2, x, p\} \) is a rank-3 simple matroid having \( \{u_1, u_2, x\} \) as a basis. An elementary check shows that for some \( i \in \{1, 2\} \), \( M |\{u, u_1, u_2, x, p\}/u \) contains a triangle. But then, by Lemma 3.8, \( \si(M/u) \) is 3-connected, contradicting the assumption that \( x \) is the only element of \( C^* \) with this property.

We now know that \( \cl(M^*) - p = C^* \). Choose a basis \( \{a, b, c\} \) of \( M \setminus C^* \). Either \( M |\{a, b, c, p\} \cong U_{1,4} \) or \( M |\{a, b, c, p\} \cong U_{1,1} \oplus U_{2,3} \). Assume that \( M |\{a, b, c, p\} \cong U_{1,1} \oplus U_{2,3} \). Then, for all \( w \in \{a, b, c\} \), \( (M/w) |\{a, b, c, p\} - w \) is a triangle. By Lemma 3.8, \( \si(M/w) \) is 3-connected. But \( \{a, b, c\} \) is contained in \( C^* \). This contradicts the assumption that there is only one element of \( C^* \) whose contraction from \( M \) gives a matroid that simplifies to a 3-connected matroid. It follows that \( M |\{a, b, c, p\} \cong U_{1,1} \oplus U_{2,3} \), that is \( M |\{a, b, c, p\} \) consists of a coloop and a triangle. Since \( p \in \cl(M^*) \setminus \{a, b, c\} \), \( p \) is not a coloop of \( M |\{a, b, c, p\} \). Assume without loss of generality that \( a \) is the coloop of \( M |\{a, b, c, p\} \). Then \( \{b, c, p\} \) is a triangle of \( M/a \), so by Lemma 3.8, \( \si(M/a) \) is connected. Therefore \( a = x \). We deduce that \( x \) is in every basis of \( M \setminus \cl(M^*) \) and, for any pair \( z_1, z_2 \) of elements of \( \cl(M^*) \), \( \{z_1, z_2, p\} \) is a triangle. It is now routinely seen that \( \cl(M^*) - x \) is a rank-2 flat of \( M \).

By (3.1.2), \( C^* = \{x, a_1, a_2, ..., a_n\} \) where \( \{a_1, a_2, ..., a_n, p\} \) is a line of \( M \). Say \( \{|a_1, a_2, ..., a_n, p\}| \geq 4 \). Consider \( a_3 \). By (2.4), \( M/a_3 \) is 3-connected. Moreover, by (3.1.1), \( M/a_1 \) has an \( N \)-minor and \( a_3 \) is in a parallel class of the minor. Hence \( M/a_1 \setminus a_3 \) has an \( N \)-minor. Thus \( M \setminus a_3 \) is 3-connected with an \( N \)-minor, contradicting the assumption that \( M \) has no elements with these properties. Thus \( C^* = \{a_1, a_2, x\} \), and \( C^* \) is a triad. Also \( \cl(C^*) = \{x, a_1, a_2, p\} \), where \( \{a_1, a_2, p\} \) is a triangle. Evidently we have the structure specified in Lemma 3.9 and it follows that there exists an element \( y \in E(M) - x \) such that \( \si(M/y) \) is 3-connected with an \( N \)-minor.

We deduce that the theorem holds if \( M \) has no element \( q \) with the property that \( M \setminus q \) is 3-connected with an \( N \)-minor. Now lift this restriction. Let \( S \) be a maximal subset of \( E(M) \) with the properties that \( r(M \setminus S) = r(M) \) and that \( M \setminus S \) is 3-connected with an \( N \)-minor. Then there is a pair of distinct elements \( x, y \in E(M \setminus S) \) such that \( \si(M \setminus S/x) \) and \( \si(M \setminus S/y) \) are 3-connected with \( N \)-minors. It is easily checked that \( \si(M/x) \) and \( \si(M/y) \) are also 3-connected with \( N \)-minors.
Theorem 3.1 does not generally hold if \( r(M) - r(N) = 1 \). For an example let \( M \) be the rank-3 matroid consisting of two disjoint 3-point lines and a point \( x \) placed freely on the plane. Let \( N = U_{2,4} \). Evidently \( M/x \cong N \), and \( x \) is the only element of \( E(M) \) with this property. Now consider Theorem 3.2.

**Proof of Theorem 3.2.** If \( r(M) \leq 4 \), then \( r(N) \leq 1 \). An easy check shows that the result holds in this case. Thus we may assume that \( r(M) \geq 5 \).

Let \( x \) be an element such that \( \si(M/x) \) is 3-connected with an \( N \)-minor. Assume that \( M \) has no element \( q \) such that both \( M'q \) and \( \si(M/x \setminus q) \) are 3-connected with \( N \)-minors. (As in the proof of Theorem 3.1, the case when there is such an element follows straightforwardly once the case when there is no such element has been established and is treated later.)

By Theorem 3.1, \( \si(M/x) \) has distinct elements \( p_1 \) and \( p_2 \) such that \( \si(M/x, p_1) \) and \( \si(M/x, p_2) \) are 3-connected with \( N \)-minors. If either \( \si(M/p_1) \) or \( \si(M/p_2) \) is 3-connected we are done. Assume that neither of these matroids is 3-connected. Then, by Lemma 3.6, there are rank-3 cocircuits \( C^*_1 \) and \( C^*_2 \) with complementary hyperplanes \( H_1 \) and \( H_2 \) respectively such that \( x \in \cl(C^*_1) \cap C^*_2 \) and \( p_1 \in \cl(C^*_1) \cap H_1 \) and \( p_2 \in \cl(C^*_2) \cap H_2 \). We distinguish two cases. We first prove

**Remark** (3.2.1) The theorem holds if \( C^*_1 = C^*_2 \).

**Proof.** Assume that \( C^*_1 = C^*_2 \). To ease notation a little, denote this cocircuit by \( C^* \) and the complementary hyperplane by \( H \). We have \( \{p_1, p_2\} \subseteq \cl(C^*) \cap H \). Consider cases that can arise. For the first case assume that there is an element \( a \in C^* \) such that either \( \{a, x, p_1\} \) or \( \{a, x, p_2\} \) is a triangle. Assume without loss of generality that \( \{a, x, p_1\} \) is a triangle. Since \( \{a, p_1\} \) is a parallel pair in \( M/x, \si(M/x, a) \cong \si(M/x, p_1) \), \( \si(M/x, a) \) is 3-connected with an \( N \)-minor. Hence, if \( \si(M/a) \) is 3-connected we are done by choosing \( \{x, y\} = \{x, a\} \). Assume that \( \si(M/a) \) is not 3-connected. Since \( \si(M/a) \) is not 3-connected, Lemma 3.8 implies that \( \cl(C^*) \) consists of precisely two lines through \( a \), which then are necessarily \( \cl(\{a, p_1\}) \) and \( \cl(\{a, p_2\}) \). Since \( p_1 \) and \( p_2 \) are in \( H \), if \( C^* \) is not a triad, then at least one of \( \cl(\{a, p_1\}) \) and \( \cl(\{a, p_2\}) \) has at least four points. Let \( z \) be a point on this line that is not in \( \{a, x, p_1, p_2\} \). By (2.4), \( M/xz \) is 3-connected. Either \( z \in \cl(\{a, x\}) \) or not. In the former case \( z \) is in a parallel class of \( M/x \), in which case \( \si(M/x \setminus z) \cong \si(M/x) \), a 3-connected matroid with an \( N \)-minor. This contradicts the assumption that \( M \) has no elements with these properties. Consider the latter case. Here \( \cl(M/x \setminus C^*(x, z)) \) contains a triangle, so by Lemma 3.8 \( \si(M/x \setminus z) \) is 3-connected. Moreover, \( z \) is in a parallel class of \( M/x, p_1 \), so \( M/x, p_1z \), and therefore \( M/x \setminus z \) has an \( N \)-minor. Again we have contradicted the assumption that there are no elements with these properties. It follows that \( C^* \) is a triad, say \( C^* = \{a, x, t\} \). But now \( \{a, t, p_2\} \) is a triangle, and thus by Lemma 3.9 there
exists an element \( y \in E(M) - x \) such that \( \text{si}(M/y) \) and \( \text{si}(M/x, y) \) are 3-connected with \( N \)-minors.

We may now assume that if \( z \in C^* \), then neither \( \{x, z, p_1\} \) nor \( \{x, z, p_2\} \) are triangles. Say \( \{x, a, b\} \) is a basis for \( M \setminus C^* \). Either \( \{a, b, p_1\} \) or \( \{a, b, p_2\} \) is independent, otherwise \( \{a, b\} \subseteq \text{cl}(\{p_1, p_2\}) \), so that neither \( a \) nor \( b \) belongs to \( C^* \). Without loss of generality assume that \( \{a, b, p_2\} \) is independent. This means that \( \{x, a, b, p_2\} \) is a circuit. We now show that \( M \setminus p_1 \) is 3-connected. Since \( \text{si}(M/p_1) \) is not 3-connected, \( \text{col}(M \setminus p_1) \) is 3-connected. Assume that \( M \setminus p_1 \) is not 3-connected. Then \( p_1 \) is in a triad \( T \) of \( M \). Since \( \{x, a, b, p_2\} \) is a circuit, any closed set that contains more than two elements of \( \{x, a, b, p_2\} \) spans \( \text{cl}(C^*) \) and hence contains \( p_1 \). Therefore the complementary hyperplane \( E(M) - T \) contains at most two elements of \( \{x, a, b, p_2\} \). Therefore \( T \subseteq \{x, a, b, p_1, p_2\} \). Hence \( \text{cl}(C^*) \) contains two distinct cocircuits. But now \( r(E - \text{cl}(C^*)) \leq r - 2 \), and \( \{\text{cl}(C^*), E - \text{cl}(C^*)\} \) is a 2-separation of \( M \), contradicting the fact that \( M \) is 3-connected. Thus \( M \setminus p_1 \) is indeed 3-connected.

