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Orthomodular Lattices Admitting No States

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ABSTRACT

The purpose of this paper is to construct a class of orthomodular lattices which admit no bounded measures.

1. INTRODUCTION

In this paper we exhibit infinitely many non-isomorphic orthomodular lattices L which admit no bounded measures. Since every Boolean lattice, in particular every sigma field of subsets of a set, is an orthomodular lattice, classical measure theory may be viewed as part of the theory of measures on orthomodular lattices. Interest in this generalization of measure theory has been stimulated by the proposed use of orthomodular lattices for the logic of quantum mechanics [6, 7, 12] and, more generally, for the logic of any empirical system [5, 11].

Because of this application to physical theory we present our results in terms of probability measures which we term *states* [1], i.e., mappings $\alpha : L \to [0, 1]$ such that $1\alpha = 1$ and $(x \vee y)\,\alpha = x\alpha + y\alpha$ whenever $x \leqslant y'$. We require only finite additivity for two reasons: first, the existence of a physically meaningful countably additive state is debatable, and, second, the non-existence of finitely additive states on L implies the non-existence of countably additive states on L.

Recall that an *orthocomplemented poset* L is a partially ordered set L having a least element 0 and a greatest element 1 together with an orthocomplementation $' : L \to L$ such that for all $a, b \in L$ (i) $(a')' = a$, (ii) $a \leqslant b$ implies $b' \leqslant a'$, and (iii) $a \vee a'$ exists and equals 1. If, moreover, L satisfies (iv) $a \leqslant b$ implies $b' \vee a$ exists and (v) $a \leqslant b$ implies $b = a \vee (b' \vee a)'$, then L is called an *orthomodular poset*. We call a maximal Boolean sub-orthomodular lattice of L a *block* [4] of L.

In constructing the stateless lattices we develop a new notation and present orthomodular lattices as unions of Boolean algebras (blocks) intertwined or pasted together in some fashion. The degree of intertwining determines whether the structure obtained is orthomodular and, in case it is, whether it is a lattice or simply a poset.

2. Building Orthomodular Lattices from Boolean Lattices

The depth of our analysis of (even finite) orthomodular lattices has long since exceeded that at which the usual Hasse diagrams yield insight. It has occurred to several researchers (e.g., [10]) to represent certain orthomodular lattices by "orthogonality spaces"; essentially, the relation $\perp (x \perp y$ if $x \leqslant y')$ is restricted to the set of atoms A (or any join dense subset) and the graph $(A, \perp)$ is depicted. Such diagrams simplify matters by replacing, for example, the 2^n elements in the Hasse diagram of the power set of an n-element set with the complete graph on n elements. The reduction in numbers of elements is considerable but the number of remaining "links" or "lines" is still too cumbersome for our purposes. We replace the complete graph on n elements by a single smooth curve (usually a straight line) containing n distinguished points. Thus we replace $n(n + 1)/2$ "links" with a single smooth curve. This representation is propitious and uncomplicated provided that the intersection of any pair of blocks contains at most one atom.

Thus the Boolean lattice 2^3 is represented by Figure 1, in which where each distinguished point (small circle) denotes an atom of 2^3. If the Hasse diagram for 2^3 is drawn as in Figure 2, then Figure 1 more readily generates Figure 2 in the mind of the viewer than if 2^3 were represented by the more usual "cube" as in Figure 3. The gestalt generated by Figure 2 in which the orthocomplement of an atom appears directly above the atom is quite valuable in visualizing the more complicated lattices which follow.

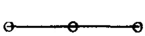

Figure 1

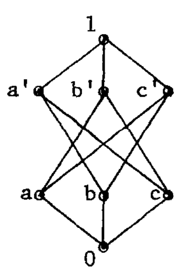

Figure 2

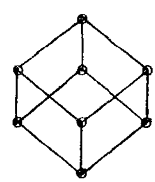

Figure 3

The 16-element lattice of Dilworth [3], D_{16}, is illustrated in the Hasse diagram given in Figure 4; this lattice is represented in our notation in Figure 5.

It is suggested that the reader train himself in the art of computing suprema and infima by reference to Figure 5 with the aid of Figure 4.

