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Gevrey's and trace regularity of a semigroup associated with beam equation and non-monotone boundary conditions ☆

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Abstract

The main aim of the paper is to present a technique that allows to infer wellposedness, trace and Gevrey's regularity of hyperbolic-like PDE's with non-monotone boundary conditions. The lack of monotonicity prevents applicability of the known semigroup methods.

In this paper we show how recently developed tools of microlocal analysis [D. Tataru, A priori estimates of Carleman's type in domains with boundary, J. Math. Pure Appl. 73 (1994) 353–387] combined with some spectral theory can be used successfully in order to obtain the needed inequalities. The method will be illustrated on a simple example of beam equation with non-monotone boundary conditions.

Keywords: Euler beam equations; Nondissipative boundary damping; Semigroup generation; Gevrey's regularity

1. Introduction

The focus of this paper is on second order in time PDE scalar equations with boundary conditions that are non-monotone. Since the boundary terms involved are not bounded by the topology

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of the underlying phase spaces, we deal with non-monotone problems which, moreover, are not amenable to perturbation or fixed point type of methods.

The aim of this work is to present technique that allows not only to prove wellposedness and appropriate energy estimates exhibited by traces of solutions, but also to infer (rather unexpected) regularity of solutions that is classified as Gevrey's class. In order to keep this paper focused and simple, we choose to illustrate the method on a simple example of beam equation. However, the methodology presented is applicable to more general, multidimensional problems.

Accordingly, we shall consider the following initial boundary value problem defined for the forced beam equation

$$u_{tt} + u_{xxxx} = f, \quad x \in \Omega = (0,1),\ t > 0, \tag{1.1}$$

with non-monotone boundary conditions given by

$$u(0,t) = u_x(0,t) = 0, \tag{1.2}$$

$$u_{xxx}(1,t) = 0, \qquad u_{xx}(1,t) = -ku_t(1,t) + b(t), \quad k \geqslant 0, \tag{1.3}$$

and initial conditions: $u(0,x) = u_0(x)$, $u_t(0,x) = u_1(x)$, $x \in \Omega$.

We are interested in wellposedness and regularity of the corresponding solutions. Wellposedness will be considered within the so called *finite energy space*, i.e. $H \equiv H^2_{cl}(\Omega) \times L_2(\Omega)$, where $H^2_{cl}(\Omega)$ is equipped with clamped boundary conditions at $x = 0$. The energy function associated with the model is standard and given by

$$E(t) = \frac{1}{2}\int_0^1 \left[\left|u_{xx}(t,x)\right|^2 + \left|u_t(t,x)\right|^2\right] dx.$$

The problem considered is a one-dimensional linear Euler–Bernoulli equation with feedback boundary conditions. This class of problems has been studied extensively in the literature, in fact in a much more challenging version when $\Omega \subset R^n$ (see [6,8] and many references therein). Thus, a natural question that arises is the following: *what is special about this particular model?* It turns out that *boundary conditions destroy natural dissipativity* of the underlying generator, thus raising fundamental question of wellposedness of finite energy solutions, and of validity of *some* energy inequality. On the other hand, this kind of boundary conditions arises naturally in modeling of rotating beams under boundary force feedback control [3]. Thus, the model is of both mathematical and physical interest.

In order to gain a better understanding of the problem and the questions raised let us recall that the *standard monotone* boundary conditions associated with (1.1) and (1.2) are the following:

$$u_{xxx}(x=1) = 0, \qquad u_{xx}(1,t) = -ku_{xt}(1,t). \tag{1.4}$$

Actually, the model (1.1) with $f = 0$, clamped end at $x = 0$ and absorbing moments as in (1.4) is a classical model of a contraction semigroup that is exponentially stable. (This property is well known not only for beams but also plates, where the analysis proves substantially more technical [4,7].) The energy identity for model (1.1), (1.2), (1.4) takes a very simple form

$$E(t) + k\int_0^t u^2_{xt}(1,s)\,ds = E(0) + \int_0^t \int_0^1 u_t(x,s) f(x,s)\,dx\,ds. \tag{1.5}$$

Thus, when $f = 0$ the dissipation rate is proportional to the square of $u_{xt}(x = 1, t)$. Instead, in the case of non-monotone boundary conditions (1.3), the situation is very different as no apparent dissipation rate emerges from the energetic calculations. Indeed, standard energy argument applied to unforced beam (with $f = 0$, $b = 0$) gives

$$E(t) + k \int_0^t u_t(1, s)u_{xt}(1, s)\,ds = E(0). \tag{1.6}$$

Thus, in contrast to (1.5), the energy relation in (1.6) does not provide (even with $f = 0$) any a priori bound for the energy. The boundary term does not seem to provide any information about an additional boundary regularity of solutions (which is the case in all problems with monotone boundary dissipation). Even more, boundary terms display a troublesome unboundedness on the boundary that is not controlled by the energy. In short, the non-monotone boundary conditions considered above do not seem to yield any dissipative law.

Based on the discussion above, one easily concludes that the problem is not within the realm of the theory of dissipative semigroups. This, of course, does not mean that there is no semigroup structure behind the model. However, should such exist it is definitely not obvious and of rather hidden structure. In fact, this issue has attracted attention of several researchers [3,11] who studied the problem, by Riesz basis techniques. On the other hand, it is well known that Riesz basis techniques, besides being computationally heavy, are limited in their applicability due to the famous "gap condition" that a priori restricts the analysis to—essentially—one-dimensional models. This has motivated our interest in studying the problem from a more intrinsic and general PDE point of view without any reliance on Riesz basis generation. Questions that are of particular interest in this study are the following: (i) what is the mechanism behind the generation and how the apparent lack of dissipation and appearance of the energy-unbounded boundary traces can be eventually mitigated by the dynamics of the problem? (ii) What is the structure of energy identity that provides information on some (rather peculiar) "smoothing" effect of boundary conditions? Finally (iii) how to quantitize an overall interior "smoothness" of the dynamics induced by the boundary conditions?