Now \( \{p_1, a\} \) is a parallel pair in \( M/x, p_2 \), so \( \text{si}(M/x, p_2 \setminus p_1) \cong \text{si}(M/x, p_2) \). But \( \text{si}(M/x, p_2) \) has an \( N \)-minor, so \( M \setminus p_1 \) has a \( N \)-minor. Therefore \( p_1 \) is an element with the property that \( M \setminus p_1 \) is 3-connected with an \( N \)-minor. By (2.5), \( \text{si}(M/x \setminus p_1) \) is 3-connected. Moreover \( p_1 \) is in a non-trivial parallel class of \( M/x, p_2 \), so \( M/x, p_2 \setminus p_1 \), and therefore \( M/x \setminus p_1 \), has an \( N \)-minor. The contradicts the assumption that \( M \) has no elements with these properties. It follows that this case is vacuous and (3.2.1) holds.

We now show

(3.2.2) **The theorem holds if** \( C^* \neq C_2^* \).

**Proof.** Assume that \( C^* \neq C_2^* \). We first examine possibilities for the rank of \( C^* \cap C_2^* \). Since \( x \in C^* \cap C_2^* \), \( r(C^* \cap C_2^*) \geq 1 \). Also \( r(C^*) = r(C_2^*) = 3 \), so by the submodular inequality, \( r(C^* \cup C_2^*) \leq r(C^*) + r(C_2^*) - r(C^* \cap C_2^*) \). Hence \( r(C^* \cup C_2^*) \in \{3, 4, 5\} \). If \( r(C^* \cup C_2^*) = 3 \), then, arguing just as in the latter part of the proof of (3.2.1), we obtain a 2-separation of \( M \), contradicting the assumption that \( M \) is 3-connected. Hence

\[
r(C^* \cup C_2^*) \in \{4, 5\}.
\]

We now show that \( C^* \cap C_2^* = \{x\} \). Assume not; say that \( z \) is another element of \( C^* \cap C_2^* \). Certainly \( z \notin \{p_1, p_2\} \). Since \( \{p_1, p_2\} \) is independent in \( M/x, \{x, p_1, p_2\} \) is independent in \( M \). Hence, either \( \{x, z, p_1\} \) or \( \{x, z, p_2\} \) is independent in \( M \). Assume without loss of generality that \( \{x, z, p_1\} \) is independent in \( M \). Evidently \( \{x, z, p_1\} \) spans \( \text{cl}(C^*) \). If \( p_1 \in C_2^* \), then \( \{x, z, p_1\} \) also spans \( C_2^* \) and hence \( r(\text{cl}(C^*) \cup \text{cl}(C_2^*)) = 3 \),
contradicting (3.1). Therefore \( p_1 \notin C^*_2 \), that is, \( p_1 \in H_2 \). But \( p_1 \in H_1 \), so \( p_1 \in H_1 \setminus H_2 \). Since \( r(C^*_1 \cup C^*_2) \neq 3 \) and \( r(C^*_1 \cap C^*_2) = 2 \), by the submodular inequality, \( r(C^*_1 \cup C^*_2) = 4 \). If \( r(H_1 \cap H_2) < r(M) - 2 \), then \( \{ C^*_1 \cup C^*_2, H_1 \cap H_2 \} \) is a 2-separation of \( M \). Hence \( r(H_1 \cap H_2) \geq r(M) - 2 \). But \( H_1 \neq H_2 \), so \( r(H_1 \cap H_2) = r(M) - 2 \). Consider \( M/x, p_1 \).

Evidently \( r_{M/x,p_1}(C^*_1 \cup C^*_2) = 2 \) and \( r_{M/x,p_1}(H_1 \cap H_2 - p_1) = r(M) - 3 \). But \( r(M) > 5 \), so \( r(M) - 3 \geq 2 \). Hence \( \{ C^*_1 \cup C^*_2, H_1 \cap H_2 - p_1 \} \) is a 2-separation of \( M/x, p_1 \) with the property that \( r_{M/x,p_1}(C^*_1 \cup C^*_2 - x) \geq 2 \) and \( r_{M/x,p_1}(H_1 \cap H_2 - p_1) \geq 2 \). This contradicts the fact that \( \text{si}(M/x, p_1) \) is 3-connected. Thus it is indeed the case that

\[
C^*_1 \cap C^*_2 = \{ x \}. \tag{3.2}
\]

By (3.2), \( C^*_1 - x \subseteq H_2 \). Now \( x \notin H_2 \), so \( \text{si}(M/x) |_{H_2} = M |_{H_2} \) and hence \( \text{si}(M/x) |_{(C^*_1 - x) = M | (C^*_1 - x)} \). But \( C^*_1 - x \) has rank 2 in \( M/x \), so \( C^*_1 - x \) has rank 2 in \( M \). Evidently the same holds for \( C^*_2 - x \) so that we have

\[
r_{ad}(C^*_1 - x) = r_{ad}(C^*_2 - x) = 2. \tag{3.3}
\]

Now assume that \( p_1 \notin C^*_2 \). Then \( (C^*_1 \cup p_1) - x \subseteq H_2 \), so applying the above argument we see that \( r_{ad}(C^*_1 \cup p_1 - x) = 2 \). If \( a \in C^*_1 - x \), then either \( a \) is in a non-trivial parallel class of \( M/x, p_1 \) or \( a \) is a loop of \( M/x, p_1 \), so that \( M/x, p_1 \) is an N-minor, that is, \( M/x | a \) is an N-minor. Recall that \( p_1 \) is on the line \( cl(C^*_1 - x) \) and not in \( C^*_1 - x \). Thus, if \( |C^*_1 - x| > 2 \), then \( a \) is in a line having at least four points, so that \( M/x | a \) is 3-connected. Moreover, \( x \) is not on this line, so that \( a \) is on a rank-2 flat of \( M/x \) containing at least four rank-1 flats, so that, by 2.5, \( \text{si}(M/x | a) \) is 3-connected. But we have assumed that \( M \) has no elements with these properties. Therefore \( |C^*_1 - x| = 2 \), so that \( C^*_1 \) is a triad. But \( (C^*_1 - x) \cup p_1 \) is a triangle. Therefore, in this case, the result follows by Lemma 3.9.

The same argument shows that the theorem holds if \( p_2 \notin C^*_2 \). For the final case assume that \( p_1 \in C^*_2 \) and \( p_2 \in C^*_1 \). Choose a basis \( \{ x, p_2, c_1 \} \) for \( C^*_1 \) and a basis \( \{ x, p_1, c_2 \} \) for \( C^*_2 \). Since \( C^*_1 \cap C^*_2 = x \), \( c_1 \neq c_2 \). Since \( \{ x, p_1, p_2, c_1, c_2 \} \) spans \( C^*_1 \cup C^*_2 \), and \( r(C^*_1 \cup C^*_2) = 4 \), \( r(\{ x, p_1, p_2, c_1, c_2 \}) = 4 \). Now \( p_1 \in cl(C^*_1) \), so \( r(\{ x, p_1, p_2, c_1 \}) = 3 \). Hence \( c_2 \) is a coloop of \( M | \{ x, p_1, p_2, c_1 \} \). Similarly \( c_1 \) is a coloop. Hence \( \{ x, p_1, p_2 \} \) is a triangle and \( \{ p_1, p_2 \} \) is dependent in \( M/x \). This contradiction shows that the case we are in is vacuous.

By (3.2.1) and (3.2.2), the theorem holds if there is no element \( q \) such that \( M \setminus q \) is 3-connected with an N-minor. Now lift this restriction. Let \( S \) be a maximal subset of \( E(M) - x \) with the properties that \( r(M \setminus S) = r(M) \), that \( M \setminus S \) is 3-connected, and that \( \text{si}(M/x \setminus S) \) is 3-connected with an
N-minor. Then there is an element $y$ in $E(M \setminus S)$ such that $\text{si}(M \setminus S; y)$ and $\text{si}(M \setminus S; x, y)$ are both 3-connected with N-minors. Then, just as in Theorem 3.1, $\text{si}(M/y)$, and $\text{si}(M/x, y)$ are also 3-connected with N-minors.

Since a matroid is 3-connected if and only if its dual is 3-connected, we can dualise Theorems 3.1 and 3.2 and Corollary 3.3 to immediately obtain the following corollaries.

**Corollary 3.10.** Let $N$ be a 3-connected minor of the 3-connected matroid $M$. If $r(M^*) \geq r(N^*) + 2$, then there exist distinct elements $x, y \in E(M)$ such that $\text{co}(M \setminus x)$ and $\text{co}(M \setminus y)$ are both 3-connected with an N-minor.

**Corollary 3.11.** Let $N$ be a 3-connected minor of the 3-connected matroid $M$ with $r(M^*) \geq r(N^*) + 3$. If $x \in E(M)$ has the property that $\text{co}(M \setminus x)$ is 3-connected with an N-minor then there exists an element $y \in E(M)$ such that $\text{co}(M \setminus y)$ and $\text{co}(M \setminus x, y)$ are both 3-connected with N-minors.

**Corollary 3.12.** Let $N$ be a 3-connected minor of the 3-connected matroid $M$. If $r(M^*) \geq r(N^*) + 3$, then there exist distinct elements $x, y \in E(M)$ such that $\text{co}(M \setminus x)$, $\text{co}(M \setminus y)$ and $\text{co}(M \setminus x, y)$ are all 3-connected with N-minors.

4. CONTRACTING OR DELETING

This section considers the results that can be obtained if we are allowed to either delete or contract elements. Most of the argument involves the case when $|E(M) - E(N)|$ is small. Unfortunately things become quite technical with numerous subcases to be considered. However the difficulties that are encountered are purely combinatorial, not conceptual.

If $|E(M) - E(N)| \geq 5$, then either $r(M) \geq r(N) + 3$, or $r(M^*) \geq r(N^*) + 3$. We therefore get as a corollary of Corollary 3.3 and Corollary 3.12:

**Corollary 4.1.** Let $N$ be a 3-connected minor of the 3-connected matroid $M$. If $|E(M) - E(N)| \geq 5$, then there exists a pair of distinct elements $x, y \in E(M)$ such that either $\text{si}(M/x)$, $\text{si}(M/y)$ and $\text{si}(M/x, y)$ are 3-connected with N-minors, or $\text{co}(M \setminus x)$, $\text{co}(M \setminus y)$ and $\text{co}(M \setminus x, y)$ are 3-connected with N-minors.