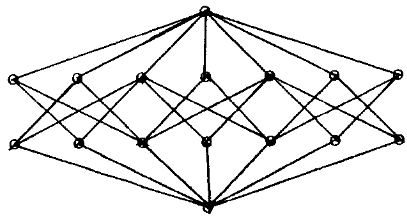

FIGURE 4

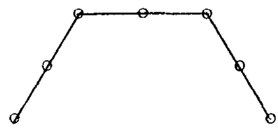

FIGURE 5

The lattice G_{32} is given by Figure 6. It provides the motivation for Theorems 2 and 3 in that (i) the intersection of any two blocks has cardinality 2 or 4 (the "lines" representing the blocks meet on at most one distinguished point) and (ii) there are no "loops" (the definition is forthcoming) consisting of fewer than five blocks.

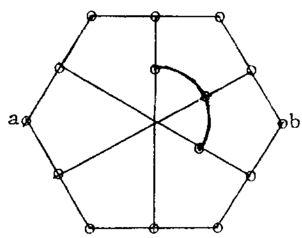

FIGURE 6

CONVENTION 1. Let $L = \bigcup \{B_\alpha : \alpha \in I\}$ be such that

(1) B_α, $\leqslant_\alpha$, $'\alpha$) is a Boolean lattice for all α in I;

(2) if $x \in B_\alpha \cap B_\beta$, $\alpha, \beta \in I$, then $x'^\beta = x'^\alpha$;

(3) if $\alpha \neq \beta$, then $B_\alpha \cap B_\beta = \{0, 1\}$ or $\{0, 1, a, a'\}$, where a is an atom of both B_α and B_β, $a' = a'^\alpha = a'^\beta$; and

(4) $B_\alpha \neq 2^1$, $B_\alpha \neq 2^2$ for all α in I.

The set $\{B_\alpha : \alpha \in I\}$ is called the set of *initial blocks* of L. For $x, y \in B_\alpha$ we write $x \vee_{B_\alpha} y$ for the supremum of x and y as computed in B_α.

DEFINITIONS. If $x, y \in L$, we define $x \leqslant y$ to mean that there exists an initial block B_α such that $\{x, y\} \subset B_\alpha$ and $x \leqslant_\alpha y$.

If $x \in L$, we define x' to be x'^α whenever $x \in B_\alpha$ where B_α is an initial block of L.

For $M, N \subset L$, we define $U_M(N) = \{m \in M : n \leqslant m \text{ for all } n \in N\}$; if $M = L$ we write $U(N)$ for $U_L(N)$; if $N = \{x\}$ we write $U(x)$ for $U(\{x\})$.

LEMMA. *If L satisfies the conditions of Convention* 1, *then $(L, \leqslant, ')$ is an orthocomplemented poset.*

Proof. The proof is straightforward and is therefore omitted.

DEFINITION. Let $n \in \mathbb{Z}$, $n \geqslant 3$. We call the set $\{B_0, B_1, \ldots, B_{n-1}\}$ of initial blocks of L an *atomistic loop of order n* in case for $0 \leqslant j < i \leqslant n-1$ we have

$$B_i \cap B_j = \begin{cases} \{0, 1, a, a'\}, & \text{if } \ i - j \in \{1, n-1\}, \\ \{0, 1\}, & \text{otherwise,} \end{cases}$$

and for $0 \leqslant k < j < i \leqslant n - 1$ we have $B_i \cap B_j \cap B_k = \{0, 1\}$.

THEOREM 2. *Let L satisfy the conditions of Convention* 1. *Then L is an orthomodular poset if and only if the order of every atomistic loop in L is at least* 4.

Proof. Let L be an orthomodular poset. Assume that there is an atomistic loop $\{B_0, B_1, B_2\}$ of order 3 in L. Let a_i be the unique atom in $B_i \cap B_{i+1}$ $(i = 0, 1, 2 (\text{mod } 3))$. The $U(\{a_0, a_1\}) \supset \{a_0 \vee_{B_1} a_1, a_2', 1\}$ but $a_0 \vee_{B_1} a_1$ and a_2' are incomparable, so that $a_0 \vee a_1$ does not exist in L. Since $a_0 \perp a_1$, L is not an orthomodular poset. Contradiction.

Conversely assume that the order of every atomistic loop in L is at least 4. We must show:

(i) If $x, y \in L$ with $x \perp y$, then $x \vee y$ exists in L.