It is the goal of this paper to develop a technique that will provide an answer to the questions raised above. Surprisingly, the methods employed are not that elementary as perhaps dictated by the simplicity of the model. The main idea is to represent the original semi-flow as a suitable "perturbation" of a "good" semi-flow generated by dissipative boundary conditions similar to these in (1.4). We say "suitable," since the perturbation is defined only at the microlocal level. The main tool for achieving this is a technique, recently developed in [12], that allows for microlocal decomposition of the traces corresponding to hyperbolic-like equations. By using the microlocal analysis tools we will be able to exhibit some dissipative law, but valid only on a *finite time* horizon. This explains the fact that the semigroup is neither contractive nor dissipative. However, finite time dissipative law exhibits an additional regularizing effect caused by the boundary conditions. Further spectral investigations of the model reaffirm this regularizing effect and, in fact, allow to prove that the resulting semigroup is of *Gevrey's class*. This is stronger regularizing effect than just differentiability recently obtained in [3] by Riesz basis techniques.

Our main results are formulated below.

Theorem 1.1. *For any $k \geqslant 0$ the model* (1.1)–(1.3) *with $f = 0$ and $b = 0$ generates a strongly continuous semigroup e^{At} on $H \equiv H^2_{cl}(\Omega) \times L_2(\Omega)$ and the following energy inequality holds for the forced equation: There exists a constant $c_t > 0$ such that for all $t > 0$,*

$$E(t)+k\big|u_t(x=1)\big|^2_{H^{1/4}(0,t)}+k\big|u_{tx}(x=1)\big|^2_{H^{-1/4}(0,t)}$$
$$\leqslant c_t\left(E(0)+\int_0^t\big|f(s)\big|^2_{L_2(0,1)}\,ds+|b|^2_{H^{1/4}(0,t)}\right). \tag{1.7}$$

Remark 1.2. Note that the inequality in (1.7) implies for $k\geqslant 0$ strong regularizing effect on the boundary. There is a gain of $\frac{1}{4}$ time derivative for the velocity component.

Our next result describes an additional interior regularity gained by the dynamics. In order to formulate our result, we introduce the following definition.

Definition 1.3. A strongly continuous semigroup e^{At} is of Gevrey's class δ for $t>t_0$ if e^{At} is infinitely differentiable for $t>t_0$ and for every compact $K\in(t_0,\infty)$ and each $\theta>0$, there exists a constant $C=C(K,\theta)$ such that

$$\big\|\big(e^{At}\big)^{(n)}\big\|_{\mathcal{L}(H)}\leqslant C\theta^n(n!)^\delta,\quad \forall t\in K,\ n=0,1,2,\ldots.$$

Theorem 1.4. *Let $0<k\neq 1$. The semigroup e^{At}, introduced above with a generator A on H, is of Gevrey's class $\delta>2$ with $t_0=0$.*

Remark 1.5. Gevrey's regularity is described in terms of the bounds on all derivatives of the semigroup. These bounds are weaker than the corresponding ones corresponding to characterization of analyticity, but they are stronger than the ones corresponding to differentiability (see [2,10,13]).

Remark 1.6. The energy inequality in (1.7) allows to study semilinear problems with both interior and boundary nonlinear terms. Because of space limitations, this topic is not pursued here.

2. Energy inequality and generation of the semigroup

The proof of Theorem 1.1 is based on microlocal analysis. The main idea goes back to the so called microlocal decomposition of traces corresponding to boundary value problems [12,13]. Indeed, the goal is to express one boundary condition in terms of the remaining three modulo a perturbation that is "smooth." Since we already know that a "good" dissipative (monotone) feedback has the form $u_{xx}=-ku_{tx}$, $x=1$, our aim is to rewrite (microlocally) the imposed nondissipative boundary conditions with the term $-ku_t$. Of course, the price for doing this is the introduction of lower order terms that destroy contractivity of the semigroup. Thus, at the end of the process we obtain good energy estimate but polluted by lower order terms.

In what follows we shall use by now classical anisotropic notation $H^s_a(\Sigma)$ and $H^s_a(Q)$, denoting anisotropic Sobolev spaces that are of anisotropic order s (see [5,7,12]). By $H^s_a(Q)$ we mean that s derivatives in Ω and $\frac{s}{2}$ derivatives in time are square integrable. (This is in line with the canonical scaling of principal part of the operator corresponding to Euler–Bernoulli, Shrodinger and heat operators.)

2.1. Preparation for the proof of Theorem 1.1

Motivated by the considerations elaborated in the introduction, it is clear that the crux of the matter and difficulty of the problem lies in the boundary behavior of the underlying PDE. Thus, our main goal is to analyze this behavior and to derive appropriate estimates for the corresponding traces. The main task and technical effort goes into proving the following trace estimates for solutions to (1.1)–(1.3).

Lemma 2.1. *For any solution to* (1.1)–(1.3), *the following a priori trace regularity is valid*: $\forall t > 0$, $\exists C_{tk} > 0$ *such that*:

$$|u_t|^2_{H_a^{1/2}(\Sigma_t)} + |u_{tx}|^2_{H_a^{-1/2}(\Sigma_t)} \leqslant C_{t,k}\left[E(0) + \int_0^t \|f\|^2\,dt + |b|^2_{H_a^{1/2}(\Sigma_t)}\right], \tag{2.1}$$

where $\|u\| \equiv |u|_{L_2(0,1)}$ *and* $\Sigma_t \equiv \{x = 1\} \times (0, t)$.

The proof of the lemma proceeds through several supporting lemmas and propositions. Before we go into technical details involving microlocal analysis, we shall explain the main idea that allows to "conjecture" the regularity of the traces postulated by Lemma 2.1.

Analysis near the boundary $x = 1$. By applying standard partition of unity and localization we may consider the half space problem. The half space problem is then microlocalised. In line with standard convention $D_x = \frac{1}{i}\frac{d}{dx}$, so that $\frac{d}{dt} \to -is$; $\frac{d}{dx} \to -i\xi$, where $s \in R$ and $\xi \in R$ are dual variables.

