Continuous dependence results for inhomogeneous ill-posed problems in Banach space

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Abstract

We apply semigroup theory and other operator-theoretic methods to prove Hölder-continuous dependence on modeling for the inhomogeneous ill-posed Cauchy problem in Banach space. The inhomogeneous ill-posed Cauchy problem is given by $\frac{du}{dt} = Au(t) + h(t), \; u(0) = \chi, \; 0 \leq t < T$; where $-A$ is the infinitesimal generator of a holomorphic semigroup on a Banach space $X$, $\chi \in X$, and $h:[0, T) \rightarrow X$. For a suitable function $f$, the approximate problem is given by $\frac{dv}{dt} = f(A)v(t) + h(t), \; v(0) = \chi$. Under certain stabilizing conditions, we prove that $\|u(t) - v(t)\| \leq \tilde{C} \beta^{1-\omega(t)} M^{\omega(t)}$ for a related norm, where $\tilde{C}$ and $M$ are computable constants independent of $\beta$, $0 < \beta < 1$, and $\omega(t)$ is a harmonic function. These results extend earlier work of Ames and Hughes on the homogeneous ill-posed problem.

Keywords: Abstract Cauchy problem; Continuous dependence on modeling; Ill-posed problems

1. Introduction

Continuous dependence results for ill-posed problems in Banach space were obtained by Ames and Hughes in [6,7]. Using $C$-semigroup theory and other operator-theoretic methods, that work extends to Banach space continuous dependence on modeling results previously obtained by Ames [3], Lattes and Lions [16], Miller [18], and Showalter [23]. In this paper, we
develop new techniques to extend the results of [7] to the inhomogeneous case, and we prove continuous dependence on modeling for the inhomogeneous ill-posed Cauchy problem.

Consider the abstract Cauchy problem

\begin{align*}
\frac{du(t)}{dt} &= Au(t), \\
  u(0) &= \chi,
\end{align*}

(1)

where \( t \geq 0 \), \( A \) is a densely-defined linear operator in a Banach space \( X \), and \( \chi \in \text{Dom}(A) \). In the event \( -A \) is the infinitesimal generator of a holomorphic semigroup, the Cauchy problem is ill-posed. We define an approximate problem that is well-posed. For example, we may approximate \( A \) by \( f(A) = A - \epsilon A^2 \) (cf. [3,16,18]). We want to ensure that a solution of the original problem, if it exists, will be appropriately close to the solution of the approximate problem. Using logarithmic convexity and stabilizing conditions, results have been obtained in Hilbert space by Ames [3], Ames and Cobb [5], Miller [18], Adelson [1], and others. Some work has been done in Banach space also, as found in [17]. We extend these results to the inhomogeneous problem in Banach space.

In Sections 2 and 3, we examine the inhomogeneous abstract Cauchy problem in Banach space. Our key idea is to regularize the data using \( C \)-semigroups. Section 2 provides the necessary background on holomorphic semigroups and \( C \)-semigroups, and in Section 3 we prove continuous dependence on modeling. In a Banach space \( X \), the inhomogeneous ill-posed problem is given by

\begin{align*}
\frac{dv}{dt} &= f(A)v(t) + h(t), \\
  v(0) &= \chi,
\end{align*}

(2)

where the operator \( -A \) is the infinitesimal generator of a holomorphic semigroup in \( X \), \( \chi \in X \), and \( h: [0, T) \rightarrow X \). We assume that \( h \) is differentiable on \( (0, T) \) and that \( h' \in L^1((0, T); X) \). We approximate this problem with

\begin{align*}
\frac{dv}{dt} &= f(A)v(t) + h(t), \\
  v(0) &= \chi,
\end{align*}

where \( f \) is a real-valued Borel function bounded above such that \( f(A) \) approximates \( A \) in a suitable sense. For \( A = -\Delta \), the choice of \( f(A) = A - \epsilon A^2 \) yields the approximate well-posed problem

\begin{align*}
\frac{\partial v}{\partial t} &= -\frac{\partial^2 v}{\partial x^2} - \epsilon \frac{\partial^4 v}{\partial x^4} + h(t), \quad \text{where } 0 < t < T, \\
  v(x, 0) &= \psi(x).
\end{align*}

As \( \epsilon \rightarrow 0 \), this approximate problem approaches the original one. Generally speaking, we prove that

\[ \|u(t) - v(t)\| \leq \tilde{C} \beta^{1-\omega(t)} M^{\omega(t)} \]

for a suitable norm, where \( u \) and \( v \) are solutions of the ill-posed and approximate problems, respectively, \( \tilde{C} \) is a constant, \( \beta \) comes from the restriction on the “distance” between \( A \) and
f(\(A\)), \(M\) is a constant resulting from the stability condition, and \(\omega(t)\) is a harmonic function. We obtain our results by defining a function following [7] and using the approach in [2]. Extending to a bent complex strip, we apply Carleman’s Inequality to obtain
\[
\|u(t) - v(t)\| \leq \tilde{C} \beta^{1-\omega(t)} M^{\omega(t)},
\]
where
\[
\|(-A + f(\(A\)))\psi\| \leq \beta \|A^{1+\delta}\psi\|
\]
for all \(\psi \in \text{Dom}(A^{1+\delta})\), with some additional restrictions on \(f\).

In Section 4 we consider some applications of our results to problems in \(L^p\) spaces. We note that in future work, we intend to extend our results to nonlinear problems (cf. [16,18]).

2. Background

We note that general background on the abstract Cauchy problem can be found in [13], while background on ill-posed problems can be found in [4,20,21]. In order to obtain results in Banach space, we introduce \(C\)-semigroups and holomorphic semigroups, following work done by deLaubenfels (cf. [11,12]). Recall the following:

**Definition 1.** [12] Let \(C\) be a bounded, injective linear operator on a Banach space \(X\). A family \(\{W(t)\}_{t \geq 0}\) of bounded operators on \(X\) is a \(C\)-semigroup if

1. \(W(t)\) is strongly continuous, i.e. for all \(x \in X\), \(W(t)x : [0, \infty) \to X\) is continuous,
2. \(W(t)W(s) = CW(t+s)\) for all \(t, s \geq 0\), and
3. \(W(0) = C\).

For the relevant background on holomorphic semigroups, their infinitesimal generators, and \(C\)-semigroups, refer to [7]. For general background on semigroups and linear operators, see [14,15].