(ii) If $x, y \in L$ with $x \leqslant y$, then $y = x \vee (y' \vee x)'$.

We may assume that $0, 1 \neq x, y$ and $x \neq y$.

Ad (i). There exists an initial block $B_1 \supset \{x, y\}$. We claim that $U(\{x, y\}) \subset B_1$. For, if there exists $c \in L - B_1$ such that $c \in U(\{x, y\})$, then x and y are necessarily atoms and c is necessarily a coatom; moreover, there exist initial blocks $B_0 \supset \{x, c\}$ and $B_2 \supset \{y, c\}$ such $\{B_0, B_1, B_2\}$ is an atomistic loop of order 3; contradiction. Hence $U(x \vee_{B_1} y) \subset B_1$, therefore

$$U(\{x, y\}) = U_{B_1}(\{x, y\}) = U_{B_1}(x \vee_{B_1} y) = U(x \vee_{B_1} y).$$

It follows that $x \vee y$ exists and equals $x \vee_{B_1} y$.

Ad (ii). Again, there exists an initial block $B_1 \supset \{x, y\}$. By (i), since

$x \perp y'$, $x \vee y'$ exists; since $x \perp (y' \vee x)'$, $x \vee (y' \vee x)'$ exists. Moreover, by (i) again,

$$x \vee (y' \vee x)' = x \vee_{B_1} (y' \vee_{B_1} x)' = y,$$

since the orthomodular identity is satisfied in B_1. The proof is complete.

By Theorem 2 and the forthcoming Theorem 3, Figure 7 is an orthomodular poset which is not a lattice. It is the orthomodular poset J_{18} given by Janowitz in [9] and is the first known such structure. By replacing the blocks in J_{18} by arbitrary atomic Boolean lattices (not necessarily of the same cardinality), we may generate an infinite family of such structures. In fact, replacement by Boolean lattices having at least two atoms would suffice.

Figure 8 and 9 are also orthomodular posets which are not lattices. Figure 8 admits states while Figure 9 does not. We shall prove this in Section 3.

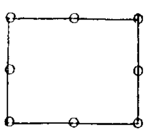

FIGURE 7

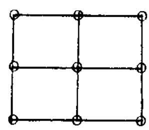

FIGURE 8

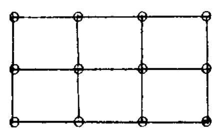

FIGURE 9

THEOREM 3. *Let L satisfy the conditions of Convention* 1. *Then L is an orthomodular lattice if and only if the order of every atomistic loop in L is at least* 5.

Proof. Let L be an orthomodular lattice. Then, by Theorem 2, L admits no loops of order 3. Suppose that L admitted a loop $\{B_0, B_1, B_2, B_3\}$ of order 4. Let a_i be the unique atom in $B_i \cap B_{i+1}$ ($i = 0, 1, 2, 3 (\mathrm{mod}\ 4)$). Then $U(\{a_0, a_2\}) \supset \{a_1', a_3', 1\}$. Since a_1' and a_3' are incomparable, $a_0 \vee a_2$ does not exist; contradiction. Hence L admits no loops of order less than 5.

Conversely assume that the order of every atomistic loop in L is at least 5. Then, by Theorem 2, L is an orthomodular poset. Let $x, y \in L$; we must prove that $x \vee y$ exists. We may assume that $0, 1 \neq x, y$ and $x \neq y$. If there is an initial block $B_\alpha \supset \{x, y\}$, then the corresponding proof in Theorem 2 yields the existence of $x \vee y$. Hence we may assume that $\{x, y\} \not\subset B_\alpha$ for every initial block B_α. Then

$$U(\{x, y\}) \subset \{c \mid c \text{ is a coatom}\} \cup \{1\}.$$

If there exist distinct coatoms $c_1, c_2 \in U(\{x, y\})$, then there exist initial blocks B_0, B_1, B_2, B_3 such that

$$\{x, c_1\} \subset B_0, \qquad \{c_1, y\} \subset B_1,$$
$$\{y, c_2\} \subset B_2, \qquad \text{and} \qquad \{c_2, x\} \subset B_3.$$