Specializing to the one-dimensional case, the characteristic polynomial corresponding to Euler–Bernoulli model represented by differential operator $P = \frac{d^2}{dt^2} + \frac{d^4}{dx^4}$ becomes $p(\xi, s) = \xi^4 - s^2$. The strategy taken from [12,13] is to decompose the symbol $p(s, \xi)$ into the product of two polynomials such that one of them has at least one root (in the variable ξ) with a negative imaginary part. It is known [13] that the corresponding pseudodifferential operator is of parabolic type, hence it induces typical parabolic smoothness of the dynamics. The remaining part of the decomposition will correspond to backward diffusion (polynomial with a root of a positive imaginary part) and two branches of conservative waves (polynomials with real roots). Then, the idea developed in [12] is to "control" the non-diffusive part of the dynamics by only "three" boundary conditions, while the diffusive part will provide for lower order terms. Putting the above program into work leads to the following decomposition of the polynomial $p(s, \xi)$. $p(s, \xi)$ can be decomposed (with respect to the normal direction ξ) into p^+ and p^- where

$$p^- \equiv \xi + i\sqrt{|s|}, \qquad p^+ \equiv \bigl(\xi^2 - |s|\bigr)\bigl(\xi - i\sqrt{|s|}\bigr). \tag{2.2}$$

The variable ξ is a dual variable corresponding to the normal (to the boundary) direction, whereas s is an anisotropic dual variable corresponding to time differentiation. Since p^- has one root with negative imaginary part, this part of the dynamics has smoothing (parabolic) like character. Instead p^+ corresponds to backward diffusions and two conservative (wave type) dynamics (see [12,13]). The idea introduced in [12] is to express (algebraically) one boundary condition in terms of the remaining three boundary conditions modulo the symbol (third order polynomial) p^+. Since the term corresponding to p^+ will provide smooth (compact) contribution, the topological properties of the system will be driven by the decomposition. In our case the troublesome boundary trace corresponds to the symbol ξ^2. Thus our aim is to express ξ^2 as a linear combination of other traces. Without loss of generality we shall assume $s > 0$ (the analysis for $s < 0$ is completely analogous, hence omitted). We shall also denote by $\Sigma = R \times \{x = 1\}$ and by $T^*(\Sigma)$ cotangent bundle to the boundary Σ—see [12,13]. Thus, for every $(x = 1, t, s) \in T^*(\Sigma)$ we have the following decomposition:

$$\xi^2 = r_2(s) + r_1(s)\xi + r_{-1}(s)\xi^3 + S_{-1}(s)p^+(\xi, s), \tag{2.3}$$

where $r_i(s)$, $i = 1, 2, -1$, are tangential PDO of anisotropic order i, $S_{-1} \in S_a^{-1}(T^*(\Sigma))$ (see [12,13]). These operators are to be determined from algebraic relations (2.3) with $p^+(\xi, s)$ replaced by (2.2). Comparing the powers of the dual variable ξ we obtain $r_{-1}(s) + S_{-1}(s) = 0$, $1 = -i\sqrt{|s|}S_{-1}(s)$, and $r_1(s) - S_{-1}(s)s = 0$, $r_2(s) + iS_{-1}(s)\sqrt{|s|}s = 0$. This gives

$$S_{-1}(s) = \frac{i}{\sqrt{|s|}} \in S_a^{-1}\big(T^*(\Sigma)\big), \qquad r_{-1}(s) = \frac{-i}{\sqrt{|s|}} \in S_a^{-1}\big(T^*(\Sigma)\big),$$
$$r_1(s) = i\frac{s}{\sqrt{|s|}} \in S_a^{1}\big(T^*(\Sigma)\big), \qquad r_2(s) = s \in S_a^{2}\big(T^*(\Sigma)\big).$$

Combining with (2.3) yields

$$\xi^2 = s + i\frac{s}{\sqrt{|s|}}\xi - \frac{i}{\sqrt{s}}\xi^3 + \frac{i}{\sqrt{|s|}}p^+(\xi, s). \tag{2.4}$$

By exploiting boundary condition $u_{xxx}(1) = 0$ and $u_{xx}(1) = -ku_t(1)$ we obtain $\xi^3 = 0$, $\xi^2 = -iks$ and substituting into (2.4) yields

$$-isk = k\xi s\frac{1}{\sqrt{|s|}(1 + ik)} - kp^+(\xi, s)\frac{1}{\sqrt{|s|}(1 + ik)}. \tag{2.5}$$

On the other hand, the symbol of u_t is equal to $-is$ and the symbol u_{tx} is equal to $-s\xi$. Since boundary conditions imply that $u_t = -\frac{u_{xx}}{k}$ at $x = 1$, the symbol corresponding to $u_{xx}(x = 1)$ can be written as iks. The equality in (2.5), at the symbolic level, gives us the relation between the velocity of the normal derivative and the velocity of the trace on the boundary $x = 1$. PDE version of (2.5) is given below

$$-ku_t(1, t) = u_{xx}(1, t) = -Du_{tx}(1, t) - (\hat{P}u)(x = 1, t), \tag{2.6}$$

where PDO (pseudodifferential) operators $D(t)$ and $\hat{P}$ are represented by the following symbols

$$D \sim d(s), \quad d(s) \equiv \frac{k}{1 + ik}\big(\sqrt{|s|}\big)^{-1},$$
$$\hat{P} \sim \hat{p}(\xi, s) = \frac{k}{\sqrt{|s|}(1 + ik)}p^+(s, \xi).$$

The relation between symbols and operators is classical [13], i.e.: $Du(t) = \int_R d(s)\mathcal{F}u(s)e^{ist}\,ds$ and $Pu(x, t) = \int_R \hat{p}(D_x, s)\mathcal{F}u(x, s)e^{ist}\,ds$ where $\mathcal{F}u$ denotes Fourier's transform with respect to tangential (time) variable).