Now we state deLaubenfels’ theorem:

**Theorem 2.** [11] Suppose \(-A\) is the infinitesimal generator of a holomorphic semigroup of angle \(\theta\), \(\theta \in (0, \pi/2]\). Then there exist \(k \in \mathbb{R}, \alpha > 0\), such that for all \(\epsilon > 0\), \(\lambda \in \mathbb{C}, \lambda A\) generates an entire \(C_\epsilon\)-group \(\{W_\epsilon(z)\}\), where \(C_\epsilon \equiv e^{-\epsilon(A-k)^\alpha}\).

This theorem is crucial to our work as it defines conditions under which the homogeneous Cauchy problem (1) is well-posed. Following [11] and [7], we have this representation:

\[
W_\epsilon(z) = \frac{e^{\lambda z}}{2\pi i} \int_{\Gamma_\phi} e^{-\epsilon w^\alpha} e^{\lambda zw} (w + k - A)^{-1} dw
\]

for a suitable contour \(\Gamma_\phi\) in \(\rho(A)\) with \(\alpha\) satisfying \(\frac{\pi}{2\alpha} > \phi > \frac{\pi}{2} - \theta\). For \(\theta \in (\frac{\pi}{4}, \frac{\pi}{2}]\), we may choose \(\alpha > 2\). Formally, \(W_\epsilon(z) = C_\epsilon e^{\lambda z} = e^{\lambda z} e^{-\epsilon(A-k)^\alpha}\).

In Banach space, we assume that the operator \(-A\) is the generator of a holomorphic semigroup. To use the above results, we need to ensure that \(f(\(A\))\) generates a holomorphic semigroup so that the approximate problem (2) is well-posed. In order to do this, we introduce the following sets.
Definition 3. [9] Let $\mathcal{S}$ denote the set of all functions $f$ of the form
\[
f(z) = az + b + \int_0^\infty \frac{z}{z+t} \, d\mu(t), \quad z \in \mathbb{C} \setminus (-\infty, 0],
\]
where $a, b \geq 0$ are constants, and $\mu$ is a nondecreasing function on $[0, \infty)$ such that
\[
\int_0^\infty \frac{d\mu(t)}{1+t} < \infty.
\]
For $f \in \mathcal{S}$, define $f(A)$ to be the closure of the closable operator defined by
\[
f(A)x = aAx + bx + \int_0^\infty \frac{1}{(t+A)^{-1}Ax} \, d\mu(t),
\]
for $x \in \text{Dom}(A)$.

Theorem 4. [9, cf. Theorem 6.1] Suppose $f \in \mathcal{S}$, $\theta \in [0, \pi/2)$, and $-A$ generates a holomorphic semigroup of angle $\theta$. Then $-f(A)$ also generates a holomorphic semigroup of angle $\theta$.

Theorem 5. [10] Let $\mathcal{P}_n$ denote the collection of $n$th degree polynomials $p$ with positive leading coefficient. Suppose $-A$ generates a holomorphic semigroup of angle $\theta$, $p \in \mathcal{P}_n$, and
\[
n \left( \frac{\pi}{2} - \theta \right) < \frac{\pi}{2}.
\]
Then $-p(A)$ generates a holomorphic semigroup of angle $\frac{\pi}{2} - n \left( \frac{\pi}{2} - \theta \right)$.

To ensure that $f(A)$ generates a holomorphic semigroup, we consider functions $f$ such that $-f \in \mathcal{S} \cup \mathcal{P}_n$, where $n$ satisfies (4).

3. Continuous dependence results in Banach space

As stated above, the inhomogeneous ill-posed problem in Banach space is given by
\[
\begin{aligned}
du(t) &= Au(t) + h(t), \\
u(0) &= \chi,
\end{aligned}
\]
where the operator $-A$ is the infinitesimal generator of a holomorphic semigroup in $X$, $\chi \in X$, and $h : [0, T) \to X$. A solution for this problem is defined as follows:

Definition 6. [22, Definition 4.2.1] A function $u : [0, T) \to X$ is a solution of (5) on $[0, T)$ if $u$ is continuous on $[0, T)$, continuously differentiable on $(0, T)$, $u(t) \in \text{Dom}(A)$ for $0 < t < T$, and (5) is satisfied on $[0, T)$.

The following theorem states conditions under which such a solution exists:

Theorem 7. [22, Corollary 4.2.10] Let $X$ be a Banach space and let $A$ be the infinitesimal generator of a $C_0$-semigroup $T(t)$ on $X$. If $h : [0, T) \to X$ is differentiable on $(0, T)$ and $h' \in$
$L^1(0, T)$, then for every $\chi \in \text{Dom}(A)$ the initial value problem (5) has a unique solution $u$ on $[0, T)$ given by

$$u(t) = T(t)\chi + \int_0^t T(t-s)h(s)\,ds.$$  

(6)

If $A$ is an unbounded operator, it is not defined everywhere and thus solutions may not exist for all $\chi$. If solutions do exist, they may not be continuously dependent on the data. In either of these cases, the problem is ill-posed. Formally, the solution of (5) has the form

$$u(t) = e^{tA}\chi + \int_0^t e^{(t-s)A}h(s)\,ds,$$

following [22].

We approximate the ill-posed problem (5) with

$$\frac{dv}{dt} = f(A)v(t) + h(t),
\quad v(0) = \chi,$$

(7)

where $f$ is a real-valued Borel function bounded above that satisfies Condition $(A)$, defined as follows:

**Definition 8.** [7] Let $-A$ be the infinitesimal generator of a holomorphic semigroup of angle $\theta$ in a Banach space $X$. Let $-f \in \mathcal{S} \cup \mathcal{P}_n$. Then $f$ satisfies Condition $(A)$ if there exist positive constants $\beta$ and $\delta$, with $0 < \beta < 1$, for which $\text{Dom}(A^{1+\delta}) \subseteq \text{Dom}(f(A))$ and

$$\|(-A + f(A))\psi\| \leq \beta\|A^{1+\delta}\psi\|$$

for $\psi \in \text{Dom}(A^{1+\delta})$.

Note that the fractional power $A^{1+\delta}$ is defined following Balakrishnan [8]. Set

$$g(\lambda) = -\lambda + f(\lambda).$$

**Lemma 9.** [7, Lemma 1] For all $t \geq 0$,

$$e^{tg(A)} = e^{-tA}e^{tf(A)}.$$

We will use this repeatedly in our proofs, together with the fact that this relationship holds for all $\alpha \in \mathbb{C}$:

$$e^{\alpha g(A)} = e^{-\alpha A}e^{\alpha f(A)}.$$

Using the theory outlined above (cf. Theorem 7), we can show that this approximate problem is well-posed if $f(A)$ generates a holomorphic semigroup and $h: [0, T) \rightarrow X$ is differentiable on $(0, T)$ with $h' \in L^1(0, T)$.