The bound on $|E(M) - E(N)|$ given by Corollary 4.1 can be improved, and we address that question now. The case $|E(M) - E(N)| = 3$ causes some problems, but when difficulties occur, a very specific structure exists.
Theorem 4.2. Let $N$ be a 3-connected minor of the 3-connected matroid $M$ with the property that either $r(M) - r(N) \geq 2$ or $r(M^*) - r(N^*) \geq 2$. Then at least one of the following holds.

(i) There is a pair of distinct elements $x, y \in E(M)$ such that $\text{si}(M/x), \text{si}(M/y)$ and $\text{si}(M/x, y)$ are all 3-connected with $N$-minors.

(ii) There is a pair of distinct elements $x, y \in E(M)$ such that $\text{co}(M\backslash x), \text{co}(M\backslash y)$ and $\text{co}(M\backslash x, y)$ are all 3-connected with $N$-minors.

(iii) $r(M) = r(N) + 2$, $|E(M) - E(N)| = 3$ and $M$ has a triad $T$ with the property that the matroid $M'$ obtained by performing a $Y - A$ exchange on $T$ has a pair of elements $x, y$ such that $\text{co}(M' \backslash x), \text{co}(M' \backslash y)$ and $\text{co}(M' \backslash x, y)$ are all 3-connected with $N$-minors.

(iv) $r(M^*) = r(N^*) + 2$, $|E(M) - E(N)| = 3$ and $M$ has a triangle $T$ with the property that the matroid $M'$ obtained by performing a $A - Y$ exchange on $T$ has a pair of elements $x, y$ such that $\text{si}(M' \backslash x), \text{si}(M' \backslash y)$ and $\text{si}(M' \backslash x, y)$ are all 3-connected with $N$-minors.

Before proving Theorem 4.2 we introduce some terminology to make the discussion less unwieldy. The pair of elements $x, y$ is a good contraction pair if $\text{si}(M/x), \text{si}(M/y)$ and $\text{si}(M/x, y)$ are all 3-connected with $N$-minors. A good deletion pair is defined in the obvious way. Loosely speaking, Theorem 4.2 says that if $M$ and $N$ are not too close to each other, then we either have a good deletion pair, a good contraction pair, or a very specific situation in which we can perform a $A - Y$ or a $Y - A$ exchange to get a matroid with a good deletion or contraction pair.

Proof of Theorem 4.2. If either $r(M) - r(N) > 2$ or $r(M^*) - r(N^*) > 2$, then by Corollary 3.3 and Corollary 3.12, $M$ has either a good deletion or a good contraction pair. Assume that $r(M) - r(N) \leq 2$, and $r(M^*) - r(N^*) \leq 2$. Then $|E(M) - E(N)| \in \{2, 3, 4\}$. An elementary check shows that the theorem holds if either $r(N) \leq 1$ or $r(N^*) \leq 1$. Thus we may assume that $r(N) > 1$ and $r(N^*) > 1$, and hence that $N$ is both simple and cosimple. We consider the three cases that arise.

Say $|E(M) - E(N)| = 2$. Then either $r(M) - r(N) = 2$ or $r(M^*) - r(N^*) = 2$. We lose no generality in assuming the latter. In this case $r(M) = r(N)$ and there exists a pair of elements $x, y$ such that $M \backslash x, y \cong N$. Clearly $M\backslash x$ and $M\backslash y$ are both 3-connected so that $\{x, y\}$ is a good deletion pair.

Assume that $|E(M) - E(N)| = 3$. This is the crucial case as it is the only one in which (iii) and (iv) may arise. Dualising if necessary we may assume without loss of generality that $r(M) = r(N) + 2$. We now specify exactly when we need to resort to (iii). This specification is a little more precise than is needed for the theorem, but it is of some independent interest. It also helps when we come to the case $|E(M) - E(N)| = 4$. 

4.2.1. Assume that $|E(M) - E(N)| = 3$, $r(M) - r(N) = 2$, and that $M$ does not have a good contraction pair. Then $M$ has an element $p$ and a triad \( \{ x, a, b \} \) with the following properties.

(a) $\cl(\{ x, a, b \}) = \{ x, a, b, p \}$.

(b) $p$ is not in a triangle.

(c) For any pair \( \{ z_1, z_2 \} \subseteq \{ x, a, b \} \), $M/\{ z_1, p \} \cong N$.

(d) For all $z$ in \( \{ x, a, b \} \), $M/z$ is 3-connected with an $N$-minor.

(e) $p$ is not in a triad.

(f) For all $z \in \{ x, a, b \}$, $\co(M \setminus z)$ is 3-connected with an $N$ minor.

(g) Let $M'$ denote the matroid obtained by performing a $Y - A$ exchange on \( \{ x, a, b \} \). Then, any pair \( \{ z_1, z_2 \} \equiv \{ x, a, b \} \) is a good deletion pair of $M'$. Moreover, each of $M' \setminus z_1$, $M' \setminus z_2$, and $M' \setminus z_1, z_2$ is 3-connected.

Proof. Certainly $M$ has a pair $x$, $p$ such that $\si(M/x)$ and $\si(M/x, p)$ are both 3-connected with $N$-minors. If $\si(M/p)$ is 3-connected, then $M$ has a good contraction pair, so $\si(M/p)$ is not 3-connected. In this case, as always, $M$ has a rank-3 cocircuit $C^*$ with complementary hyperplane $H$ such that $x \in C^*$ and $p \in \cl(C^*) \cap H$. Now $\cl(C^*) = \{ x, p \}$ is a rank-1 flat of $M/x, p$. Since $\si(M/x, p)$ has an $N$-minor, $|E(\si(M/x, p))| \geq |E(N)|$. It follows that $|\cl(C^*) - \{ x, p \}| \leq 2$. From this we deduce that $C^*$ is a triad, say $C^* = \{ x, a, b \}$, and $p$ is the only element of $\cl(C^*) \cap H$. Thus $C^*$ satisfies (a).

Assume that $p$ is in a triangle with a subset of $\{ a, b, x \}$. There are two cases. For the first assume that $\{ p, a, x \}$ is a triangle. Since $\si(M/p)$ is not 3-connected, it follows by (2.6) that $\co(M\setminus p)$ is 3-connected. Since $\{ p, a, x \}$ is a triangle of $M$, $\{ a, p \}$ is a parallel pair of $M/x$. It is now routinely seen that $M/x, p \setminus a \cong M/x, a \setminus p \cong N$ so that $M\setminus p$ has an $N$ minor. Assume that $M\setminus p$ is 3-connected. Then $|E(M\setminus p) - E(N)| = 2$, and $r(M\setminus p) - r(N) = 2$. It is shown above that in this case $M\setminus p$ has a good contraction pair and it follows that $M$ also has such a pair. Thus $M\setminus p$ is not 3-connected. Then, since $\co(M\setminus p)$ is 3-connected, $p$ is in a triad. Since a triad that meets a triangle meets it in two points, this triad contains either $x$ or $a$. Say the triad contains $a$. Then $x$ is not in the triad, but $x$ is in the closure of the triad, so it follows by Lemma 3.5 that $\si(M/x)$ is not 3-connected; contradicting the assumption made at the start of the proof that this matroid is 3-connected. Therefore the triad contains $x$. Say that $\{ p, c, x \}$ is a triad. Since $\{ a, x, p \}$ is a triangle, and $b \notin \cl(\{ a, x, p \})$, $\{ a, x, p \}$ is a triangle of $M/b$. Thus, by Lemma 3.8, $\si(M/b)$ is 3-connected. Moreover, by Lemma 3.7, $\si(M/b, p)$ is 3-connected with an $N$-minor. Recall that $M\setminus p$ is not 3-connected. It now follows from Lemma 3.9 that $M$ has an element $y$ such that $\si(M/y)$ and $\si(M/b, y)$ are both 3-connected.
with \(N\)-minors. Therefore \(\{b, y\}\) is a good contraction pair. This contradiction shows that \(\{p, a, x\}\) is not a triangle. An identical argument shows that \(\{p, b, x\}\) is not a triangle. If \(\{a, b, p\}\) is a triangle, then, by Lemma 3.9, \(M\) has a good contraction pair, so \(\{a, b, p\}\) is not a triangle. Thus \(p\) is in no triangle with members of \(C^*\).

Since \(M\) \(\{a, b, p, x\}\) has no triangles, \(\{a, b\}\) is a parallel class of \(M/p, x\), so \(M/p, x \cup a \cong M/p, x \cup b\). Since \(si(M/p, x)\) has an \(N\)-minor, \(M/p, x \cup a\) and \(M/p, x \cup b\) both have \(N\)-minors. Thus, since \(|E(M) - E(N)| = 3\), we have

\[
M/x, p \cup a \cong M/x, p \cup b \cong N. \quad (4.1)
\]

The fact that \(p\) is in no other triangle follows easily from (4.1) and the 3-connectivity of \(N\), and we conclude that (b) holds.

Now say that \((z_1, z_2, z_3)\) is a permutation of \((a, b, x)\). By Lemma 3.7, \(M/z_1, p\) has an \(N\)-minor. Since \(\{a, b, x, p\}\) has no triangles, \(\{z_2, z_1\}\) is a parallel class of \(M/z_1, p\) and, arguing as in (4.1), we conclude that \(M/z_1, p \cup z_2 \cong N\). This proves (c).

Certainly \(M/z_1\) has an \(N\)-minor. Moreover, since \(\{z_2, z_3, p\}\) is a triangle of \(M/z_1\), it follows from Lemma 3.8 that \(si(M/z_1)\) is 3-connected. Any triangle of \(M\) using \(z_1\) must contain exactly two points of the triad \(\{z_1, z_2, z_3\}\). We know that there are no such triangles. Hence \(z_1\) belongs to no triangles. It follows that \(M/z_1\) has no parallel pairs. Hence \(M/z_1\) is 3-connected. This proves (d).