Since $\Re d(s) = \frac{k}{1+k^2}(\sqrt{|s|})^{-1}$, $d(s) \in S_a^{-1}(T^*(\Sigma))$ and $\hat{p}(\xi, s) \in S_a^2(Q)$, where $Q \equiv \Sigma \times (0, 1)$. In other words, D is a tangential operator of anisotropic order -1 and $\hat{P}$ is a second order PDO operator (also anisotropic) in all variables. In particular, due to microlocal Garding's inequality [13] the following coercivity property holds

$$\Re\int_R \big\langle Dz(s), z(s)\big\rangle\,ds \geqslant Ck|z|^2_{H_a^{-1/2}(R)} = Ck|z|^2_{H^{-1/4}(R)}, \tag{2.7}$$

where we denote complex inner products by $\langle u, v\rangle \equiv u\bar{v}$ and $(u, v) \equiv \int_0^1 u(x)\bar{v}(x)\,dx$. Since the symbol p^+ corresponds to smooth part of dynamics, the regularity of the solutions is driven by the boundary conditions

$$u_{xx}(1,t) = -Du_{tx}(1,t), \tag{2.8}$$

where D is a positive PDO operator with the symbol $\frac{k}{1+ik}(\sqrt{s})^{-1}$. The above problem is dissipative, hence the energy of the unforced equation is non-increasing. However, the original problem is "polluted" by the trace at $x = 1$ of the operator $\hat{P}(x,t)$. However, this operator corresponds to forward diffusion represented by p^-, hence it is smoothing. In order to quantify this last statement and to proceed rigorously with our program we need to introduce the backward adjoint problem. The reason for this is that the original variable $u(t)$ is not naturally defined for $t < 0$ (unless the initial conditions were zero, in which case we could extend (in t) solution to the entire real line). Instead, the adjoint variable will have a natural extension to the entire real line. This allows for rigorous application of the strategy explained above and based on microlocal analysis tools.

2.2. *The backward adjoint problem*

In what follows we shall perform microlocal analysis on the following adjoint problem. Let $T > 0$ be fixed. We consider:

$$\begin{aligned}
&z_{tt} + z_{xxxx} = 0, \quad \text{in } (0,1)\times(-\infty,T),\\
&z(x=0,t) = z_x(x=0,t) = 0, \qquad z_{xx}(x=1,t) = l(t),\\
&z_{xxx}(x=1,t) = -kz_{xt}(x=1,t) + g(t),\\
&z(x,T) = z_t(x,T) = 0, \quad \text{on } (0,1).
\end{aligned} \tag{2.9}$$

First of all we extend the variable z by zero for $t > T$, so $z(t)$ is defined for $t \in R$.

Step 1: *Decomposition on the boundary.* Microlocal decomposition of the boundary operators proceeds as follows:

$$\xi^3 = r_3(s) + r_2(s)\xi + r_1(s)\xi^2 + S_0(s)p^+(\xi,s). \tag{2.10}$$

Comparing the powers of the dual variable ξ we obtain

$$S_0(s) = 1, \qquad r_1(s) = i\sqrt{|s|}, \qquad r_2(s) = s, \qquad r_3(s) = -is\sqrt{|s|}.$$

The above gives, after accounting for the boundary condition $z_{xx}(t) = l(t)$,

$$\xi^3 = -is\sqrt{|s|} + s\xi - i\sqrt{|s|}l(s) + p^+(s,\xi). \tag{2.11}$$

Exploiting boundary conditions $z_{xxx}(1) = -kz_{tx}(1) + g(t)$ yields

$$i\xi^3 = ks\xi + g(s). \tag{2.12}$$

Substituting into (2.11) yields $ks\xi + g(s) = s\sqrt{|s|} + is\xi + \sqrt{|s|}l(s) + ip^+(\xi,s)$. Consequently

$$s\xi = \frac{k+i}{k^2+1}\bigl[s\sqrt{|s|} - g(s) + \sqrt{|s|}l(s) + ip^+(\xi,s)\bigr],$$

$$i\xi^3 = k\frac{k+i}{k^2+1}\bigl[s\sqrt{|s|} - g(s) + \sqrt{|s|}l(s) + ip^+(s,\xi)\bigr] + g(s), \tag{2.13}$$

$$z_{xxx}(1,t) = D_1 z_t(1,t) + P_1 z(x=1,t) + c_k g(t) + D_k l(t), \tag{2.14}$$

where the symbols of respective operators are given by $d_1(s) = k\frac{ki-1}{k^2+1}\sqrt{|s|}$ and $\tilde{p}_1(\xi, s) = k\frac{ki-1}{k^2+1}p^+(\xi, s)$, $c_k = \frac{ik}{k^2+1}$, $d_k = k\frac{k+i}{k^2+1}\sqrt{|s|}$. Similarly, from (2.13)

$$z_{tx}(1, t) = \frac{k+i}{k^2+1}g(t) + \frac{1}{k}D_k l(t) + \frac{1}{k}(P_1 z)(x = 1, t) + \frac{1}{k}D_1 z_t(1, t). \tag{2.15}$$

We note that $\Re\, d_1(s) = -\frac{k}{k^2+1}\sqrt{|s|}$. By Garding inequality [13]

$$-\Re \int_R \langle D_1 z(t), z(t)\rangle\, dt \geqslant C_1 k|z|^2_{H_a^{1/2}(R)} \geqslant C_1 k|z|^2_{H^{1/4}(R)} \tag{2.16}$$

as expected, after comparing to the analysis of u problem. The above inequality indicates "smoothing" of the $\frac{1}{4}$ time derivative. From (2.15) after noting that $d_1 \in S_a^1(T^*(\Sigma))$ and $d_k \in S_a^1(T^*(\Sigma))$, we also obtain

$$|z_{tx}|_{H_a^{-1/2}(R)} \leqslant C_k\big[|g|_{H_a^{-1/2}(R)} + |l|_{H_a^{1/2}(R)} + \big|(P_1 z)(x = 1)\big|_{H_a^{-1/2}(R)} + |z_t|_{H_a^{1/2}(R)}\big]. \tag{2.17}$$

Step 2: *Localization of z problem near the boundary.* We localize the adjoint equation (2.9) in the nbh of the boundary $x = 1$. This is done with a help of smooth cutoff functions $\phi(x)$ such that $\phi = 1$ in the nbh of $x = 1$ and has support in say $[1/2, 1]$. Since $\Box z = 0$ where $\Box \equiv -D_t^2 + \Delta^2 = \frac{\partial^2}{\partial t^2} + \Delta^2$, we have $\Box(\phi z) = [\Box, \phi]z \equiv R(z)$. Here $[A, B]$ denotes the commutator of differential operators A, B. With the above notation we can write down the Euler–Bernoulli equation in the form

$$P^- P^+ \phi z = R(z), \quad \text{for } (x, t, \xi, s) \in Q \times R^2,$$

where the commutator $R(z)$ is a third order operator in x. Denoting by $v = P^+\phi z$ we obtain that v is a solution to a "parabolic" problem with respect to normal direction. This is because p^- has a root with a negative imaginary part, so we will be solving $P^- v = R(z)$. Noting that v has a compact support at $x = 0$ we are in a position to apply parabolic energy estimates [12] (see [12, p. 372]). This yields

$$|v|_{H_a^{-\alpha}(\Sigma)} + |v|_{L_2(J; H_a^{-\alpha+1/2}(\Sigma))} \leqslant C\big|R(z)\big|_{L_2(J, H_a^{-\alpha-1/2}(\Sigma))}. \tag{2.18}$$

Here (in line with the notation in [12]; see also [7]) we denote $J = [1/2, 1]$. Parameter α is any real positive number allowing for rescaling in the inequality tangential derivatives.