Set $g(A)x = -Ax + f(A)x$ for $x \in \text{Dom}(A) \cap \text{Dom}(f(A))$. If $-A$ is the infinitesimal generator of a holomorphic semigroup of angle $\theta$ and $-f \in \mathcal{S} \cup \mathcal{P}_n$, then $f(A)$ also generates a holomorphic semigroup of some possibly different angle. We can choose an appropriate angle
\( \theta' < \frac{\pi}{2} \) such that \(-A\) and \(f(A)\) both generate holomorphic semigroups of angle \(\theta'\). As explained in [7], \(e^{-\zeta A}\) and \(e^{\xi f(A)}\) commute for \(\zeta \in S_{\theta'}\) because the resolvents of their generators commute. Thus for \(\zeta \in S_{\theta'}\), \(e^{-\zeta A}e^{\xi f(A)}\) defines a holomorphic semigroup of angle \(\theta'\). This semigroup has generator \(g(A)\), the closure of \(-A + f(A)\). Following [7], denote this semigroup by \(e^{\xi g(A)}\).

Assume \(u(t)\) and \(v(t)\) are solutions of (5) and (7), respectively, with \(u(0) = v(0) = \chi\), and consider the \(C_\epsilon\)-semigroup \(\{W_\epsilon(z)\}\), with \(C_\epsilon = e^{-\epsilon A}\). Choose \(\theta'\) so that both \(-A\) and \(f(A)\) generate holomorphic semigroups of angle \(\theta'\), as discussed above. For \(\zeta \in S_{\theta'}, \zeta = s + re^{i\theta}\), set

\[
\begin{align*}
u_\epsilon(t) &= e^{\epsilon r f(A)} C_\epsilon v(t), \\
and set \\
h_\epsilon(s) &= C_\epsilon h(s).
\end{align*}
\]

Lemma 10. For \(\epsilon > 0\) and \(t \geq 0\),

\[
u_\epsilon(t) = C_\epsilon u(t).
\]

Proof. By definition,

\[
u_\epsilon(t) = W_\epsilon(0)u(t) = C_\epsilon u(t).\]

\]

Lemma 11. For \(\epsilon > 0\) and \(t \geq 0\),

\[
u_\epsilon(t) = C_\epsilon v(t).
\]

Proof. Again, for \(\zeta = t\) we have \(r = 0\) and so by definition

\[
u_\epsilon(t) = C_\epsilon v(t).\]

Lemma 12. [7, cf. Lemma 4] For \(\epsilon > 0\) and \(\zeta \in S_{\theta'}\),

\[
W_\epsilon(re^{i\theta})\chi = W_{\frac{\epsilon}{2}}(re^{i\theta})C_{\frac{\epsilon}{2}}\chi.
\]

Proof. First, note that \(C_{\frac{\epsilon}{2}}\) commutes with \(W_{\frac{\epsilon}{2}}(\zeta)\), as discussed above. Next, we show that \(W_{\frac{\epsilon}{2}}(\zeta)C_{\frac{\epsilon}{2}}\) is a \(C_\epsilon\)-semigroup with generator \(A\). Let \(\zeta_1, \zeta_2 \in S_{\theta'}\).

\[
W_{\frac{\epsilon}{2}}(\zeta_1)C_{\frac{\epsilon}{2}}W_{\frac{\epsilon}{2}}(\zeta_2)C_{\frac{\epsilon}{2}} = [W_{\frac{\epsilon}{2}}(\zeta_1 + \zeta_2)C_{\frac{\epsilon}{2}}]C_{\frac{\epsilon}{2}}C_{\frac{\epsilon}{2}}
\]

\[
= C_\epsilon [W_{\frac{\epsilon}{2}}(\zeta_1 + \zeta_2)C_{\frac{\epsilon}{2}}]
\]

\[
and W_{\frac{\epsilon}{2}}(0)C_{\frac{\epsilon}{2}} = C_{\frac{\epsilon}{2}}C_{\frac{\epsilon}{2}} = C_\epsilon,\so by definition W_{\frac{\epsilon}{2}}(\zeta)C_{\frac{\epsilon}{2}} is a C_\epsilon-semigroup. We know W_\epsilon(\zeta) is a C_\epsilon-semigroup with generator A and both \(W_{\frac{\epsilon}{2}}(\zeta)\) and \(W_{\frac{\epsilon}{2}}(\zeta)\) solve the same differential equation, so we must have \(W_{\frac{\epsilon}{2}}(\zeta)C_{\frac{\epsilon}{2}}\chi = W_\epsilon(\zeta)\chi\). \]

\]

Lemma 13. [7, cf. Lemma 5] For \(\epsilon > 0\) and \(\zeta \in S_{\theta'}\),

\[
e^{\epsilon r f(A)} C_\epsilon \chi = e^{\epsilon r g(A)} W_\epsilon(re^{i\theta})\chi.
\]
Proof. By definition, \( e^{re^{i\theta}g(A)} = e^{-re^{i\theta}A}e^{re^{i\theta}f(A)} \). Using Theorem 2.4 from [12], for \( \chi \in \text{Dom}(A) \) and \( \zeta = re^{i\theta} \)

\[
\frac{d}{d\zeta} [e^{-\zeta A} W_\epsilon(\zeta) \chi] = e^{-re^{i\theta}A}AW_\epsilon(re^{i\theta})\chi + (-Ae^{-re^{i\theta}A})W_\epsilon(re^{i\theta})\chi = 0.
\]

Since \( W_\epsilon(0) = C_\epsilon \), we must have

\[
e^{-re^{i\theta}A}W_\epsilon(re^{i\theta})\chi = C_\epsilon \chi.
\]

Recall that \( e^{-\zeta A} \) and \( e^{zf(A)} \) commute, and thus

\[
e^{re^{i\theta}g(A)}W_\epsilon(re^{i\theta})\chi = e^{re^{i\theta}f(A)}e^{-re^{i\theta}A}W_\epsilon(re^{i\theta})\chi = e^{re^{i\theta}f(A)}C_\epsilon \chi. \]