Assume that \(p\) is in a triad. Evidently this triad contains exactly one element of \(\{z_1, z_2, z_3\}\), so that without loss of generality we have a triad \(\{p, z_1, y\}\) where \(y \notin C^*\). The situation we are in is similar to, but not quite the same, as that of Lemma 3.9. We know that \(M/z_2, p \cup z_1 \cong N\), and \(\{p, y\}\) is a series pair of \(M \cup z_1\), so \(M/z_2, y \cup z_1 \cong N\). Thus \(si(M/z_2, y)\) is 3-connected with an \(N\)-minor. Consider \(M/y, M/p\) is not 3-connected, so if \(M/y\) is not 3-connected, then, by (2.7), \(M\) has a triangle containing \(p\), contradicting the fact that no such triangle exists. Hence \(M/y\) is 3-connected. It now follows that \(\{z_2, y\}\) is a good contraction pair, contradicting the fact that \(M\) has no such pairs. Hence \(p\) is in no triad and (e) holds.

Consider \(co(M/z_1)\). Since \(\{z_2, z_3\}\) is a series pair of \(M \cup z_1\), we have \(co(M \cup z_1) \cong co(M/z_1 \cup z_2)\). Assume that \(M/z_1 \cup z_2\) is not 3-connected. Then this matroid has a 2-separation \(\{P, Z\}\) where \(p \in P\). But \(M/z_1 \cup z_2/p \cong N\), a 3-connected matroid. This can only happen if \(P\) is a series pair of \(M/z_1 \cup z_2\). It follows that \(P \cup z_2\) is a triad of \(M/z_1\) and, indeed, of \(M\), contradicting the fact that \(M\) has no triads containing \(p\). This contradiction shows that \(co(M \cup z_1)\) is 3-connected. Of course, \(co(M \cup z_2)\) and \(co(M \cup z_3)\) are also 3-connected, and (f) holds.

Let \(M'\) denote the matroid obtained by performing a \(Y-A\) exchange on the triad \(\{z_1, z_2, z_3\}\). Recall the labelling of the ground set of a matroid
obtained by a $Y-A$ exchange from Section 2. It is easily checked that $M_1\cong M_2$, a 3-connected matroid with an N-minor. Moreover $M_1\cong co(M_2)$, a 3-connected matroid with an N-minor. This establishes (g).

There is one more case to cover.

(4.2.2) If $|E(M) - E(N)| = 4$ and $r(M) - r(N) = 2$, then $M$ has either a good deletion pair or a good contraction pair.

Proof. It is straightforward to verify that if $M$ is a wheel or a whirl, then $M$ has both a good deletion pair and a good contraction pair. Assume that $M$ is neither a wheel nor a whirl. Then, by the splitter Theorem, $M$ has an element $y$ such that either $M\setminus y$ or $M/y$ is 3-connected with an N-minor. We lose no generality in assuming that $M\setminus y$ is 3-connected with an N-minor. If $M\setminus y$ has a good contraction pair, then $M$ certainly has a good contraction pair. Assume that $M\setminus y$ does not have a good contraction pair. Then there is a set $\{p, x, a, b\} \subseteq E(M\setminus y)$ with the properties given by (4.2.1). Certainly $co(M\setminus y)$ is 3-connected with an N-minor. What about $co(M\setminus x)$? If $co(M\setminus x)$ is 3-connected, then $\{x, y\}$ is a good deletion pair. Assume that $co(M\setminus x)$ is not 3-connected. We now show that under this hypothesis $\{a, b, y\}$ is a triangle. We first show that $\{a, b\}$ is the only series pair of $M\setminus x, y$.

First note that by (4.2.1)(e), $p$ is not in a triad of $M\setminus y$. Hence $p$ is not in a series pair of $M\setminus x, y$. Assume that $M\setminus x, y$ has a series pair other than $\{a, b\}$. Say that $\{s_1, s_2\}$ is a series pair distinct from $\{a, b\}$. Without loss of generality we can assume that $a \notin \{s_1, s_2\}$. Then $\{s_1, s_2\}$ is a series pair of $M\setminus x, y/a, p$. (This is easily seen by thinking of the dual. If $\{p_1, p_2\}$ is a parallel pair of a matroid $M'$, and $T \subseteq E(M') - \{p_1, p_2\}$, then $\{p_1, p_2\}$ is a parallel pair of $M\setminus T$.) But $M\setminus x, y/a, p \cong N$ by (4.2.1)(c). Therefore $N$ has a series pair. Certainly $r(M\setminus y) > 3$, otherwise $si(M\setminus y/p)$ is 3-connected. Hence $r(N) \geq 2$. The only 3-connected matroid with rank at least two and a series pair is $U_{2,3}$. But, if $N = U_{2,3}$, then $|E(M\setminus y) - \{p, a, b, x\}| = 2$. One deduces from this that $M\setminus y$ has a series pair, contradicting the fact that $M\setminus y$ is 3-connected. Hence $\{a, b\}$ is indeed the only series pair of $M\setminus x, y$. It now follows that, since $co(M\setminus x)$ is not 3-connected, $y \in cl(\{a, b\})$. Certainly $y$ is not parallel to either $a$ or $b$. Hence $\{a, b, y\}$ is a triangle.

Now consider $M\setminus y, a$. Applying the above argument to this case we deduce that either $\{a, y\}$ is a good deletion pair, or $\{b, x, y\}$ is a triangle. Say both $\{a, b, y\}$ and $\{b, x, y\}$ are triangles. Then, the fact that $\{a, b, x\}$ has rank 3, and the submodular inequality imply that $\{b, y\}$ is a parallel pair. We deduce from this contradiction that $\{a, y\}$ is a good deletion pair.
We have covered all cases where $|E(M) - E(N)| \leq 4$ and Theorem 4.2 follows.

To illustrate the necessity of the case covered by Theorem 4.2(iii) we consider an example. Choose the Fano matroid $F_7$ with one of its triangles labelled $\{x, a, b\}$. Extend $F_7$ by placing a point $p$ freely on the line $\{x, a, b\}$. Let $M$ be the matroid obtained by performing a $A - Y$ exchange on the triangle $\{x, a, b\}$ of the extended matroid, and choose $N = U_{2,5}$. If $z \in E(M)$, then $\text{si}(M/z)$ is 3-connected with an $N$-minor if and only if $z \in \{x, a, b\}$. But, for any pair $\{z_1, z_2\}$ of elements of $\{x, a, b\}$, $M/z_1, z_2$ does not have an $N$-minor. Hence $M$ does not have a good contraction pair.

In one sense Theorem 4.2 is not satisfactory. Say that $\{x, y\}$ is a good deletion pair of the matroid $M$. While $\text{co}(M \setminus x, y)$ is 3-connected, if $\{x, y\}$ is contained in a triad, then $M \setminus x, y$ has a coloop, so that $M \setminus x, y$ is not connected. It is certainly worth knowing when we can avoid the problem. To risk bad terminology say that $\{x, y\}$ is a fine deletion pair of $M$ if it is a good deletion pair with the property that $M \setminus x, y$ is connected. A fine contraction pair is the dual of a fine deletion pair. Theorem 4.2 shows that if it suffices to get a good deletion or contraction pair, then we can almost always do it: we only have to resort to $A - Y$ or $Y - A$ exchanges in a few cases. The next theorem shows that if our priority is to get a fine deletion or contraction pair, then we can still usually do it, but we have a few more cases in which $A - Y$ or $Y - A$ exchanges have to be resorted to.

A matroid obtained from a 3-connected matroid by a $Y - A$ exchange is 3-connected unless some triangle uses two elements of the triad being exchanged, in which case it is 3-connected up to parallel pairs. There is no such triangle in the case described by Theorem 4.2(iii). But in one case that arises in Theorem 4.3 we may have such a triangle and a single 2-element parallel class can occur. In what follows we adopt the convention that to perform a $Y - A$ exchange we first exchange that triad for a triangle and then simplify the resulting matroid. Evidently such an exchange preserves 3-connectivity. A dual comment applies to $A - Y$ exchanges.

**Theorem 4.3.** Let $N$ be a 3-connected minor of the 3-connected matroid $M$ such that either $r(M) \geq r(N) + 2$ or $r(M^*) \geq r(N^*) + 2$. Then at least one of the following holds.

(i) There is a pair of distinct elements $x, y \in E(M)$ such that $M/x, y$ is connected, and $\text{si}(M/x), \text{si}(M/y)$ and $\text{si}(M/x, y)$ are all 3-connected with $N$-minors.

(ii) There is a pair of distinct elements $x, y \in E(M)$ such that $M \setminus x, y$ is connected, and $\text{co}(M \setminus x), \text{co}(M \setminus y)$ and $\text{co}(M \setminus x, y)$ are all 3-connected with $N$-minors.
(iii) \(|E(M) - E(N)| \in \{3, 4\}\) and it is possible to perform a single \(A - Y\) exchange to obtain a matroid satisfying (i).

(iv) \(|E(M) - E(N)| \in \{3, 4\}\) and it is possible to perform a single \(Y - A\) exchange to obtain a matroid satisfying (ii).

We first note a lemma. We omit the routine proof.

Lemma 4.4. Let \(C^*\) be an independent triad of the matroid \(M\) with corresponding hyperplane \(H\). If \(\text{si}(M|H)\) is 3-connected, then \(\text{si}(M)\) is 3-connected.

Proof of Theorem 4.3. Assume that \(M\) has neither a good deletion pair nor a good contraction pair. Then, by Theorem 4.2, \(|E(M) - E(N)| = 3\) and it is possible to perform a \(A - Y\) or a \(Y - A\) exchange to obtain a matroid with a good deletion or contraction pair. Moreover it is easily checked that the pair obtained in the proof of Theorem 4.2 is a fine deletion or contraction pair respectively. Thus, in this case, either (iii) or (iv) holds.