Now $\{x, y\} \not\subset B_0$, so that $B_0 \neq B_1$; similarly $B_0 \neq B_2$, $B_1 \neq B_3$, and $B_2 \neq B_3$. If $B_0 = B_3$ and $B_1 = B_2$, then $B_0 \cap B_2$ contains two distinct coatoms; contradiction. Hence one of the two equalities fails to hold, say $B_0 \neq B_3$. If $B_1 = B_2$, then $\{B_0, B_1, B_3\}$ is an atomistic loop of order 3, a contradiction. If $B_1 \neq B_2$, then $\{B_0, B_1, B_2, B_3\}$ is an atomistic loop of order 4, a contradiction. We may therefore conclude that no such distinct c_1, c_2 exist, so that

$$U(\{x, y\}) = \begin{cases} \{1\} \\ \text{or} \\ \{c, 1\} & \text{where } c \text{ is a coatom.} \end{cases}$$

In either case $x \vee y$ exists. The proof is complete.

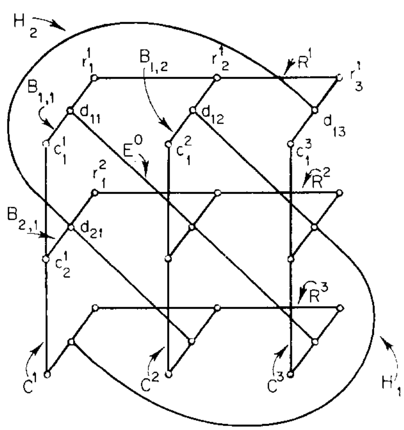

FIGURE 10. $G_{3,3}$

Figure 10, which we call $G_{3,3}$, may be considered as the lattice version of Figure 8. The "3 $\times$ 3 matrix" of distinguished points in Figure 8 is replaced by a "3 $\times$ 3 matrix" of copies $B_{i,j}$ of 2^3 in Figure 10. The other blocks, determined by the R^i, C^i, and H_i, are inserted to ensure that each atom is in exactly two blocks. (Note that in the diagram only the atoms of these blocks are depicted.) $G_{3,3}$ satisfies the conditions of Convention 1 and by Theorem 3 is an orthomodular lattice. In a similar fashion we exhibit the lattice version of Figure 9 in Figure 11. This orthomodular lattice, $G_{3,4}$, admits no states. A general construction of a class of such lattices together with a proof that they admit no states appears in Section 3.

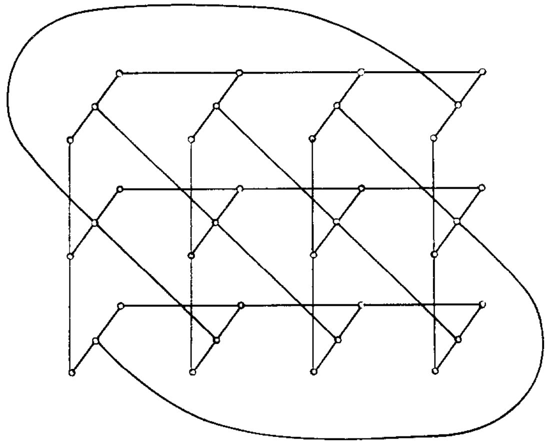

FIGURE 11. $G_{3,4}$

We now concern ourselves with the question: Are the initial blocks of L the only blocks? Note that, in an orthomodular lattice satisfying Convention 1, any chain containing at least two non-distinguished elements ($\neq 0, 1$) is contained in a unique initial block.

LEMMA. *Let L be an orthomodular lattice satisfying the conditions of Convention* 1. *If B is a Boolean suborthomodular lattice of L which contains at least eight elements, the $B \subset B_\alpha$ for some unique initial block B_α.*

Proof. Let x, y, z, x', y', z' be distinct elements of $B - \{0, 1\}$. We must prove that there is exactly one initial block containing $\{x, y, z\}$. We have

$$x = (x \wedge y) \vee (x \wedge y'),$$
$$x = (x \wedge z) \vee (x \wedge z'),$$

and

$$y = (y \wedge z) \vee (y \wedge z').$$

One of each of the following groups of three must occur.