Lemma 14. [7] Let \( -A \) be the generator of a holomorphic semigroup of angle \( \theta \) in a Banach space \( X \), where \( \theta \in (\frac{\pi}{4}, \frac{\pi}{2}) \). Let \( \theta' = \min(\theta, \frac{\pi}{2} - n(\frac{\pi}{2} - \theta)) \). Consider the bent strip \( S = \{\zeta = s + re^{i\theta'} \mid 0 \leq s \leq T, r \geq 0\} \), and replace \( \theta' \) with \( \theta \) for simplicity. Set \( \alpha > 2 \), and without loss of generality let \( k = 0 \) in the definition of \( W_\epsilon(\zeta) \). Then

\[
re^{\gamma} e^{-2sr \cos \theta - r^2} \| W_\frac{\gamma}{2} (s + re^{i\theta}) \| \leq C,
\]

where \( \gamma \) is a constant and \( C \) is a constant that depends on \( \epsilon \) but is independent of \( r \geq 0, 0 \leq s \leq T \).

Proof. Let \( \lambda \in \Gamma_\phi \) as in the definition of \( W_\epsilon(\zeta) \). \( \lambda = |\lambda|e^{\pm i\phi} \). \( re^{\gamma} e^{-\frac{r^2}{2}} \) is bounded for \( r \geq 0 \), so consider

\[
\left\| e^{-2sr \cos \theta - \frac{r^2}{2}} W_\frac{\gamma}{2} (s + re^{i\theta}) \right\|.
\]

Taking \( \alpha > 2 \), and without loss of generality, \( k = 0 \) in the definition of \( W_\epsilon(\zeta) \), the above quantity can be written as

\[
\left\| e^{-2sr \cos \theta - \frac{r^2}{2}} \left( \frac{1}{2\pi i} \right) \int_{\Gamma_\phi} e^{-\frac{1}{2} \lambda^2} e^{(s + r e^{i\theta})\lambda} R(\lambda; A) d\lambda \right\|
\]

and

\[
\left\| e^{-2sr \cos \theta - \frac{r^2}{2}} \left( \frac{1}{2\pi i} \right) \int_{\Gamma_\phi} e^{-\frac{1}{2} \lambda^2} e^{(s + r e^{i\theta})\lambda} R(\lambda; A) d\lambda \right\|
\]

\[
\leq C e^{-2sr \cos \theta - \frac{r^2}{2}} \left( \frac{1}{2\pi i} \right) \int_{\Gamma_\phi} \left| e^{-\frac{1}{2} \lambda^2} e^{(s + r e^{i\theta})\lambda} \right| \| R(\lambda; A) \| |d\lambda|
\]

\[
\leq C e^{-2sr \cos \theta - \frac{r^2}{2}} \left( \frac{1}{2\pi i} \right) \int_{\Gamma_\phi} \left| e^{-\frac{1}{2} \lambda^2} |\lambda| e^{i\phi} e^{(s + r e^{i\theta})|\lambda| e^{i\phi}} \right| \| R(\lambda; A) \| |d\lambda|
\]

\[
\leq C e^{-2sr \cos \theta - \frac{r^2}{2}} \left( \frac{1}{2\pi i} \right) \int_{\Gamma_\phi} \left| e^{-\frac{1}{2} \lambda^2} |\lambda| \cos(\alpha\phi) e^{(s + r \cos \theta + ri \sin \theta)|\lambda| (\cos \phi + i \sin \phi)} \right| \| R(\lambda; A) \| |d\lambda|
\]

\[
\leq C e^{-2sr \cos \theta - \frac{r^2}{2}} \left( \frac{1}{2\pi i} \right) \int_{\Gamma_\phi} \left| e^{-\frac{1}{2} \lambda^2} |\lambda| \cos(\theta + \phi) \right| R(\lambda; A) |d\lambda|
\]
Lemma 15. \[ \psi(z) \text{ be a complex function with } z = \xi + \eta e^{i\theta}. \text{ Assume } \psi(z) \text{ is continuous and bounded on the strip } S = \{ \zeta = s + r e^{i\theta} | 0 \leq s \leq T, r \geq 0 \}. \text{ Define} \]
\[
\Phi(\zeta) = -\frac{1}{\pi} \int_{S} \int_{S} e^{-2i\phi} e^{\frac{1}{z - \zeta}} \dd \phi \dd \zeta \left( \frac{1}{z - \zeta} + \frac{1}{z + 1 + \zeta} \right) \dd \xi \dd \eta.
\]
Then $\Phi(\zeta)$ is absolutely convergent, $\bar{\partial}\Phi(\zeta) = \psi(z)$, and
\[
\int_{-\infty}^{\infty} \left| \frac{1}{z - \zeta} + \frac{1}{z + 1 + \zeta} \right| \, d\eta \leq K \left( 1 + \log \frac{1}{|\xi - s|} \right)
\] (8)
if $\xi \neq s$.

Now we state our result.

**Theorem 16.** Suppose $-A$ is the infinitesimal generator of a holomorphic semigroup of angle $\theta$, $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$ on a Banach space $X$. Let $f$ be a function for which $-f \in \mathcal{S} \cup \mathcal{P}_n$, where $n(\frac{\pi}{2} - \theta) < \pi$, and assume that $f$ satisfies Condition (A). Assume there exists a constant $\gamma$, independent of $\beta$ and $\omega$, such that $\|e^{\frac{\gamma(A)}{1+\delta}}\| \leq e^{\gamma|\zeta|}$ for $\zeta \in S_{\theta'}$, $\theta' = \min\{\theta, \frac{\pi}{2} - n(\frac{\pi}{2} - \theta)\}$.

Assume in addition that $h : [0, T) \to H$ is continuously differentiable with $h'(t) \in L^1(0, T)$ and $h(t) \in \text{Dom}(A^{1+\delta})$ for all $t \in [0, T)$. Let $u(t)$ and $v(t)$ be solutions of (5) and (7), respectively, with $u(0) = v(0) = \chi$, and assume $\|u(T)\| \leq M_1$, $\|A^{1+\delta} \chi\| \leq M_2$, and $\|A^{1+\delta} h(t)\| \leq M_3$. Then there exist constants $\tilde{C}$ and $M$, independent of $\beta$, such that for $0 \leq t < T$,
\[
\|u(t) - v(t)\|_1 \leq \tilde{C} \beta^{1-\omega(t)} M^{\omega(t)}
\]
where $\|\cdot\|_1 = \|C_{e^{\cdot}} \cdot\|$ and $\omega(\tau)$ is a harmonic function on the bent strip $S = \{\zeta = s + re^{\pm i\theta'} | 0 \leq s \leq T, r \geq 0\}$.