Assume that \(M\) has either a good deletion or contraction pair. By duality we lose no generality in assuming that \(M\) has a good deletion pair; say that \(\{e, f\}\) is such a pair. If this pair is not in a triad, then it is a fine deletion pair. Thus we may assume that \(\{e, f\}\) is in a triad. Let \(\{a, e, f\}\) be a such a triad. Let \(H\) denote the hyperplane \(E(M) - \{a, e, f\}\). Clearly \(\text{co}(M|H) \cong \text{co}(M'|e, f)\) so that \(M|H\) has an \(N\)-minor. In what follows we frequently claim that certain minors of \(M\) have an \(N\)-minor. Invariably the unstated justification for this claim is that the minor referred to has a submatroid isomorphic to \(M|H\). We now prove

(4.3.1) Neither \(\{a, e\}\), nor \(\{a, f\}\) is contained in a triangle. Moreover, either \(M/e\) or \(M/f\) is 3-connected.

Proof. Consider \(M\setminus f\). This matroid has \(\{a, e\}\) as a series pair. Thus, in obtaining \(\text{co}(M\setminus f)\), at least one of \(a\) and \(e\) is contracted. If \(\{a, e\}\) is contained in a triangle, then \(\text{co}(M\setminus f)\) would have either a parallel pair or a loop, and hence \(\text{co}(M\setminus e)\) is not 3-connected. But \(\{e, f\}\) is a good deletion pair, so \(\text{co}(M\setminus f)\) is 3-connected. From this contradiction we deduce that \(\{a, e\}\), and similarly \(\{a, f\}\), is not contained in a triangle. It now follows from (2.7) that either \(M/e\) or \(M/f\) is 3-connected. 

Assume that \(\{e, f\}\) is in a triangle. Note that \(M/a\) has an \(N\)-minor. Moreover, by Lemma 3.8, \(\text{si}(M/a)\) is 3-connected. By (4.3.1) \(a\) is not in a triangle, so \(M/a\) is 3-connected. On the other hand, if \(\{e, f\}\) is not in a triangle, then by (2.7), there is an element \(x\) in \(\{a, e, f\}\) such that \(M/x\) is 3-connected. Moreover, \(M/x\) has an \(N\)-minor. In either case there is an element \(x \in \{a, e, f\}\) such that \(M/x\) is 3-connected with an \(N\)-minor.
We consider case based on the rank of $M$, disposing of the straightforward cases first. Assume that $r(M) \geq r(N) + 3$. The, by Theorem 3.2, there is an element $y \in E(M)$ such that $\{x, y\}$ is a good contraction pair. But $M/x$ is 3-connected, so no triangle of $M$ contains $x$. Hence $\{x, y\}$ is a fine contraction pair. Given the hypotheses that we are under, the case $r(M) = r(N)$ does not occur.

Assume that $r(M) = r(N) + 1$. Then $M \mid H$ is an extension of $N$ since $M \mid H$ has an $N$-minor and $r(M \mid H) = r(N)$. Assume that $M \mid H = N$. Let $M'$ denote the matroid obtained by performing a $Y - A$ exchange on $\{a, e, f\}$. For all $A \subset \{a, e, f\}$, $M' \backslash A$ is an extension of the 3-connected matroid $N$, and hence $M' \backslash A$ is either 3-connected or 3-connected up to parallel pairs. In either case $M' \backslash A$ is connected and $\mathrm{si}(M' \backslash A)$ is 3-connected with an $N$-minor. Thus any 2-element subset of $\{a, e, f\}$ is a fine deletion pair of $M'$. Since $|E(M) - E(N)| = 3$, $M$ satisfies (iv). Now assume that $M \mid H \neq N$. In this case there is an element $y \in H$ such that $(M \mid H) \backslash y$ has an $N$-minor. Since $(M \mid H) \backslash y$ is a (possibly trivial) extension of a 3-connected matroid, $\mathrm{si}(M' \mid H \backslash y)$ is 3-connected. Since $M$ is 3-connected, $M \mid H$ has no parallel pairs. Thus $(M \mid H) \backslash y$ is 3-connected. By Lemma 4.4, $M' \mid y$ is 3-connected. Consider $M' \backslash y$, $e$. This has a series pair $\{a, f\}$. Also $M' \backslash y, e/f$ is a single-element extension of $(M \mid H) \backslash y$, a 3-connected matroid. Hence $M' \backslash y, e/f$ is 3-connected unless $a$ is in a non-trivial parallel class of this matroid. By (4.3.1), $\{a, f\}$ is not in a triangle, so that $a$ is not in a parallel class of $M' \backslash y, e/f$. Hence $M' \backslash y, e/f$ is 3-connected and it follows that $M' \backslash y, e/f \cong \mathrm{co}(M' \backslash y, e)$. Thus $\mathrm{co}(M' \backslash y, e)$ is 3-connected with an $N$-minor. Finally we note that $M' \backslash y, e$ is a series extension of the connected matroid $M' \backslash y, e/f$ so that $M' \backslash y, e$ is connected. Hence $\{y, e\}$ is a fine deletion pair.

Now assume that $r(M) = r(N) + 2$. Certainly there exists an element $z \in H$ such that $(M \mid H)/z$ has an $N$-minor. Since $r((M \mid H)/z) = r(N)$, $(M \mid H)/z$ is an extension of $N$. Therefore $\mathrm{si}(M' \mid H)/z$ is 3-connected. Assume that there exists an element $z$ with the above properties such that $z \notin \mathrm{cl}(\{a, e, f\})$. Consider $M/z$. The set $H - z$ is a hyperplane of this matroid whose corresponding cocircuit is $\{a, e, f\}$. Since $\mathrm{si}(M/z)(H - z) = (M \mid H)/z$, $\mathrm{si}(M/z)(H - z)$ is 3-connected. Since $z \notin \mathrm{cl}(\{a, e, f\})$, $\{a, e, f\}$ is an independent triad of $M/z$. It follows by Lemma 4.4 that $\mathrm{si}(M/z)$ is 3-connected. By (4.3.1) we can assume without loss of generality that $M/e$ is 3-connected. Moreover $M/e, e$ is an extension of $N$. Hence $\mathrm{si}(M/z, e)$ is 3-connected. Since $M/e$ is 3-connected, $M/e, z$ is connected. Therefore $\{e, z\}$ is a fine contraction pair.

Assume that $z' \in H$, and $(M \mid H)/z'$ has a $N$-minor, then $z' \in \mathrm{cl}(\{a, e, f\})$.

Now we show that, under this assumption,

(4.3.2) $M \mid H$ is 3-connected.
Proof. As before choose \( z \in H \) such that \((M \mid H)/z\) has an \( N \)-minor. By assumption \( z \in \text{cl}_M(\{a, e, f\}) \). Assume that \( M \mid H \) is not connected. Then it has a separation \( \{U, Z\} \), where \( z \in Z \). But \((M \mid H)/z\) is connected, so \( Z = \{z\} \). Moreover \( r(U) = r(H) - 1 = r(M) - 2 \). Also \( z \in \text{cl}(\{a, e, f\}) \), so \( r_M(\{a, e, f, z\}) = 3 \). Now,
\[
r_M(H - z) + r_M(\{a, e, f, z\}) = (r(M) - 2) + 3 = r(M) + 1.
\]

Hence \( \{H - z, \{a, e, f, z\}\} \) is a 2-separation of \( M \), contradicting the fact that \( M \) is 3-connected. Thus \( M \mid H \) is connected.

Assume that \( M \mid H \) is not 3-connected. Then it has a 2-separation \( \{U, Z\} \) where \( z \in Z \). If \( z \in \text{cl}_M(U) \), then \((M \mid H)/z\) is not connected, so \( z \notin \text{cl}_M(U) \). It is easily checked that if \( \{A, B\} \) is a 2-separation of a connected matroid without any parallel pairs, then so too is \( \{A - \text{cl}(B), \text{cl}(B)\} \). It follows that we lose no generality in assuming that \( U \) is a flat. Since \( \text{si}((M \mid H)/z) \) is 3-connected, \( r_M(Z) \leq 2 \). But \( M \mid H \) is simple, and, by the definition of 2-separation, \( |Z| \geq 2 \). Hence \( r_M(Z) = 2 \), and \( r_M(U) = r_M(H) - 1 \). So \( U \) is a hyperplane of \( M \mid H \), and \( Z \) is a rank-2 cocircuit of this matroid. Choose \( y \in Z - z \). Evidently \( \text{si}(M \mid H)/y) \cong \text{si}((M \mid H)/z) \). Hence \((M \mid H)/y\) has an \( N \)-minor. We have assumed that all elements with this property are in \( \text{cl}_M(\{g, e, f\}) \). Therefore \( Z \equiv \text{cl}_M(\{a, e, f\}) \). We now have,
\[
r_M(U) + r_M(\text{cl}_M(\{a, e, f\})) - U) = r(M) - 2 + 3 = r(M) + 1,
\]
contradicting the fact that \( M \) is 3-connected. We deduce that \( M \mid H \) is indeed 3-connected. \( \blacksquare \)

Assume that there is an element \( q \in H \) such that \((M \mid H)/q\) has an \( N \)-minor. There are two possibilities.

(4.3.3) If \((M \mid H)/q\) is not 3-connected, then \( M \) has a fine contraction pair.

Proof. Since \((M \mid H)/q\) is not 3-connected, it has a 2-separation \( \{U, Z\} \). There exists an element \( z \in \text{El}(H \mid H)/q) \) such that \((M \mid H)/q)/z\) is an extension of \( N \). Without loss of generality assume that \( z \in Z \). Then, since \( \text{si}((M \mid H)/q)/z) \) is 3-connected, we can deduce, just as in the proof of (4.3.2), that \( r_M(Z) \leq 2 \) and \( z \notin \text{cl}(U) \). Moreover we may assume without loss of generality that \( U \) is a flat. Again arguing as in (4.3.2), we deduce that \( \{U, Z \cup \{a, e, f\}\} \) is a 2-separation of \( M \mid q \). Moreover it is quickly checked that for any pair \( x, y \in \{a, e, f\} \), \( \text{si}(M/x, y) \cong \text{si}((M \mid H)/z) \), so \( \text{si}(M/x, y) \) is 3-connected with an \( N \)-minor. For all \( x \in \{a, e, f\} \), \( M/x \) is an extension of \( M \mid H \), a 3-connected matroid. Thus \( \text{si}(M/x) \) is 3-connected. By (4.3.1), we may assume without loss of generality that \( M/e \) is 3-connected. Therefore \( M/e, f \) is connected. It follows that \( \{e, f\} \) is a fine contraction pair. \( \blacksquare \)
On the other hand.