(1) $x \wedge y = 0$, (4) $x \wedge z = 0$,
(2) $x \wedge y' = 0$, (5) $x \wedge z' = 0$,
(3) $x \wedge y \neq 0 \neq x \wedge y'$; (6) $x \wedge z \neq 0 \neq x \wedge z'$;
(7) $y \wedge z = 0$,
(8) $y \wedge z' = 0$,
(9) $y \wedge z \neq 0 \neq y \wedge z'$.

By renaming y, y' and z, z' we may assume that (1) and (4) do not occur. We break the proof into two parts depending on whether (2) or (3) occurs.

Case I: (2) occurs. Then $0 < x < y < 1$ so that there exists a unique initial block $B_1 \supset \{x, y\}$. One of (5) or (6) occurs.

Assume (5) occurs. Then $0 < x < z < 1$ so that there exists a unique initial block $B_2 \supset \{x, z\}$. Suppose $B_1 \neq B_2$. Then $x = y \wedge z$. The occurrence of (7) or (8) forces $x = 0$ or $x = y$, respectively, so that (9) must occur; it follows that $y \wedge z$ and $y \wedge z'$ are both atoms of $B_1 \cap B_2$ and hence are equal, which is impossible. But one of (7), (8), or (9) must occur; therefore $B_1 = B_2 \supset \{x, y, z\}$.

Assume (6) occurs. Then

$$0 < x \wedge z < x < y < 1 \quad \text{and} \quad 0 < x \wedge z' < x < y < 1.$$

Hence $x \wedge z \in B_1$, $x \wedge z' \in B_1$; moreover there exists a unique initial block $B_3 \supset \{x \wedge z, z\}$. Suppose $B_1 \neq B_3$. The occurrence of (7) or (8) forces $x \wedge z = 0$ or $x \wedge z' = 0$, respectively, so that (9) must occur; it follows that $x \wedge z$ and $x \wedge z'$ are both atoms of $B_1 \cap B_3$ and hence are equal, which is impossible, therefore $B_1 = B_3 \supset \{x, y, z\}$.

Case II: (3) occurs. Then $0 < x \wedge y \leqslant y \leqslant x' \vee y < 1$ so that there exists a unique initial block $B_1 \supset \{x \wedge y, y, x' \vee y\}$. It follows that $x \in B_1$. The remainder of the proof is similar to that of Case I and is therefore omitted.

The following theorem is now obvious:

Theorem 4. *Let L be an orthomodular lattice satisfying the conditions of Convention* 1. *Then the only blocks in L are the initial blocks.*

3. States on Orthomodular Lattices

Definitions. Let L be an orthomodular lattice. A *state* on L (sometimes called a *probability measure on L*) is a mapping $\alpha : L \to [0, 1]$ such that

(i) $0\alpha = 0$,

(ii) $1\alpha = 1$,

(iii) if $x, y \in L$ with $x \perp y$, then $(x \vee y)\,\alpha = x\alpha + y\alpha$.

Let S_L denote the set of all states on L. Note that, if $x, y \in L$ with $x \leqslant y$, then $x\alpha \leqslant y\alpha$ for all $\alpha \in S_L$. S_L is said to be *full* in case $x, y \in L$ with $x\alpha \leqslant y\alpha$ for all $\alpha \in S_L$ implies $x \leqslant y$.

M. K. Bennett [2] has shown that G_{32} does not admit a full set of states. (The idea of the proof is this: if $\alpha \in S_{G_{32}}$ then $a\alpha \leqslant b'\alpha$ (cf. Figure 6); hence, were $S_{G_{32}}$ full, we would have $a \leqslant b'$ which is false.) Thus, although there are infinitely many states on G_{32}, the intertwining of the blocks creates enough of a restriction that the state space is not full.

There is another way of looking at states. Let $\mathscr{B}_L$ denote the set of all blocks in the orthomodular lattice L; for $B \in \mathscr{B}_L$, let S_B denote the set of all states on B. Note that, if $\alpha \in S_L$, then $\alpha\,|_B \in S_B$. Moreover, we may regard $\alpha \in S_L$ as a selection function

$$\alpha : \mathscr{B}_L \to \bigcup_{B \in \mathscr{B}_L} S_B$$

such that

(i) if $B \in \mathscr{B}_L$, then $B\alpha \in S_B$, and

(ii) if $B_1, B_2 \in \mathscr{B}_L$, then $(B_1\alpha)\,|_{B_1 \cap B_2} = (B_2\alpha)\,|_{B_1 \cap B_2}$.