**Proof.** Here, our goal is to define an analytic function and apply Carleman’s Inequality (cf. [19]). For $\zeta \in S$, set
\[
\phi_e(\zeta) = u_e(\zeta) - v_e(\zeta).
\]
Apply the Cauchy–Riemann operator $\bar{\partial}$ to $\phi_e(\zeta)$, where
\[
\bar{\partial} = \frac{1}{2i\sin\theta} \left( e^{i\theta} \frac{\partial}{\partial s} - \frac{\partial}{\partial r} \right).
\]
We have
\[
\bar{\partial}\phi_e(\zeta) = \frac{1}{2i\sin\theta} \left( e^{i\theta} \frac{\partial}{\partial s}(u_e(\zeta) - v_e(\zeta)) - \frac{\partial}{\partial r}(u_e(\zeta) - v_e(\zeta)) \right)
\]
\[
= \frac{1}{2i\sin\theta} \left[ e^{i\theta} \frac{\partial}{\partial s}(W_e(r e^{i\theta}) u(s) - e^{i\theta} f(A) C_e v(s)) - \frac{\partial}{\partial r}(W_e(r e^{i\theta}) u(s) - e^{i\theta} f(A) C_e v(s)) \right]
\]
\[
= \frac{1}{2i\sin\theta} \left[ e^{i\theta} (W_e(r e^{i\theta}) (Au(s) + h(s)) - e^{i\theta} f(A) C_e (f(A)v(s) + h(s))) \right]
\]
\[
= \frac{1}{2i\sin\theta} \left[ e^{i\theta} (AW_e(r e^{i\theta}) u(s) - e^{i\theta} f(A)e^{i\theta} f(A) C_e v(s)) \right]
\]
Now, define

$$\Phi_\epsilon(\zeta) = -\frac{1}{\pi} \int_S e^{-e^{-2i\phi_\epsilon^2}} \partial \phi_\epsilon(z) \left( \frac{1}{z - \zeta} + \frac{1}{\bar{z} + 1 + \zeta} \right) d\xi d\eta,$$

where $z = \xi + \eta e^{i\phi}$. Note that the results of Lemma 15 hold for $\Phi_\epsilon$. For $\zeta \in S$ and $x^* \in X^*$, the dual space of $X$, define a new function $w_\epsilon$ as follows:

$$w_\epsilon(\zeta) = e^{-e^{-2i\phi_\epsilon^2}} x^* \phi_\epsilon(\zeta) - x^* \Phi_\epsilon(\zeta),$$

with $\alpha > 2$ and without loss of generality, $k = 0$ in the definition of $W_\epsilon(\zeta)$. We claim that $w_\epsilon$ is holomorphic on the interior of the bent strip $S$ and bounded and continuous on $S$, and hence that $w_\epsilon$ meets the criteria of Carleman’s Inequality. Again, we use the Cauchy–Riemann operator $\bar{\partial}$.

From Lemma 15, $\bar{\partial} \Phi_\epsilon(\zeta) = e^{-e^{-2i\phi_\epsilon^2}} \bar{\partial} \phi_\epsilon(\zeta)$. Then for $\zeta$ in the interior of $S$,

$$\bar{\partial} w_\epsilon(\zeta) = \bar{\partial} (e^{-e^{-2i\phi_\epsilon^2}} x^* \phi_\epsilon(\zeta) - x^* \Phi_\epsilon(\zeta))
= \bar{\partial} \left( e^{-e^{-2i\phi_\epsilon^2}} x^* \phi_\epsilon(\zeta) \right) - \bar{\partial} (x^* \Phi_\epsilon(\zeta))
= x^* \left( e^{-e^{-2i\phi_\epsilon^2}} \bar{\partial} (\phi_\epsilon(\zeta)) + [\bar{\partial} (e^{-e^{-2i\phi_\epsilon^2}})] \phi_\epsilon(\zeta) - e^{-e^{-2i\phi_\epsilon^2}} \bar{\partial} \phi_\epsilon(\zeta) \right)
= 0,$$

so $w_\epsilon$ is holomorphic on $S$.

For $\zeta = s + re^{i\theta}$,

$$|e^{-e^{-2i\phi_\epsilon^2}}| = e^{-s^2 \cos 2\theta - 2sr \cos \theta - r^2},$$

so we have

$$|w_\epsilon(\zeta)| = |e^{-e^{-2i\phi_\epsilon^2}} x^* \phi_\epsilon(\zeta) - x^* \Phi_\epsilon(\zeta)|
\leq e^{-s^2 \cos 2\theta - 2sr \cos \theta - r^2} \|x^*\| \|\phi_\epsilon(s + re^{i\theta})\| + \|x^*\| \|\Phi_\epsilon(s + re^{i\theta})\|.$$

Note that

$$A^{1+\delta} W_\epsilon(re^{i\theta}) u(s) = A^{1+\delta} W_\epsilon^2(re^{i\theta}) C_2 u(s)
= A^{1+\delta} W_\epsilon^2(re^{i\theta}) u_2^\epsilon(s)
= W_\epsilon^2(re^{i\theta}) A^{1+\delta} u_2^\epsilon(s),$$

using the definition of $W_\epsilon(\zeta)$, the fact that $A^{1+\delta}$ is closed, and the fact that $u_2^\epsilon(s) \in \text{Dom}(A^{1+\delta})$ since $W_\epsilon(\zeta)$ is an entire $C_\epsilon$-semigroup.