(4.3.4) If \((M \setminus H) \setminus q\) is 3-connected, then \(M\) has a fine deletion pair.

**Proof.** By Lemma 4.4, \(M \setminus q\) is 3-connected. We know that \(\text{co}(M \setminus \varnothing)\) is 3-connected. Now \(\{a, f\}\) is a series pair of \(M \setminus q, e\), and \(M \setminus q, e/f\) is an extension of \(M \setminus H\). Hence, either \(M \setminus q, e/f\) is 3-connected or it has a parallel pair. Since \(\text{co}(M \setminus \varnothing)\) is 3-connected, the latter does not happen. Thus \(M \setminus q, e/f\) is 3-connected. Moreover, it is now clear that this matroid is equal to \(\text{co}(M \setminus q, e)\). Also \(M \setminus q, e/f\) has an \(N\)-minor since it is an extension of \((M \setminus H) \setminus q\). Finally, since \(M \setminus q\) is 3-connected, \(M \setminus q, e\) is connected. We deduce that \(\{e, q\}\) is a fine deletion pair.

For the last case assume that there is no element \(q\) such that \((M \setminus H) \setminus q\) has an \(N\)-minor. Then \((M \setminus H)/z \cong N\), and \(|E(M) - E(N)| = 4\). Recall that we are under the convention that to perform a \(Y - A\) exchange we first exchange that triad for a triangle and then simplify the resulting matroid. Perform such an exchange on the triad \(\{a, e, f\}\). We obtain a matroid \(\text{si}(M')\) where \(M'\) is the intermediate matroid with possible parallel classes. By (4.3.1) neither \(\{a, e\}\) nor \(\{a, f\}\) are contained in triangles. Hence neither \(f\) nor \(e\) have an element parallel to them in \(M'\). Thus each of \(\text{si}(M')/e\), \(\text{si}(M')/f\) and \(\text{si}(M')/e, f\) are isomorphic to extensions of \(M \setminus H\). Since none of these matroids has loops or non-trivial parallel classes, each of them is 3-connected. We deduce that \(\{e, f\}\) is a fine deletion pair of \(\text{si}(M')\), and \(M\) satisfies (iv).

For an example to illustrate the necessity of Theorem 4.3(iv), take the example given after the proof of Theorem 4.2 and add a point parallel to \(x\) before doing the \(A - Y\) exchange. The dual of this matroid illustrates the necessity of Theorem 4.3(iii).

5. STABILIZERS

In this section we focus on matroids that are representable over a partial field \(F\). We always assume that representations are in *standard form*, that is, in the form \([I | A]\) where \(I\) is an identity matrix. Adopting a common practice we delete reference to the identity matrix and simply say that \(M\) is represented by \(A\), where the \(i\)th row of \(A\) represents the element represented by the \(i\)th column of \(I\). Two representations of \(M\) are equivalent if one can be obtained from the other by the following operations: performing a pivot on a non-zero element of \(A\); permuting rows or columns (along with their labels); multiplying a row or a column by a non-zero scalar; and applying an automorphism of \(F\) to the entries of \(A\). Note
that when one pivots, columns and labels are interchanged at the end so that the result is still in the form $[I | A']$ and we are focusing on just $A'$.

As we have noted before, frequently rank-2 matroids are treated as an exceptional case, and all representations of such a matroid are regarded as being equivalent. We remind the reader again that this exception is not in place here and representations of a rank-2 matroid may well be inequivalent.

We now describe a situation that crops up frequently in matroid representation theory. Say that $M$ is an $F$-representable matroid, that $x, y \in E(M)$, and that $A$ is a representation of $M \setminus x, y$ that extends to a representation of $M$. Consider extensions of $A$ to representations of $M \setminus x$ and $M \setminus y$. For inductive arguments one wants knowledge of these representations to give us information about representations of $M$. There are two natural questions to ask.

(a) When are we assured that all extensions of $A$ to representations of $M$ are equivalent?

(b) When are we assured that if $[A | x]$ and $[A | y]$ are representations of $M \setminus y$ and $M \setminus x$ respectively, then $[A | x, y]$ represents $M$?

To be assured of a positive answer to either (a) or (b), it clearly helps if all extensions of $A$ to representations of $M \setminus x$ are equivalent and that the same holds for extensions to representations of $M \setminus y$. But even this does not give a guarantee as the following examples show.

Choose $F = GF(5)$ and $M = U_{2,4}$, so that $M \setminus x, y = U_{2,2}$. Then the $2 \times 2$ identity matrix $I_2$ represents $M \setminus x, y$. All extensions of $M \setminus x, y$ to representations of $M \setminus y$ are equivalent and, of course, the same holds for extensions to representations of $M \setminus x$. But even this does not give a guarantee as the following examples show.

Choose $F = GF(4)$ with elements $\{0, 1, \omega, \omega^2\}$. Let $M = U_{2,5}$. By elementary field theory, $\omega$ and $\omega^2$ are automorphically equivalent. Consider $M \setminus x, y = U_{2,2}$. Represent $M \setminus x, y$ over $GF(4)$ by the matrix.

$$[I | A] = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}.$$

Evidently all extensions of $A$ to representations of $M \setminus x$ or $M \setminus y$ are equivalent. Choose $x = [\omega]$, $y = [\omega^2]$. Then $[A | x]$ and $[A | y]$ represent $M \setminus x$ and $M \setminus y$ respectively, but $[A | x, y]$ does not represent $M$, so that (b) does not hold. Consider (a). Let $M$ be $I_2,4 \oplus U_{2,4}$, and $x$ and $y$ be
elements such that \( M \setminus x, y = U_{2, 3} \oplus U_{2, 3} \). Then one routinely checks that a quaternary representation of \( M \setminus x, y \) extends uniquely to quaternary representations of \( M \setminus x \) and \( M \setminus y \) respectively, but that there exist inequivalent representations of \( M \) that agree on a representation of \( M \setminus x, y \). Thus in this case (a) does not hold.

The last example indicates that the role played by automorphisms in matroid equivalence is somewhat different from that played by matrix operations. This motivates the following definition: two representations of a matroid are \textit{strongly equivalent} if one can be obtained from the other by the standard matrix operations described above without applying an automorphism. With the notion of strong equivalence we can define “stabilizer.” At this stage we need to be clear about the distinction between isomorphism and equality. To say that a matroid has an \( N \)-minor always means, of course, that \( M \) has a minor isomorphic to \( N \). When we refer to a \textit{fixed} \( N \)-minor we mean a fixed minor that is isomorphic to \( N \). The matroid \( N \) is an \( F \)-stabilizer for \( M \) (or \( N \) stabilizes \( M \) over \( F \)) if, whenever \( N' \) is a fixed \( N \)-minor of \( M \), any two representations of \( M \) that induce strongly-equivalent representations of \( N' \) are strongly equivalent. In other words, \( N \) stabilizes \( M \) if all extensions of a representation of a fixed \( N \)-minor to a representation of \( M \) are strongly equivalent. If no danger of ambiguity exists we at times omit reference to the field.

We wish to show that the task of checking that a matroid stabilizes a class is reducible to a finite case check. We begin by developing some elementary properties. For a matrix \( A \), let \( B[A] \) denote the bipartite graph whose vertices are the index sets of the rows and columns of \( A \) respectively. An edge joins vertices \( i \) and \( j \) if and only if the \((i, j)\)th entry of \( A \) is not equal to zero. Brylawski and Lucas [4] have shown that if \( A \) represents \( M \) over a field \( F \), then \( M \) is connected if and only if \( B[A] \) is connected. This fact generalizes immediately to partial fields. For fields, the next lemma is an immediate consequence of results of [4]. The generalization to partial fields is entirely routine and we omit the proof.

\textbf{Lemma 5.1.} Let \( A \) and \( A' \) be matrices representing the same matroid \( M \) over a partial field \( F \) such that rows and columns of \( A \) represent the same elements of \( M \) as corresponding rows and columns of \( A' \). Then \( B[A] = B[A'] \). Assume that \( F \) is a spanning forest of \( B[A] \), having the property that, for each edge \( f \in F \), the entries in \( A \) and \( A' \) corresponding to \( f \) are equal. Then \( A \) and \( A' \) are strongly equivalent representations of \( M \) if and only if \( A = A' \).

\textbf{Lemma 5.2.} Let \( x \) be an element of the connected \( F \)-representable matroid \( M \). Assume that \( M \setminus x \) is connected and that \( M \setminus x \) stabilizes \( M \). If \( A \)
represents \( M \setminus x \) over \( F \), and \( [A \mid x] \) and \( [A \mid x'] \) both represent \( M \) over \( F \), then \( x \) is a scalar multiple of \( x' \).

**Proof.** Certainly a coordinate of \( x \) is nonzero if and only if the corresponding coordinate of \( x' \) is nonzero, so that \( B[A \mid x] = B[A \mid x'] \). If \( x \) is a loop, then the result is immediate. Assume that \( x \) and \( x' \) are scaled so that their leading nonzero entries are equal to one. Since \( M \setminus x \) is connected, \( B[A] \) is connected, so it has a spanning tree. Adding the edge corresponding to the leading nonzero entry of \( x \) produces a spanning tree of \( B[A \mid x] = B[A \mid x'] \). For an edge \( f \) of this tree, the entries of \( [A \mid x] \) and \( [A \mid x'] \) corresponding to \( f \) are clearly equal. Since \( M \setminus x \) stabilizes \( M \), \( [A \mid x] \) and \( [A \mid x'] \) are strongly equivalent. It now follows from Lemma 5.1 that \( x = x' \). Removing the assumption that \( x \) and \( x' \) have been scaled leads immediately to the conclusion that they are scalar multiples.