Thus, a state may be viewed as selection function whose images coincide on their intersection—a rather strong condition. The condition is in fact so strong that there exist orthomodular lattices L which admit no states whatsoever. The following construction exhibits an infinite class of such lattices.

Let m and n be integers with $n \geqslant m \geqslant 3$; for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, let $\mathscr{B} = \{B_{ij}\}$ be a family of copies of the Boolean lattice 2^3 which are disjoint except for a common 0 and 1. Label the atoms of each $B_{i,j}$ as follows: c_i^j, $d_{i,j}$, and r_j^i.

Consider the set $\bar{R}^i = \{r_j^i : 1 \leqslant j \leqslant n\}$. Let $\hat{R}^i$ be the Boolean lattice whose generating atoms are $\bar{R}^i$. For each j, identify the 0, 1, and $(r_j^i)'$ of $\hat{R}^i$ with the 0, 1, and $(r_j^i)'$ of $B_{i,j}$ thereby obtaining a Boolean lattice R^i having 0, 1, r_j^i, and $(r_j^i)'$ in common with $B_{i,j}$. Write

$$R^i = [r_j^i : 1 \leqslant j \leqslant n].$$

Now consider the set $\bar{C}^j = \{c_i^j : 1 \leqslant i \leqslant m\}$. As above, let $\hat{C}^j$ be the Boolean lattice whose generating atoms are $\bar{C}^j$. For each i, identifying the 0, 1, and $(c_i^j)'$ of $\hat{C}^j$ with the 0, 1, and $(c_i^j)'$ of $B_{i,j}$ thereby obtaining a Boolean lattice C^j having 0, 1, c_i^j, and $(c_i^j)'$ in common with $B_{i,j}$. Write $C^j = [c_i^j : 1 \leqslant i \leqslant m]$.

Similarly we define the Boolean lattices D^k, E^k, F^k, H_1, and H_2 as follows:

for $1 \leqslant i \leqslant m$, $R^i = [r_j^i : 1 \leqslant j \leqslant n]$,

for $1 \leqslant j \leqslant n$, $C^j = [c_i^j : 1 \leqslant i \leqslant m]$,

for $1 \leqslant k \leqslant m - 3$, $D^k = [d_{i,j} : 1 \leqslant i \leqslant m - k$ and $j = n - m + k + i]$,

for $0 \leqslant k \leqslant n - m$, $E^k = [d_{i,j} : 1 \leqslant i \leqslant m$ and $j = k + i]$,

for $1 \leqslant k \leqslant m - 3$, $F^k = [d_{i,j} : k + 1 \leqslant i \leqslant m$ and $j = i - k]$,

$$H_1 = [d_{1,n-1}, d_{2,n}, d_{m,1}],$$
$$H_2 = [d_{1,n}, d_{m-1,1}, d_{m,2}].$$

Let

$$\mathscr{D} = \{R^i : 1 \leqslant i \leqslant m\} \cup \{C^j : 1 \leqslant j \leqslant n\} \cup \{D^k : 1 \leqslant k \leqslant m - 3\} \cup \{E^k : 0 \leqslant k \leqslant n - m\} \cup \{F^k : 1 \leqslant k \leqslant m - 3\} \cup \{H_1, H_2\}.$$

Note that $^{\#}\mathscr{D} = 2n + 2m - 3$. Let $G_{m,n}$ be the orthocomplemented poset defined by taking as initial blocks the Boolean lattices $\bigcup \mathscr{B} \cup \bigcup \mathscr{D}$. Note that $G_{m,n}$ satisfies the conditions of Convention 1. $G_{3,3}$ and $G_{3,4}$ are given in Figures 10 and 11, respectively. Theorem 3 yields the following.

RESULT 5. For $n \geqslant m \geqslant 3$, $G_{m,n}$ is an orthomodular lattice.

An *atom* is an orthomodular poset L is an element $a \in L$ such that $0 < a$ and if $b \leqslant a$ then $b = 0$ or $b = a$. L is *atomic* if every non-zero element of L dominates an atom. Assume that L is an atomic orthomodular poset and let A denote the set of atoms in L. We say that a collection $\mathscr{B}_1$ of blocks of L *covers the atoms* of L in case $A \subseteq \bigcup \mathscr{B}_1$. $\mathscr{B}_1$ is an *exact covering* of A in case $B_1 \cap B_2 \cap A = \varnothing$ for all distinct $B_1, B_2 \in \mathscr{B}_1$.