We need to show that $w_\epsilon(\zeta)$ is bounded on the bent strip $S$. First, consider $e^{-s^2 \cos 2\theta - 2sr \cos \theta - r^2} \|\phi_\epsilon(\zeta)\|$ on the sides of the strip. When $s = 0$ we have $\zeta = re^{i\theta}$, and so

$$\|\phi_\epsilon(\zeta)\| = \|u_\epsilon(re^{i\theta}) - v_\epsilon(re^{i\theta})\|
= \|W_\epsilon(re^{i\theta}) u(0) - e^{re^{i\theta} f(A)} C_\epsilon v(0)\|
= \|W_\epsilon(re^{i\theta}) x - e^{re^{i\theta} f(A)} C_\epsilon x\|
= \|W_\epsilon(re^{i\theta}) x - e^{re^{i\theta} g(A)} W_\epsilon(re^{i\theta}) x\|
= \|[I - e^{re^{i\theta} g(A)}] W_\epsilon(re^{i\theta}) x\|.$$
\[
= \left\| \int_0^r e^{te^{i\theta}} g(A) e^{(r+\Re e^{i\theta})t} \chi \, dt \right\|
\leq e^{\gamma r} \int_0^r \| g(A) e^{(r+\Re e^{i\theta})t} \chi \| \, dt
\leq re^{\gamma r} \beta \| A^{1+\delta} W \| \| e^{(r+\Re e^{i\theta})t} \chi \|
\leq re^{\gamma r} \beta \| W \| \| e^{(r+\Re e^{i\theta})t} \chi \| \| A^{1+\delta} C \chi \|,
\]
using Condition (A), Lemma 13, and the assumption on \( \| e^{\xi g(A)} \| \). Combining this result with Lemma 14, we have
\[
\left| e^{-e^{-2i\theta} (T+re^{i\theta})^2} \left\| \phi_e(T+re^{i\theta}) \right\| \leq e^{-r^2} e^{\gamma r} \beta \| W \| \| e^{(r+\Re e^{i\theta})t} \chi \| \| A^{1+\delta} C \chi \|
\leq C \beta \| A^{1+\delta} C \chi \|,
\]
which is bounded as a function of \( \xi \).

On the right-hand side of the strip, \( s = T \) and thus \( \xi = T + re^{i\theta} \), so we have
\[
\| \phi_e(\xi) \| = \left\| u_e(T + re^{i\theta}) - v_e(T + re^{i\theta}) \right\|
\leq \left\| W_e(re^{i\theta})u(T) - e^{re^{i\theta}f(A)} C_v v(T) \right\|
\leq \left\| W_e(re^{i\theta})u(T) - e^{re^{i\theta}g(A)} W_e(re^{i\theta}) v(T) \right\|
\leq \left\| W_e(re^{i\theta}) \| u(T) \| - e^{re^{i\theta}g(A)} \| v(T) \| \right\|
\leq \left\| W_e(re^{i\theta}) \| \| u(T) \| + e^{\gamma r} \| v(T) \| \right\|.
\]
From our stabilizing conditions and the fact that \( (g(A)\psi, \psi) \leq \gamma (\psi, \psi) \), we have that \( \| v(T) \| \) is bounded since
\[
\| v(T) \| = \left\| e^{Tf(A)} \chi + \int_0^T e^{(T-s)f(A)} h(s) \, ds \right\|
\leq \left\| e^{Tg(A)} e^{TA} \chi + \int_0^T e^{(T-s)g(A)} e^{(T-s)A} h(s) \, ds \right\|
\leq e^{T\gamma} \left\| e^{TA} \chi \right\| + \int_0^T e^{(T-s)\gamma} \| e^{(T-s)A} h(s) \| \, ds
\leq e^{T\gamma} L + e^{T\gamma} T \tilde{N}.
\]
Thus
\[
\left| e^{-e^{-2i\theta}(T+re^{i\theta})^2} \left\| \phi_e(T+re^{i\theta}) \right\|
\leq e^{-T^2 \cos 2\theta - 2Tr \cos \theta - r^2} \left\| W_e(re^{i\theta}) \| \| u(T) \| + e^{\gamma r} \| v(T) \| \right\|,
\]
(9)
which is bounded using our stabilizing conditions and the fact that \( \| v(T) \| \) is bounded, as shown in (9). So, on the sides of the strip, we have

\[
\| e^{-2i\phi} \phi_\epsilon(\zeta) \| \leq C_\beta \| A^{1+\delta} C_\zeta \| + \| W_\epsilon(\epsilon e^{i\theta}) \| \| u(T) \| + e^{yr} \| v(T) \|.
\]

To show that \( \Phi_\epsilon \) is bounded on the strip, we will examine \( \| \partial \phi_\epsilon(\zeta) \| \).

\[
\| \partial \phi_\epsilon(\zeta) \| = \left\| \frac{1}{2 \sin \theta} \left[ W_\epsilon(\epsilon e^{i\theta}) h(s) - e^{r e^{i\theta}} f(A) C_\epsilon h(s) \right] \right\|
\]

\[
= \left\| \frac{1}{2 \sin \theta} \left[ W_\epsilon(\epsilon e^{i\theta}) h(s) - e^{r e^{i\theta}} g(A) W_\epsilon(\epsilon e^{i\theta}) h(s) \right] \right\|
\]

\[
= \left\| \frac{1}{2 \sin \theta} \left[ I - e^{r e^{i\theta}} g(A) \right] W_\epsilon(\epsilon e^{i\theta}) h(s) \right\|
\]

\[
= \left\| \frac{1}{2 \sin \theta} \left( \int_0^r e^{r e^{i\theta}} g(A) W_\epsilon(\epsilon e^{i\theta}) h(s) \, dt \right) \right\|
\]

\[
\leq \frac{1}{\sqrt{2}} e^{yr} \left\| \int_0^r g(A) W_\epsilon(\epsilon e^{i\theta}) h(s) \, dt \right\|
\]

\[
\leq \frac{1}{\sqrt{2}} e^{yr} \beta \| A^{1+\delta} W_\epsilon(\epsilon e^{i\theta}) h(s) \|
\]

\[
\leq \frac{1}{\sqrt{2}} e^{yr} \beta \| W_2(\epsilon e^{i\theta}) \| A^{1+\delta} h_2(s) \|
\]

Again using Lemma 14, we obtain

\[
\| e^{-2i\phi} \partial \phi_\epsilon(\zeta) \| \leq \| e^{-2i\phi} \| \| \partial \phi_\epsilon(\zeta) \|
\]

\[
\leq \left( e^{-s^2 \cos 2\theta - 2sr \cos \theta - r^2} \right)^{1/2} e^{yr} \beta \| W_2(\epsilon e^{i\theta}) \| A^{1+\delta} h_2(s) \|
\]

\[
\leq \frac{1}{\sqrt{2}} e^{-s^2 \cos 2\theta} \beta e^{yr} e^{-2sr \cos \theta - r^2} \| W_2(\epsilon e^{i\theta}) \| A^{1+\delta} h_2(s) \|
\]

\[
\leq \frac{1}{\sqrt{2}} e^{-s^2 \cos 2\theta} \beta C \| A^{1+\delta} h_2(s) \|
\]