Applying (2.18) with $\alpha = 1/2$ yields the following inequality for the "smooth" part P_1,

$$\big|\big(P^+\phi z\big)(x = 1)\big|_{H_a^{-1/2}(\Sigma)} + \big|P^+\phi z\big|_{L_2(J\times\Sigma)} \leqslant C\big|R(z)\big|_{L_2(J, H_a^{-1}(\Sigma))} \tag{2.19}$$

and since $\tilde{p}_1 = c_k p^+$, the above inequality gives

$$\big|P_1(t)\phi z(x = 1)\big|_{H_a^{-1/2}(\Sigma)} \leqslant C\big|R(z)\big|_{L_2(J, H_a^{-1}(\Sigma))}. \tag{2.20}$$

Now, let us analyze the effect of the commutator $R(z)$. Direct computations give $R(z) = [\Box, \phi]z = -[\frac{d^4}{dx^4}, \phi]z$. Thus, the principal part of this commutator is driven by $D_x^3 z$. This leads to

$$\big|R(z)\big|_{L_2(J, H_a^{-1}(\Sigma))} \leqslant C|z_{xxx}|_{L_2(J, H_a^{-1}(\Sigma))}. \tag{2.21}$$

On the other hand, from (2.9) $z_t \in L_2(J \times \Sigma) \Rightarrow D_x^4 z \in H^{-1}(R; L_2(J))$. Interpolating the above with $D_x^2 z \in L_2(J \times \Sigma)$ gives

$$\|z_{xxx}\|_{L_2(J, H_a^{-1}(\Sigma))} \leqslant C\big[\|z_t\|_{L_2(J\times\ \Sigma)} + \|z_{xx}\|_{L_2(J\times\Sigma)}\big]. \tag{2.22}$$

Estimates (2.21), (2.22) give

$$\big|R(z)\big|_{L_2(J, H_a^{-1}(\Sigma))} \leqslant C\big[|z_t|_{L_2(J\times\Sigma)} + |z_{xx}|_{L_2(J\times\Sigma)}\big] \tag{2.23}$$

and from (2.20)

$$\big|\big(P^+\phi\big)(x=1)\big|_{H_a^{-1/2}(\Sigma)} \leqslant C\big[|z_t|_{L_2(J\times\Sigma)} + |z_{xx}|_{L_2(J\times\Sigma)}\big]. \tag{2.24}$$

Step 3: *Energy inequality for the z problem.* Let $\Sigma_T = (t, T) \times \{x = 1\}$.

Lemma 2.2. *Let z be a solution to* (2.9). *Then for any* $0 \leqslant t \leqslant T$ *the following a priori inequality holds.*

$$\begin{aligned}
&\big\|z_t(t)\big\|^2 + \big\|z_{xx}(t)\big\|^2 + \big|z_t(1)\big|^2_{H_a^{1/2}(\Sigma_T)} \leqslant C_t\big[|g|^2_{H_a^{-1/2}(\Sigma_T)} + |l|^2_{H_a^{1/2}(\Sigma_T)}\big],\\
&\big|z_{xxx}(1)\big|^2_{H_a^{-1/2}(\Sigma_T)} + \big|z_{tx}(1)\big|^2_{H_a^{-1/2}(\Sigma_T)} \leqslant C_t\big[|g|^2_{H_a^{-1/2}(\Sigma_T)} + |l|^2_{H_a^{1/2}(\Sigma_T)}\big].
\end{aligned} \tag{2.25}$$

Proof. We begin with the estimate defined on $R \times (0, 1)$. To this end we introduce new variable $\hat z \equiv z(t)e^{\gamma t}$ for sufficiently large $\gamma > 0$. We also note that the estimates in Lemma 2.2 are invariant with respect to addition to the equation of lower order term, say Nz, where N will be chosen suitably large. Thus, the equation for the new variable becomes

$$\hat z_{tt} + \hat z_{xxxx} - 2\gamma \hat z_t + \big(\gamma^2 + N\big)\hat z = 0 \tag{2.26}$$

with terminal condition $\hat z(T) = \hat z_t(T) = 0$ and boundary conditions at $x = 1$ given by $\hat z_{xxx}(1) = -k\hat z_{tx} + k\gamma \hat z_x + \hat g$, $\hat z_{xx}(1) = \hat l$.

The following estimate is valid for $\hat z$ with N sufficiently large.

Proposition 2.3.

$$\begin{aligned}
&\int_R \big(\|\hat z_t\|^2 + \|\hat z_{xx}\|^2\big)\,dt + k\big|\hat z_t(1)\big|^2_{H_a^{1/2}(R)} + k\big|\hat z_{xxx}(1)\big|^2_{H_a^{-1/2}(R)} + k\big|\hat z_{tx}(1)\big|^2_{H_a^{-1/2}(R)}\\
&\quad \leqslant C\big[|\hat g|^2_{H_a^{-1/2}(R)} + |\hat l|^2_{H_a^{1/2}(R)}\big].
\end{aligned} \tag{2.27}$$

Proof. Standard energy identity along with zero values at T give

$$\int_R \big[2\gamma\|\hat z_t\|^2 - \big\langle \hat z_{xxx}(1), \hat z_t(1)\big\rangle + \big\langle \hat l, \hat z_{tx}(1)\big\rangle\big]\,dt = 0. \tag{2.28}$$