LEMMA 6. *Let L be a finite orthomodular poset, let A be the set of atoms in L, let α be a state on L, and let $\mathscr{B}_1$ and $\mathscr{D}_1$ be exact coverings of A. Then $^{\#}\mathscr{B}_1 = {}^{\#}\mathscr{D}_1$.*

Proof. For any set $M \subseteq A$ let $M\alpha = \sum \{m\alpha : m \in M\}$. Note that L atomic and $\sum B\alpha = 1$ for any block B of L. Now

$$^{\#}\mathscr{B}_1 = \sum_{B \in \mathscr{B}_1} B\alpha = A\alpha = \sum_{B \in \mathscr{D}_1} B\alpha = {}^{\#}\mathscr{D}_1.$$

COROLLARY. *The poset given in Figure 9 admits no states.*

THEOREM 7. *For $n \geqslant m \geqslant 3$, $S_{G_{m,n}} \neq \varnothing$ if and only if $m = n = 3$.*

Proof. If $m = n = 3$, then the function which maps each atom of $G_{3,3}$ to $1/3$ generates a state. (In fact there are infinitely many states on $G_{3,3}$.) Conversely, assume that there exists a state α on $G_{m,n}$. Let A denote the atoms of $G_{m,n}$. Then $\mathscr{B}$ and $\mathscr{D}$ are exact coverings of A; $^{\#}\mathscr{B} = mn$ and $^{\#}\mathscr{D} = 2m + 2n - 3$. By Lemma 6 we have

$$mn = 2m + 2n - 3. \tag{I}$$

Write $n = m + k$ for some non-negative integer k. Hence, substituting $m + k$ for n in (I) and transposing we obtain,

$$m^2 + (k - 4)\, m + (3 - 2k) = 0. \tag{II}$$

Hence

$$m = \frac{4 - k \pm \sqrt{k^2 + 4}}{2}, \tag{III}$$

so that there exists a positive integer p such that

$$(p + k)(p - k) = 4. \tag{IV}$$

Hence $k = 0$, $p = 2$ so that $m = 1$ or 3. But $m \geqslant 3$, so $m = 3$. Substituting back in (I) we obtain $m = n = 3$. Hence the result.

Note that the assumption $n \geqslant m \geqslant 3$ was made only for convenience in defining $G_{m,n}$. A similar construction may be made if $m > n \geqslant 3$; this involves changes only in the indices appearing in the definition of "diagonal blocks," the D^k's, E^k's, and F^k's.

If $m = n = \infty$, we may construct an orthomodular lattice $G_{\infty,\infty}$ by an analogous procedure. (In this case, H_1 and H_2 do not appear.) The diagram for $G_{\infty,\infty}$ is quite symmetric and appears in Figure 12. $G_{\infty,\infty}$ also admits no states but, of course, the previous argument breaks down. It may be replaced by the following argument:

THEOREM. $S_{G_{\infty,\infty}} = \varnothing$.

Proof. Suppose $\alpha \in S_{G_{\infty,\infty}}$. Then there exists N such that, for $i = 1, 2,..., 10$ and for $j \geqslant N$, $r_j{}^i\alpha < 1/5$. Also, for each

$$j = N, N + 1,..., N + 9,$$

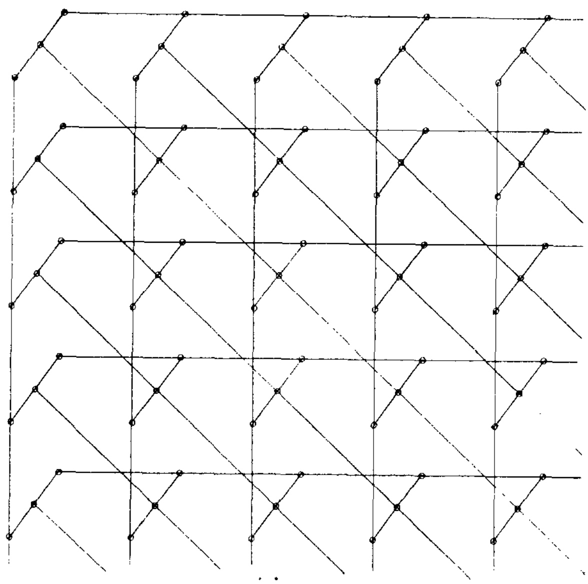

FIGURE 12. $G_{\infty,\infty}$

at least 6 of the 10 elements $\{c_i^j \mid 1 \leqslant i \leqslant 10\}$ are mapped by α to a number $<1/4$. Therefore at most 40 of