Therefore, for \( z = \xi + \eta e^{i\phi} \),

\[
\| \Phi_\epsilon(\zeta) \| = \left\| -\frac{1}{\pi} \int_S e^{-2i\phi} \frac{1}{2} \partial \phi_\epsilon(\zeta) \left( \frac{1}{z - \zeta} + \frac{1}{\bar{z} + 1 + \zeta} \right) d\xi \, d\eta \right\|
\]

\[
\leq K \int_0^T \left\| e^{-2i\phi} \frac{1}{2} \partial \phi_\epsilon(\zeta) \right\| \left( 1 + \log \frac{1}{|\xi - s|} \right) d\xi
\]

\[
\leq K \int_0^T \frac{1}{\sqrt{2}} e^{-s^2 \cos 2\phi} \beta C \| A^{1+\delta} h_2(\xi) \| \left( 1 + \log \frac{1}{|\xi - s|} \right) d\xi
\]
Note that \( h(\xi) \in \text{Dom}(A^{1+\delta}) \), so \( \|A^{1+\delta}h_2(\xi)\| = \|W_2 A^{1+\delta}h(\xi)\| \leq C\|A^{1+\delta}h(s)\| \). From our stability conditions we have \( \|A^{1+\delta}h(s)\| \leq M_3 \), so the above integral is bounded. Then by the maximum modulus principle, \( w_\epsilon \) is bounded on \( S \). We can also show that \( w_\epsilon \) is continuous on \( S \), so we have met the conditions for Carleman’s Inequality.

Using Carleman’s Inequality on \( w_\epsilon \), we obtain

\[
|w_\epsilon(s)| \leq \tilde{C} M(0)^{1-\omega(s)} M(T)^{\omega(s)},
\]

where \( M(s) = \sup_{r \geq 0} |w_\epsilon(s + re^{i\theta})| \), and \( \omega(s) \) is a harmonic function on \( S \). We have

\[
M(0) = \sup_{r \geq 0} |w_\epsilon(re^{i\theta})| \leq e^{-r^2} \|x^*\| \|\phi_\epsilon(re^{i\theta})\| + \|x^*\| \|\Phi_\epsilon(re^{i\theta})\|
\]

\[
\leq C\beta \|A^{1+\delta}C_2\chi\| \|x^*\| + \beta \left[ \tilde{K} \int_0^T \|A^{1+\delta}h_2(\xi)\| \left(1 + \log \frac{1}{|\xi|}\right) d\xi \right] \|x^*\|
\]

\[
\leq \beta \left[ C \|A^{1+\delta}C_2\chi\| + \tilde{K} \int_0^T \|A^{1+\delta}h_2(\xi)\| \left(1 + \log \frac{1}{|\xi|}\right) d\xi \right] \|x^*\|
\]

and

\[
M(T) = \sup_{r \geq 0} |w_\epsilon(T + re^{i\theta})| \leq e^{-T^2 \cos 2\theta - 2Tr \cos \theta - r^2} \|x^*\| \|\phi_\epsilon(T + re^{i\theta})\| + \|x^*\| \|\Phi_\epsilon(T + re^{i\theta})\|
\]

\[
\leq e^{-T^2 \cos 2\theta - 2Tr \cos \theta - r^2} \|W_\epsilon(re^{i\theta})\| \left[ \|u(T)\| + e^{\gamma r} \|v(T)\| \right] \|x^*\|.
\]

Thus for \( \epsilon > 0 \),

\[
|w_\epsilon(s)| \leq \tilde{C} \|x^*\| \left\{ \beta \left[ C \|A^{1+\delta}C_2\chi\| + \tilde{K} \int_0^T \|A^{1+\delta}h_2(\xi)\| \left(1 + \log \frac{1}{|\xi|}\right) d\xi \right] \right\}^{1-\omega(s)}
\]

\[
\times \left\{ e^{-T^2 \cos 2\theta - 2Tr \cos \theta - r^2} \|W_\epsilon(re^{i\theta})\| \left[ \|u(T)\| + e^{\gamma r} \|v(T)\| \right] \right\}^{\omega(s)}.
\]

Taking the supremum over \( x^* \in X^*, \|x^*\| \leq 1 \), we obtain

\[
\|w_\epsilon(s)\| \leq \tilde{C} \left\{ \beta \left[ C \|A^{1+\delta}C_2\chi\| + \tilde{K} \int_0^T \|A^{1+\delta}h_2(\xi)\| \left(1 + \log \frac{1}{|\xi|}\right) d\xi \right] \right\}^{1-\omega(s)}
\]

\[
\times \left\{ e^{-T^2 \cos 2\theta - 2Tr \cos \theta - r^2} \|W_\epsilon(re^{i\theta})\| \left[ \|u(T)\| + e^{\gamma r} \|v(T)\| \right] \right\}^{\omega(s)}.
\]

Note that \( \chi \in \text{Dom}(A^{1+\delta}) \), and so \( \|A^{1+\delta}C_2\chi\| = \|C_\epsilon A^{1+\delta}\chi\| \leq C\|A^{1+\delta}\chi\| \). As discussed above, \( \|v(T)\| \) is bounded, say by \( K \), and \( h(\xi) \in \text{Dom}(A^{1+\delta}) \), so \( \|A^{1+\delta}h_2(\xi)\| \leq M_3 \). Thus we have
\[ \| w_\epsilon(s) \| \leq \tilde{C} \left\{ \beta \left[ C_1 \| A^{1+\delta} \chi \| + \tilde{K} \int_0^T C M_3 \left( 1 + \log \frac{1}{|\xi|} \right) d\xi \right] \right\}^{1-\omega(s)} \times \left\{ e^{-T^2 \cos 2\theta - 2Tr \cos \theta - r^2} \| W_\epsilon (r e^{i\theta}) \| \left[ M_1 + e^{\gamma r K} \right] \right\}^{\omega(s)} \leq \tilde{C} \beta \left[ C_1 \| A^{1+\delta} \chi \| + C_2 \right]^{1-\omega(s)} \times \left\{ e^{-T^2 \cos 2\theta - 2Tr \cos \theta - r^2} \| W_\epsilon (r e^{i\theta}) \| \left[ M_1 + e^{\gamma r K} \right] \right\}^{\omega(s)} \leq \tilde{C} \beta^{1-\omega(s)} \left\{ e^{-T^2 \cos 2\theta - 2Tr \cos \theta - r^2} \| W_\epsilon (r e^{i\theta}) \| \left[ M_1 + e^{\gamma r K} \right] \right\}^{\omega(s)} \leq \tilde{C} \beta^{1-\omega(s)} M^{\omega(s)} \]

for a possibly different value of \( \tilde{C} \). It remains to show that \( \| u(t) - v(t) \|_1 \leq C \beta^{1-\omega(t)} M^{\omega(t)} \).