By using (2.14) where we replace g by $\hat g + k\gamma\hat z_x(1)$ we rewrite the first boundary integral as follows:

$$\begin{aligned}
\int_R \big\langle \hat z_{xxx}(1), \hat z_t(1)\big\rangle\,dt = \int_R &\big(\big\langle D_1\hat z_t(1), \hat z_t(1)\big\rangle + \big\langle P_1\hat z(x=1), \hat z_t(1)\big\rangle\\
&+ \big\langle c_k\big(k\gamma\hat z_x(1) + \hat g\big) + D_k\hat l(t), \hat z_t(1)\big\rangle\big)\,dt
\end{aligned} \tag{2.29}$$

combining with (2.26) gives

$$2.30 \int_R \left[2\gamma \|\hat{z}_t\|^2 - \langle D_1\hat{z}_t(1), \hat{z}_t(1)\rangle - \langle P_1\hat{z}(x=1), \hat{z}_t(1)\rangle\right] dt = \int_R \langle c_k\big(k\gamma\hat{z}_x(1) + \hat{g}\big) + D_k\hat{l}, \hat{z}_t(1)\rangle\, dt. \tag{2.30}$$

Exploiting (2.16) and (2.21) applied with $\hat{z}$ and taking real parts gives

$$\begin{aligned} 2\gamma \int_R \|\hat{z}_t\|^2\, dt + ck\big|\hat{z}_t(1)\big|^2_{H_a^{1/2}(R)} &\leqslant \big|P_1\hat{z}(x=1)\big|_{H_a^{-1/2}(R)}\big|\hat{z}_t(1)\big|_{H_a^{1/2}(R)} + |l|_{H_a^{1/2}(R)}\big|z_{tx}(1)\big|_{H_a^{-1/2}(R)} \\ &\quad + c_k\big(k\gamma\big|\hat{z}_x(1)\big|_{H_a^{-1/2}(R)}\big|\hat{z}_t(1)\big|_{H_a^{1/2}(R)} + |\hat{g}|_{H_a^{-1/2}(R)}\big|\hat{z}_t(1)\big|_{H_a^{1/2}(R)}\big) \end{aligned} \tag{2.31}$$

and by (2.17), (2.24) along with $|\hat{z}_x(1)| \leqslant \epsilon\|\hat{z}_{xx}\| + C_\epsilon\|\hat{z}\|$, where we take $\epsilon \sim \frac{1}{\gamma}$,

$$\begin{aligned} 2\gamma \int_R \|\hat{z}_t\|^2\, dt + ck\big|\hat{z}_t(1)\big|^2_{H_a^{1/2}(R)} &\leqslant C_k \int_R \left[\|\hat{z}_t\|^2 + \|\hat{z}_{xx}\|^2 + \gamma^2\|z|^2\right] dt \\ &\quad + C_k\big(|\hat{g}|^2_{H_a^{-1/2}(R)} + |\hat{l}|^2_{H_a^{-1/2}(R)}\big). \end{aligned} \tag{2.32}$$

Our last step is the estimate for the potential energy. This is achieved by exploiting "equipartition" of the energy.

Multiplying (2.26) by $\hat{z}$ and integrating by parts we obtain

$$\int_R \big(\|\hat{z}_{xx}\|^2 + \big(\gamma^2 + N\big)\|\hat{z}\|^2 - \|\hat{z}_t\|^2 + \langle\hat{z}_{xxx}(1), \hat{z}(1)\rangle - \langle l, \hat{z}_x(1)\rangle\big)\, dt = 0. \tag{2.33}$$

We estimate the boundary term by using boundary conditions and integrating by parts in t,

$$\int_R \hat{z}_{xxx}(1)\hat{z}(1)\, dt = \int_R \left[k\hat{z}_x(1)\hat{z}_t(1) + \gamma\hat{z}_x(1)\hat{z}(1) + \hat{g}\hat{z}(1)\right] dt \tag{2.34}$$

hence, by Sobolev's embeddings, interpolation inequalities and trace theorem along with $|\hat{z}_x(1)| \leqslant \epsilon\|\hat{z}_{xx}\| + C_\epsilon\|\hat{z}\|$,

$$\int_R \big|\hat{z}_{xxx}(1)\hat{z}(1)\big|\, dt \leqslant \epsilon\big[|\hat{z}_t|^2_{H_a^{1/2}(R)} + \|\hat{z}_{xx}\|^2\big] + C_\epsilon|\hat{g}|^2_{H_a^{1/2}(R)} + C_{\gamma,\epsilon}\int_R \|\hat{z}\|^2\, dt \tag{2.35}$$

and combining with (2.33) after taking ϵ small gives

$$\int_R \left[\|\hat{z}_{xx}\|^2 + \big(\gamma^2 + N - C_{\epsilon,\gamma}\big)\|\hat{z}\|^2 - \|\hat{z}_t\|^2\right] dt \leqslant C\left[|\hat{g}|^2_{H_a^{1/2}(R)} + |\hat{l}|^2_{H_a^{1/2}(R)}\right]. \tag{2.36}$$

The above inequality along with (2.32) after taking γ large (to absorb $\|\hat{z}_t\|^2$) and N large (to absorb $\|\hat{z}\|^2$) yields the estimate for the first three terms in Proposition 2.3. The estimate for $|z_{tx}|_{H_a^{-1/2}(R)}$ and $|z_{xxx}|_{H_a^{-1/2}(R)}$ follows now from (2.14), (2.15) and (2.36), (2.32). This concludes the proof of Proposition 2.3. □

An immediate consequence of Proposition 2.3 is that the same estimates hold on every finite interval $[t, T]$. Even more, we can replace $\hat{z}$ by the original z at the price of adding to the RHS the term $N\int_t^T \|z\|^2$. To continue with the proof of Lemma 2.2, we apply energy method to the backward problem, after taking advantage of Proposition 2.3.

$$\begin{aligned}\|z_t(t)\|^2 + |z_{xx}(t)|^2 &= 2\int_t^T z_{xxx}(1,t)z_t(1,t)\,dt \\ &\leqslant 2|z_{xxx}(1,t)|_{H_a^{-1/2}(\Sigma_T)}|z_t(1,t)|_{H_a^{1/2}(\Sigma_T)} \\ &\leqslant C_k\left[|g|^2_{H_a^{-1/2}(\Sigma_T)} + |l|^2_{H_a^{1/2}(\Sigma_T)} + C\int_t^T \|z\|^2\,dt\right]. \end{aligned} \tag{2.37}$$