**Lemma 5.3.** Let \( x \) and \( y \) be distinct elements of the \( F \)-representable matroid \( M \). If \( M \setminus x, y \) is connected and stabilizes both \( M \setminus x \) and \( M \setminus y \), then \( M \setminus x, y \) stabilizes \( M \). Moreover, if \( A \) represents \( M \setminus x \) and \( [A \mid x] \) and \( [A \mid y] \) represent \( M \setminus y \) and \( M \setminus x \) respectively, then \( [A \mid x, y] \) represents \( M \).

**Proof.** Let \( [A \mid x', y'] \) be a matrix that represents \( M \), and \( [A \mid x, y] \) be a matrix that has the property that \( [A \mid x] \) and \( [A \mid y] \) represent \( M \setminus y \) and \( M \setminus x \) respectively. (Note that any extension of \( A \) to a potentially inequivalent representation of \( M \) has the properties of \( [A \mid x, y] \).) Since \( M \setminus x, y \) stabilizes \( M \setminus y \) and \( M \setminus x \) respectively, it follows from Lemma 5.2 that \( x' \) is a scalar multiple of \( x \) and \( y' \) is a scalar multiple of \( y \). We deduce immediately that \( [A \mid x, y] \) and \( [A \mid x', y'] \) are strongly equivalent and also that \( [A \mid x, y] \) represents \( M \).

Versions of cases of Lemma 5.3 appear either explicitly or implicitly in a number of papers, see for example [6, 15].

One point to note about stabilizers is the following. If \( N \) stabilizes \( M \), then it follows from the definition of stabilizer that all extensions of a representation of a fixed \( N \)-minor of \( M \) to a representation of \( M \) are strongly equivalent. Of course there is no guarantee that a given representation of \( N \) does extend to a representation of \( M \). For example it is easily seen that \( U_{2,4} \) stabilizes the non-Fano matroid \( F^* \) over the real numbers, but of the infinite number of inequivalent representations of a fixed \( U_{2,4} \)-minor of \( F^* \), only one extends to a representation of \( F^* \).

We say that a class of matroids is well closed if it is minor closed and closed under duals and isomorphism. Let \( \mathcal{M} \) be a well-closed class of matroids representable over some partial field \( F \). A matroid \( N \in \mathcal{M} \) stabilizes \( \mathcal{M} \) over \( F \) if \( N \) stabilizes every 3-connected matroid in \( \mathcal{M} \) with an \( N \)-minor.
The point of the definition is that one can use stabilizers to bound the number of inequivalent $F$-representations for matroids in certain classes. Recall that a matroid $M$ has $k$ inequivalent representations over $F$ if there are $k$ equivalence classes of $F$-representations of $M$ using the usual notion of equivalence of representations, and that $M$ is uniquely representable if all $F$-representations of $M$ are equivalent, again using the usual notion of equivalence. The following proposition is an immediate consequence of the above definition.

**Proposition 5.4.** Let $N$ be an $F$-stabilizer for the class $\mathcal{M}$ of $F$-representable matroids. If $N$ has $k$ inequivalent $F$-representations, then any 3-connected matroid in $\mathcal{M}$ has at most $k$ inequivalent $F$-representations.

Specializing Proposition 5.4 we get

**Corollary 5.5.** Let $N$ be an $F$-stabilizer for the class $\mathcal{M}$ of $F$-representable matroids. If $N$ is uniquely representable over $F$, then every 3-connected matroid in $\mathcal{M}$ with an $N$-minor is uniquely representable over $F$.

Our main goal is to show that the task of checking that a matroid is an $F$-stabilizer for a class is finite. The next proposition notes some basic properties of stabilizers. We omit the elementary proof.

**Proposition 5.6.** Let $F$ be a partial field, $N$ be a 3-connected $F$-representable matroid, and $\mathcal{M}$ be a well-closed class of $F$-representable matroids containing $N$.

(i) If $M$ is $F$-representable with an $N$-minor, then $N$ stabilizes $M$ if and only if $N^*$ stabilizes $M^*$.

(ii) $N$ stabilizes $\mathcal{M}$ over $F$ if and only if $N^*$ stabilizes $\mathcal{M}$ over $F$.

(iii) Let $N_1$ and $N_2$ be $F$-representable matroids such that $N$ is a minor of $N_1$ and $N_1$ is a minor of $N_2$. Assume that $N$ stabilizes $N_1$. Then $N$ stabilizes $N_2$ if and only if $N_1$ stabilizes $N_2$.

(iv) If $N$ is an $F$-stabilizer for $\mathcal{M}$, and $N'$ is a 3-connected matroid in $\mathcal{M}$ with an $N$-minor, then $N'$ is an $F$-stabilizer for $\mathcal{M}$.

We also need to note some elementary facts about $A - Y$ and $Y - A$ exchanges.

**Lemma 5.7.** Let $M'$ be a matroid obtained from the matroid $M$ by a single $A - Y$ exchange, and let $F$ be a partial field.

(i) $M'$ is $F$-representable if and only if $M$ is.
If $M$ is $F$-representable, then strong-equivalence classes of representations of $M$ are in one-to-one correspondence with strong-equivalence classes of $M'$.

(iii) If $M$ and $M'$ both have $N$-minors, then $N$ is an $F$-stabilizer for $M$ over $F$ if and only if $N$ is an $F$-stabilizer for $M'$ over $F$.

**Proof.** For fields (i) is proved in [1]. The generalization to partial fields is straightforward. We briefly outline the technique since it leads to the proof of (ii). Recall the definition of $A-Y$ exchange. There exists a triangle $\{a, b, c\}$ of $M$, such that $M'$ is obtained by taking a generalized parallel connection of $M$ with $M(K_4)$ across the triangle $\{a, b, c\}$ and then deleting the triangle. It is straightforward to check that such a parallel connection preserves representability over any partial field. Moreover, it is shown in [12] that binary matroids are uniquely representable over any partial field. Via this unique representation we obtain a canonical bijection between $F$-representations of $M$ and $M'$. Thus (ii) holds. Part (iii) is an immediate consequence of (ii). □

Finally we can prove

**Theorem 5.8.** Let $\mathcal{M}$ be a well-closed class of matroids representable over the partial field $F$ and let $N$ be a 3-connected matroid in $\mathcal{M}$. Then $N$ stabilizes $\mathcal{M}$ if and only if $N$ stabilizes each 3-connected matroid $M$ in $\mathcal{M}$ of the following type:

(i) $M$ has an element $x$ such that $M\setminus x = N$.

(ii) $M$ has an element $y$ such that $M/y = N$.

(iii) $M$ has elements $x$ and $y$ such that $M\setminus xy = N$, and both $M\setminus x$ and $M\setminus y$ are 3-connected.

**Proof.** The theorem is non-trivial in one direction only. Consider this direction. Let $M$ be a 3-connected matroid in $\mathcal{M}$. Assume that $M$ has an $N$-minor. If $|E(M) - E(N)| = 1$, then it follows by (i) or (ii) that $N$ stabilizes $M$. Assume that $|E(M) - E(N)| = 2$. If $r(M) = r(N)$, then it follows by Lemma 5.3 that $N$ stabilizes $M$. If $r(M^*) = r(N^*)$, then it follows by Proposition 5.6 and Lemma 5.3 that $N$ stabilizes $M$. Otherwise we have a pair $\{x, y\}$ such that $M\setminus xy \geq N$. If both $M\setminus x$ and $M\setminus y$ are 3-connected, then $N$ stabilizes $M$ by (iii) above. Assume that one of these matroids is not 3-connected. Taking duals if necessary and using Proposition 5.6 we may assume without loss of generality that $M\setminus x$ is not 3-connected. Since $M\setminus x$ is connected, but not 3-connected, and $M\setminus y$ is 3-connected, $y$ is in a series pair of $M\setminus x$. Hence $M$ has a triad containing $x$ and $y$. Assume that $\{x, y, a\}$ is such a triad. Let $M'$ be that matroid obtained by performing a $Y-A$ exchange on this triad. It is easily checked that $M'\setminus y, a \cong N$. 


Moreover, \( r(M') = r(N) \). Thus, if \( M' \) is 3-connected, then \( N \) stabilizes \( M' \) by Lemma 5.3. If \( M' \) is not 3-connected, then either \( y \) or \( a \) is in a parallel class of \( M' \). Evidently a matroid stabilizes any matroid obtained by adding a parallel element. Using this fact and (i), we deduce that in this case too, \( N \) stabilizes \( M' \). It now follows from Lemma 5.7 that \( N \) stabilizes \( M \).

This establishes the base case for the induction argument. Assume then, that \( |E(M) - E(N)| > 2 \) and that the result holds for all matroids satisfying the hypotheses of the theorem with ground sets having cardinality less than \( |E(M)| \).

By Theorem 4.3, \( M \) has either a fine deletion pair, a fine contraction pair, or it is possible to perform a \( A - Y \) or a \( Y - A \) exchange to obtain a matroid with a fine deletion or contraction pair. Thus, by dualising and performing a \( A - Y \) or a \( Y - A \) exchange if necessary, we can assume without loss of generality that \( M \) has a fine deletion pair \( \{x, y\} \). If \( M' \backslash x \), \( y \) is 3-connected, then, by the induction assumption \( N \) stabilizes \( M' \backslash x \), \( y \), \( M' \backslash x \) and \( M' \backslash y \). Assume that \([A \mid x, y]\) and \([A' | \ x', y']\) are representations of \( M \) that agree on a representation of a common \( N \)-minor of \( M' \backslash x \), \( y \). Then \( A \) and \( A' \) are strongly equivalent and hence \([A'' | \ x'', y'']\) is strongly equivalent to a matrix of the form \([A | x', y']\). But \( N \) stabilizes \( M' \backslash y \), so \([A' | x]\) is strongly equivalent to \([A | x']\). Hence \( x \) is a scalar multiple of \( x' \). Similarly \( y \) is a scalar multiple of \( y' \). Therefore \([A | x', y']\) is strongly equivalent to \([A' | x, y]\), so that \([A'' | \ x'', y'']\) is strongly equivalent to \([A | x, y]\). We deduce that \( N \) stabilizes \( M \).