Recall that

\[ \| \Phi_\epsilon(\xi) \| \leq \beta \left[ \tilde{K} \int_0^T \| A^{1+\delta} h_\xi(\xi) \| \left( 1 + \log \frac{1}{|\xi - s|} \right) d\xi \right]. \]

We have

\[ \| e^{-2i\theta \epsilon^2} \Phi_\epsilon(t) \| = \| w_\epsilon(t) - \Phi_\epsilon(t) \| \leq \tilde{C} \beta^{1-\omega(t)} M^{\omega(t)} + \beta \left[ \tilde{K} \int_0^T \| A^{1+\delta} h_\xi(\xi) \| \left( 1 + \log \frac{1}{|\xi - s|} \right) d\xi \right] \leq \tilde{C} \beta^{1-\omega(t)} M^{\omega(t)} + \beta C_2 \leq \tilde{C} \beta^{1-\omega(t)} M^{\omega(t)} \]

for a possibly different value of \( \tilde{C} \) since \( \beta < 1 \). Hence we have

\[ \| u_\epsilon(t) - v_\epsilon(t) \| \leq \tilde{C} \beta^{1-\omega(t)} M^{\omega(t)}, \]

or

\[ \| u(t) - v(t) \|_1 \leq \tilde{C} \beta^{1-\omega(t)} M^{\omega(t)}. \]

4. Examples and applications

Consider the following example:

\[ \frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + h(t), \quad \text{where } 0 < x < c, \ 0 < t < T, \]
\[ u(0, t) = u(c, t) = 0, \quad 0 < t < T, \]
\[ u(x, 0) = k(x), \quad 0 < x < c, \]

(10)

the initial-boundary value problem for the backwards heat equation on the Banach space \( L^p(0, c) \), where \( 1 \leq p < \infty \). Here, \( A \) is given by \(-\Delta\), where \( \Delta \) is the Laplace operator, and
$k(x)$ is the initial data. $\text{Dom}(A)$ is the Sobolev space $W^{2,p}(0,c)$ with the added condition that for $w \in \text{Dom}(A)$, $w(0,t) = w(c,t) = 0$. In [3,7,16,18], this ill-posed problem is compared to the following well-posed problem:

$$\frac{\partial v}{\partial t} = -\frac{\partial^2 v}{\partial x^2} - \epsilon \frac{\partial^4 v}{\partial x^4} + h(t), \quad \text{where } 0 < x < c, \ 0 < t < T,$$

$$v(0,t) = v(c,t) = 0 \quad \text{for } 0 < t < T, \quad \text{and}$$

$$v(x,0) = k(x) \quad \text{for } 0 < x < c,$$

for $\epsilon > 0$. Following [7], let $f(\lambda) = \lambda - \epsilon \lambda^2$ for $0 < \epsilon < 1$. Note that $f$ is continuous and thus $f$ is a Borel function. We want to show $f$ satisfies Condition (A), so we need to show there exists $\omega \in \mathbb{R}$ such that $f(\lambda) \leq \omega$ for all $\lambda \in [0,\infty)$, and there exist positive constants $\beta, \delta$, with $0 < \beta < 1$, for which $\text{Dom}(A^{1+\delta}) \subseteq \text{Dom}(f(A))$ and

$$\|(-A + f(A))\psi\| \leq \beta \|A^{1+\delta}\psi\|$$

for all $\psi \in \text{Dom}(A^{1+\delta})$.

Note that $f$ satisfies Condition (A) with $\omega = \frac{1}{4\epsilon}$, $\beta = \epsilon$, and $\delta = 1$. We also need $-f \in \mathcal{S} \cup \mathcal{P}_n$ for $f$ to satisfy Condition (A). For $f(\lambda) = \lambda - \epsilon \lambda^2$, it is clear that $-f \in \mathcal{P}_2$. Also, $g(A) = -\epsilon A^2$ generates a semigroup of contractions, so we may choose some $\gamma \leq 0$ such that $(g(A)\psi, \psi) \leq \gamma' (\psi, \psi)$.

Following [23], we also may use

$$\frac{\partial v}{\partial t} = -\frac{\partial^2 v}{\partial x^2} + \epsilon \left(\frac{\partial^2 v}{\partial x^2}\right) \left(\frac{\partial v}{\partial t}\right) + h(t), \quad \text{where } 0 < x < c, \ 0 < t < T,$$

$$v(0,t) = v(c,t) = 0 \quad \text{for } 0 < t < T, \quad \text{and}$$

$$v(x,0) = k(x) \quad \text{for } 0 < x < c$$

as our approximate problem. Again, we have $A = -\Delta$ on the Banach space $L^p(0,c)$. Since

$$\frac{\partial v}{\partial t} = -\frac{\partial^2 v}{\partial x^2} + \epsilon \left(\frac{\partial^2 v}{\partial x^2}\right) \left(\frac{\partial v}{\partial t}\right) + h(t)$$

is equivalent to

$$\frac{\partial v}{\partial t} = \left(-\frac{\partial^2}{\partial x^2}\right) \left[\left(I - \epsilon \frac{\partial^2}{\partial x^2}\right)^{-1}\right] v(t) + h(t),$$

let $f(\lambda) = \lambda(1 + \epsilon \lambda)^{-1}$ (cf. [7]), where $\epsilon > 0$. $f$ satisfies Condition (A) with $\omega = \frac{1}{2}$, $\beta = \epsilon$, and $\delta = 1$ and $-f \in \mathcal{S}$. Again, $g(A)$ generates a semigroup of contractions, so we can choose $\gamma' \leq 0$. Hence for both $f(\lambda) = \lambda - \epsilon \lambda^2$ and $f(\lambda) = \lambda(1 + \epsilon \lambda)^{-1}$, $f$ meets Condition (A) and so by Theorem 16 we have

$$\|u(t) - v(t)\| \leq \tilde{C} \beta^{1-\omega(t)} M^{\omega(t)},$$

where $\omega(t)$ is a harmonic function.

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