Cronwall's inequality provides the estimate for the energy terms. Combining this with Proposition 2.3 completes the proof of Lemma 2.2. □

2.3. *Proper proof of Lemma 2.1*

The argument to be used exploits (2.25) along with duality. The goal is to obtain trace estimates for the original problem u given by (1.1)–(1.3).

$$\begin{aligned}\int_t^T (f, z_t)\,ds &= \int_t^T \big[(\Box u, z_t) + (u_t, \Box z)\big]\,ds = \int_t^T \frac{d}{dt}\big[(u_t, z_t) + (u_{xx}, z_{xx})\big]\,ds \\ &\quad + \int_t^T \big[u_t(1,t)z_{xxx}(1,t) - u_{xx}(1,t)z_{tx}(1,t) - u_{tx}(1)z_{xx}(1)\big]\,ds \\ &= -\big(u_t(t), z_t(t)\big) + \big(u_{xx}(t), z_{xx}(t)\big) + \int_t^T \big[u_t(1,t)(-k)z_{tx}(1,t) - u_{tx}(1)l \\ &\quad + ku_t(1,t)z_{tx}(1,t) + u_t(1,t)g(t) + z_{tx}(1,t)b(t)\big]\,dt. \end{aligned} \tag{2.38}$$

Taking $t = 0$ in (2.38) and evoking (2.25), (2.15)

$$\begin{aligned}&\left|\int_0^T u_t(1,t)g(t)\,dt - \int_0^T u_{tx}l(t)\,dt\right| \\ &\quad \leqslant \|u_t(0)\|\,\|z_t(0)\| + \|u_{xx}(0)\|\,\|z_{xx}(0)\| + \left|\int_0^T (f, z_t)\,dt + \int_0^T z_{tx}(1)b\,dt\right| \\ &\quad \leqslant C\big(|g|_{H_a^{-1/2}(\Sigma_T)} + |l|_{H_a^{1/2}(\Sigma_T)}\big)\left(\|u_t(0)\| + \|u_{xx}(0)\| + \int_0^T \|f\|\,dt + |b|_{H_a^{1/2}(\Sigma_T)}\right). \end{aligned} \tag{2.39}$$

Thus, by Riesz representation theorem, arbitrariness of l and g and arbitrariness of T we obtain from (2.39) the estimate (2.1) stated in Lemma 2.1

2.4. Completion of the proof of Theorem 1.1

Lemma 2.1 provides the estimate for the boundary traces. In order to complete the proof of energy inequality in (1.7) we need to estimate the interior norms. This can be done now by standard energy estimate applied to u problem. Indeed, multiplying (1.1) by u_t and integrating by parts gives

$$\|u_t(t)\|^2 + \|u_{xx}(t)\|^2 - 2\int_0^t \langle u_{xx}(1), u_{xt}(1)\rangle\,dt = 2\int_0^t (f, u_t)\,dt + \|u_t(0)\|^2 + \|u_{xx}(0)\|^2. \tag{2.40}$$

Exploiting boundary conditions

$$\|u_t(t)\|^2 + \|u_{xx}(t)\|^2 \leqslant 2\big|ku_t(1) + b\big|_{H_a^{1/2}(\Sigma_t)}\big|u_{xt}(1)\big|_{H_a^{-1/2}(\Sigma_t)} + 2\int_0^t \|f\|\|u_t\|\,dt + \|u_t(0)\|^2 + \|u_{xx}(0)\|^2 \tag{2.41}$$

and recalling the result of Lemma 2.1

$$E(t) \leqslant C_{tk}\left[E(0) + |b|^2_{H_a^{1/2}(\Sigma_t)} + \int_0^t \|f\|^2\,dt + \int_0^t E(\tau)\,d\tau\right], \tag{2.42}$$

application of Gronwall's lemma together with Lemma 2.1 completes the proof of energy inequality (1.7).

The proof of generation of the semigroup e^{At} on H, due to linearity of the problem, is now standard. We exploit a priori estimates for the original and the adjoint problem. See the proof of Theorem 3 in [9].

3. Gevrey's regularity

Applying Laplace transform to the initial boundary value problem (1.1)–(1.3) yields the boundary value problem

$$u_{xxxx} + \lambda^2 u = 0, \quad u(0) = u_x(0) = u_{xxx}(1) = u_{xx}(1) + \lambda k u(1) = 0, \tag{3.1}$$

that may be rewritten as a system

$$u_{xxxx} + \lambda v = 0, \qquad v - \lambda u = 0, \tag{3.2}$$

$$u(0) = u_x(0) = u_{xxx}(1) = u_{xx}(1) + kv(1) = 0. \tag{3.3}$$

It is straightforward to verify that the generator A has the form

$$A \equiv \begin{pmatrix} 0 & I \\ -D_x^4 & 0 \end{pmatrix}$$

with $D(A) \equiv \{(u,v) \in H,\ v \in H^2_{cl}(\Omega),\ u \in H^4(\Omega),\ u_{xx}(1) = -kv(1),\ u_{xxx}(1) = 0\}$. Let $R(\lambda, A) = (\lambda I - A)^{-1}$ denote resolvent operator corresponding to the generator A. The crux of the proof of Theorem 1.4 is based on the following estimate for the resolvent valid on the imaginary axis.

Lemma 3.1. *Let $0 < k$ and $k \neq 1$. Then the following estimate takes place*:

$$\lim_{|\tau|\to\infty} \left\| R(i\tau, A)\right\|^2_{\mathcal{L}(H)} \leqslant \frac{C}{|\tau|}, \quad \tau \in R.$$

Indeed, Lemma 3.1 along with Theorem 1.1 and Theorem T.4 in [1] (see also [2, Theorem 1.1]) imply that e^{At} has Gevrey's regularity of order $\delta > 2$, $t_0 = 0$.