Assume that \( M' \backslash x \), \( y \) is not 3-connected. Since \( \{x, y\} \) is a fine deletion pair, \( M' \backslash x \), \( y \) is a connected matroid that cosimplifies to a 3-connected matroid with an \( N \)-minor. Again assume that \([A | x, y]\) and \([A' | \ x', y']\) are representations of \( M \) that agree on a common \( N \)-minor of \( M' \backslash x \), \( y \). We can assume that \( N \) is a minor of \( \text{co}(M' \backslash x, y) \). An elementary argument shows that representations of \( M' \backslash x \), \( y \) are strongly equivalent if and only if representations of \( \text{co}(M' \backslash x, y) \) are strongly equivalent. Hence, as before, \([A'' | \ x'', y'']\) is strongly equivalent to a matrix of the form \([A | x', y']\). To prove the theorem it suffices to show that \( x' \) and \( y' \) are scalar multiples of \( x \) and \( y \) respectively. We complete the proof by showing that \( x' \) is a multiple of \( x \).

There is a set \( S \subseteq E(M' \backslash x, y) \) such that \( \text{co}(M' \backslash x, y) = M' \backslash x, y/S \). We have shown above that \( x' \) is a multiple of \( x \) if \( |S| = 0 \). Say that \( |S| = k > 0 \) and make the obvious induction assumption. Note that we are currently under two inductive hypotheses. Say that \( s \in S \). Then \( s \) is in a series pair \( \{s, t\} \) of \( M' \backslash x, y \). We may assume without loss of generality that \( s \) and \( t \) represent the first two rows of \( A \) respectively. Let \( A_s \) and \( A_t \) denote the matrices obtained by deleting the first and second rows of \( A \) respectively. Then \( A_s \) and \( A_t \) represent \( M' \backslash x, y/s \) and \( M' \backslash x, y/t \) respectively. Using both inductive hypotheses we see that the vector that extends \( A_s \) to a representa-
tion of $M \setminus y/s$ is unique up to scalar multiples. Hence $[x_2, x_3, ..., x_r]' = k[x_2', ..., x_r']'$ for some $k \in F$. What about $A_t$? A minor niggle has to be removed here. It might be the $N$-minor that we have chosen uses $t$. But it is easily seen that in this case we can choose another $N$-minor to guarantee that $[x_1, x_3, ..., x_r]' = k'[x_1', x_3', ..., x_r']'$. If $x$ is a loop of $M \setminus y/s, t$, then $\{x, s, t\}$ is a triangle of $M$ and $\{y, s, t\}$ is a triad of $M$. But then $x$ is in a non-trivial parallel class of $\text{co}(M \setminus y)$ contradicting the fact that $\text{co}(M \setminus y)$ is 3-connected. It follows that $x$ is not a loop of $M \setminus y/s, t$. Hence $[A^r | x^r, y^r]$ and $[A | x, y]$ are strongly equivalent.

6. REPROOFS

In this section we consider some applications of stabilizers. Most of the applications are reproofs of known results. On purely aesthetic criteria the original proofs of these theorems are probably more attractive—they all use arguments based on geometric insight, while arguments using stabilizers amount to routine case checking. The point is to show that stabilizers give a general technique in matroid representation theory. If $N$ is a minor of $M$ we will say that $M$ is a major of $N$. The matroid $P_6$ is obtained by performing a single $A-Y$ exchange on $U_{2,6}$.

**Lemma 6.1.** (i) For any partial field $F$, $U_{2,4}$ is an $F$-stabilizer for the class of $F$-representable matroids with no $U_{2,5}$- or $U_{3,5}$-minor.

(ii) $U_{2,4}$ is a $\text{GF}(4)$-stabilizer for the class of $\text{GF}(4)$-representable matroids.

(iii) For any partial field $F$, $U_{2,5}$ is an $F$-stabilizer for the class of $F$-representable matroids with no $U_{2,6}$, $P_6^*$, $U_{4,6}$- or $U_{3,6}$-minor.

(iv) $U_{2,5}$ is a $\text{GF}(5)$-stabilizer for the class of $\text{GF}(5)$-representable matroids.

**Proof.** By Theorem 5.8, to check that a matroid $N$ is a stabilizer for $\mathcal{M}$, we need only check 3-connected matroids in $\mathcal{M}$ that are either (a) single-element extensions of $N$, (b) single-element coextensions of $N$, or (c) majors $M$ of $N$ with a pair of elements $\{x, y\}$ such that $M \setminus x/y = N$ and such that $M \setminus x$ and $M \setminus y$ are both 3-connected. In an obvious way we will refer to a matroid $M \in \mathcal{M}$ as being a major of type (a), (b), or (c).

Consider part (i). It is easily checked that $U_{2,4}$ has no majors in the class of type (a) or (b) and hence none of type (c), so that, remarkably enough, (i) holds by a vacuous case check.
For (ii), $U_{2,5}$ is the only type-(a) major of $U_{2,4}$. It is clear that $U_{2,4}$ stabilizes $U_{2,5}$ over $GF(4)$. (This is not true for other fields.) The only type-(b) major of $U_{2,4}$ is $U_{3,5}$. But $U_{3,5} = U_{2,5}^*$ so $U_{2,4}$ stabilizes $U_{3,5}$. Type-(c) majors are single-element extensions of $U_{3,5}$. It is easily checked that the only 3-connected quaternary extensions of $U_{3,5}$ are $U_{3,6}$ and the matroid $Q_6$, obtained by placing a point on the intersection of two lines of $U_{3,5}$. Clearly $U_{3,5}$ stabilizes $Q_6$ over $GF(4)$ (in fact over any field). An easy check shows that $U_{3,5}$ stabilizes $U_{3,6}$ over $GF(4)$. By Proposition 5.6(iii), the relation of stabilization is transitive so that $U_{2,4}$ stabilizes all type-(c) extensions over $GF(4)$. Thus (ii) holds.

Consider (iii). Since $U_{2,6}$ is excluded, $U_{2,5}$ has no type-(a) majors in the class, and hence no type-(c) majors. Up to duality, a type-(b) major is a type-(a) extension of $U_{3,5}^*$, that is, of $U_{3,5}$. We have already noted that the only 3-connected quaternary extensions of $U_{3,5}$ are $Q_6$ and $U_{3,6}$. It is easy to further check that the only other 3-connected extensions of $U_{3,5}$ is $P_6$. Thus $Q_6$ is the only type (b) extension of $U_{3,5}$ that is in the class. But as noted above $U_{3,5}$ stabilizes $Q_6$ over any field.

The argument for (iv) is somewhat more complicated. In essence it is covered in [11] and we omit it.

In the next theorem, (i) is proved in [4], (ii) is an unstated immediate consequence of results in [13], (iii) is proved in [21], (iv) is a theorem of Kahn [8], (v) is proved in [11], while (vi) is new.

**Theorem 6.2.** (i) Ternary matroids are uniquely representable over $GF(3)$.

(ii) If $q$ is a prime other than 2 or 3, then a 3-connected matroid with no $U_{2,5}$- or $U_{3,5}$-minor has at most $q - 2$ inequivalent representations over $GF(q)$.

(iii) If $q$ is a prime power other than 2 or 3, then a 3-connected ternary matroid has at most $q - 2$ inequivalent representations over $GF(q)$.

(iv) 3-connected quaternary matroids are uniquely representable over $GF(4)$.

(v) 3-connected $GF(5)$-representable matroids have at most six inequivalent representations over $GF(5)$.

(vi) If $q$ is a prime power greater than 3, then a 3-connected quaternary matroid with no $U_{3,6}$-minor has at most $(q - 2)(q - 3)$ inequivalent representations over $GF(q)$.

**Proof.** We frequently use the fact that binary matroids are uniquely representable over any field. This is shown in [4]. It is also a consequence of the fact that $U_{2,5}$ stabilizes the class of binary matroids representable over any field.
Consider (i). Let $M$ be a ternary matroid. We may assume that $M$ is non-binary, so that $M$ has a $U_{2,4}$-minor. Certainly $U_{2,4}$ is uniquely representable over $GF(3)$. If $M$ is 3-connected, then by Corollary 5.5 and Lemma 6.1(i), $M$ is uniquely representable over $GF(3)$. If $M$ is not 3-connected, then the result follows from the facts that 3-connected matroids are uniquely representable and that $GF(3)$ has no non-trivial automorphisms.

Evidently, $U_{2,4}$ has at most $q - 2$ inequivalent representations over $GF(q)$. Note that if $q$ is prime, $U_{2,4}$ has exactly $q - 2$ inequivalent representations, otherwise it has fewer. Part (ii) now follows by Proposition 5.4 and Lemma 6.1(i). Part (iii) follows immediately from part (ii). Since $U_{2,4}$ is uniquely representable over $GF(4)$, part (iv) follows from Corollary 5.5 and Lemma 6.1(ii).

Let $M$ be a 3-connected $GF(5)$-representable matroid. If $M$ has no $U_{2,5}$- or $U_{3,5}$-minor, then $M$ has at most three inequivalent $GF(5)$-representations by (ii). Assume that $M$ has a $U_{2,5}$-minor. It is easily checked that $U_{2,5}$ has six inequivalent $GF(5)$-representations. Thus by Proposition 5.4 and Lemma 6.1(iv), $M$ has at most six inequivalent $GF(5)$-representations. Thus (v) holds.

Consider part (vi). It is well known that no member of $\{U_{2,6}, U_{4,6}, P_6\}$ is quaternary. Hence, by Lemma 6.1(iii), $U_{2,5}$ is a $GF(q)$-stabilizer for the class of $GF(q)$-representable quaternary matroids with no $U_{3,6}$-minor. Part (iv) now follows from the easily checked fact that $U_{2,5}$ has at most $(q - 2)(q - 3)$ inequivalent representations over $GF(q)$.

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