$$X = \{c_i^j \mid 1 \leqslant i \leqslant 10, N \leqslant j \leqslant N+9\}$$

are mapped by α to a number $\geqslant 1/4$.

CLAIM: of the (at least 60) elements $c_i^j \in X$ such that $c_i^j\alpha < 1/4$ there exist at least two of the form

$$c_i^j, c_{i+k}^{j+k} \quad (1 \leqslant i \leqslant i+k \leqslant 10, \quad N \leqslant j < j+k \leqslant N+9).$$

For, if this were not so, then at least $9-p$ of

$$c_1^{N+p}, c_2^{N+p+1}, \ldots, c_{10-p}^{N+9} \quad (0 \leqslant p \leqslant 8)$$

are mapped by α to a number $\geqslant 1/4$. Hence at least 45 $c_i^j \in X$ are mapped by α to a number $\geqslant 1/4$; but there are at most 40 such elements; contradiction. Hence the claim.

Let $c_i^{\,j}$, c_{i+k}^{j+k} be the two elements obtained. Note that $r_j^{\,i} \perp c_i^{\,j}$ and $r_{j+k}^{i+k} \perp c_{i+k}^{j+k}$. Also there exist unique orthogonal elements $d_{i,j}$, $d_{i+k,j+k}$ such that $\{r_j^{\,i}, c_i^{\,j}, d_{i,j}\}$ and $\{r_{j+k}^{i+k}, c_{i+k}^{j+k}, d_{i+k,j+k}\}$ are blocks. It follows that

$$d_{i,j}\alpha = 1 - r_j^{\,i}\alpha - c_i^{\,j}\alpha \geqslant 1 - 1/5 - 1/4 = 11/20$$

and, similarly,

$$(d_{i+k,j+k})\,\alpha \geqslant 11/20.$$

But $d_{i,j}\alpha + d_{i+k,j+k}\alpha \leqslant 1$, which yields a contradiction. Hence no such α exists.

4. Comments

Following S. S. Holland, Jr. [8], we define a *measure* on a complete orthomodular lattice L to be a function $m : L \to [0, \infty]$ such that $m(0) = 0$ and $m(\bigvee a_\beta) = \sum m(a_\beta)$ for *any* orthogonal family $\{a_\beta\} \subseteq L$. A measure m is *semifinite* if every non-zero element of L majorizes a non-zero element b with $m(b) < \infty$, and *finite* if $m(a) < \infty$ for all a in L.

Now every semifinite measure on L is finite if L contains a finite block. Hence an argument similar to that of Theorem 7 proves the following.

THEOREM. *For integers $m, n \geqslant 3$ there exists a semifinite measure on $G_{m,n}$ if and only if $m = n = 3$.*

Of course the trivial measure which maps every non-zero element to ∞ is a measure on any (complete) orthomodular lattice.

We conclude with an unpublished observation of Arlan Ramsay: $G_{m,n}$ *admits (non-trivial) bounded signed measures,* i.e., mappings $m : L \to [a, b] \subseteq \mathbb{R}$ such that $m(0) = 0$ and $m(\bigvee x_\alpha) = \sum m(x_\alpha)$ for any orthogonal family $\{x_\alpha\}$ in L. (Note that our argument for the non-existence of states on $G_{m,n}$ shows that, for any such measure on $G_{m,n}$, $m(1) = 0$ and hence $m(x') = -m(x)$ for all $x \in G_{m,n}$.) The existence of bounded signed measures on $G_{m,n}$ is important because it indicates that additional conditions are needed on an orthomodular lattice in order to obtain a decomposition of a signed measure corresponding to the Jordan decomposition of classical measure theory.

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