Proof of Lemma 3.1. Let us begin by writing down explicitly the resolvent operator. We find from (3.2):

$$(u,v) = R(\lambda, A)(f,g) \iff u - v = f, \qquad \lambda v + u_{xxxx} = g \tag{3.4}$$

with the boundary conditions

$$u(0) = u_x(0) = 0, \qquad u_{xx}(1) = -kv(1), \qquad u_{xxx}(1) = 0. \tag{3.5}$$

This is equivalent solving

$$\begin{aligned} &u_{xxxx} + \lambda^2 u = g + \lambda f, \\ &u(0) = u_x(0) = 0, \qquad u_{xx}(1) = -kv(1), \qquad u_{xxx}(1) = 0. \end{aligned} \tag{3.6}$$

Thus, the result of Lemma 3.1 is established as soon as we prove:

$$\|u_{xx}\|^2_{L_2(0,1)} + \|v\|^2_{L_2(0,1)} \leqslant \frac{C}{|\lambda|}\left[\|f\|^2_{H^2_{cl}(0,1)} + \|g\|^2_{L_2(0,1)}\right], \tag{3.7}$$

where the estimate is uniform for all $\lambda = i\tau$ with $|\tau|$ large. The rest of the paper is devoted to the proof of this inequality. All absolute constants that appear in the proof will be denoted by the same letter C with different indexes. Introduce the following selfadjoint Sturm–Liouville problem,

$$w'''' - \lambda^2 w = 0, \quad x \in (0,1), \qquad w(0) = w'(0) = w'''(1) = w''(1) = 0.$$

The standard methods of the operator theory imply that the problem has a discrete positive spectrum $\{\lambda_n^2\}$ with the only point of accumulation at infinity, and the eigenfunctions $\{w_n(x)\}$ form an orthogonal basis in both $L_2(0,1)$ and $H^2(0,1)$, $\int w_i'' w_m'' = \int w_i w_m = 0$ if $i \neq m$, $\int (w_m'')^2 = \lambda_m^2 \int w_m^2$. It is easy to find the following form of the eigenfunction,

$$w_n(x) = N_n \times \left(\sin\sqrt{\lambda_n}x - \sinh\sqrt{\lambda_n}x - Q(\lambda_n)(\cos\sqrt{\lambda_n}x - \cosh\sqrt{\lambda_n}x)\right), \tag{3.8}$$

$$Q(\lambda_n) = \frac{\sinh\sqrt{\lambda_n} + \sin\sqrt{\lambda_n}}{\cos\sqrt{\lambda_n} + \cosh\sqrt{\lambda_n}}. \tag{3.9}$$

Here we introduce the normalization factor N_n so that

$$\int w_n^2 = 1, \quad \text{and hence} \quad \int \left(w_n''\right)^2 = \lambda_n^2. \tag{3.10}$$

It may be shown that

$$N_n^2 = \left(\frac{1-\cos\sqrt{\lambda_n}}{1+\cos\sqrt{\lambda_n}}\right)^{\sigma_n} = \frac{1-\sigma_n\cos\sqrt{\lambda_n}}{1+\sigma_n\cos\sqrt{\lambda_n}}$$
$$\text{with } \sigma_n = 1, \text{if } \tan\sqrt{\lambda_n} > 0,\ \sigma_n = -1,\ \text{if } \tan\sqrt{\lambda_n} < 0. \tag{3.11}$$

The equation for eigenvalues has the form

$$1 + \cos\sqrt{\lambda_n}\cosh\sqrt{\lambda_n} = 0. \tag{3.12}$$

An elementary asymptotic analysis of (3.12) shows that the eigenvalues have the following form:

$$\sqrt{\lambda_n} \asymp \pi n + \frac{\pi}{2} + 2(-1)^n e^{-(\pi n+\pi/2)} \quad \text{as } n \to \infty, \tag{3.13}$$

and hence N_n are bounded below and above. The following identities directly follow from (3.8) and (3.12),

$$\cosh\sqrt{\lambda_n} = -\frac{1}{\cos\sqrt{\lambda_n}}, \qquad \sinh\sqrt{\lambda_n} = \tan\sqrt{\lambda_n}\,\sigma_n,$$
$$w_n(1) = \frac{2N_n}{\sin\sqrt{\lambda_n}}(1+\sigma_n\cos\sqrt{\lambda_n}), \qquad w_n'(1) = -2N_n\sqrt{\lambda_n}\,\sigma_n. \tag{3.14}$$

The following estimates directly follow from (3.14):

$$\left|w_n'(1)\right| \leqslant C_1\sqrt{\lambda_n}, \qquad \left|w_n(1)\right| \leqslant C_2. \tag{3.15}$$

Multiplying the equations for resolvent (3.4) by w_n, integrating over (0, 1), integrating by parts, and using boundary conditions (3.5) yields

$$kv(1)w_n'(1) + \lambda_n^2\int uw_n + \lambda\int vw_n = \int gw_n,$$
$$-\int vw_n + \lambda\int uw_n = \int fw_n,$$

which implies

$$\int uw_n = \frac{1}{\lambda_n^2+\lambda^2}\left[-kv(1)w_n'(1) + \int gw_n + \lambda\int fw_n\right],$$
$$\int vw_n = \frac{1}{\lambda_n^2+\lambda^2}\left[-\lambda kv(1)w_n'(1) + \lambda\int gw_n - \lambda_n^2\int fw_n\right].$$

Represent u, v, f, g as the (orthogonal) Fourier series with respect to the basis $\{w_n\}$, with Fourier coefficients denoted respectively by u_n, v_n, f_n, g_n. The normalization of the eigenfunctions (3.10) implies $u_n = \int uw_n$, $v_n = \int vw_n$, etc., so that $\|u_{xx}\|^2 = \sum_n \lambda_n^2|u_n|^2$, $\|v\|^2 = \sum_n |v_n|^2$,

$$\|f_{xx}\|^2 = \sum_n \lambda_n^2|f_n|^2, \qquad \|g\|^2 = \sum_n |g_n|^2. \tag{3.16}$$

We find:

$$u(x) = \sum_n \frac{1}{\lambda_n^2+\lambda^2} \times \left[-kv(1)w_n'(1) + \int gw_n + \lambda\int fw_n\right]w_n(x), \tag{3.17}$$

$$v(x) = \sum_n \frac{1}{\lambda_n^2+\lambda^2} \times \left[-\lambda kv(1)w_n'(1) + \lambda\int gw_n - \lambda_n^2\int fw_n\right]w_n(x). \tag{3.18}$$

The last equation implies

$$v(1) = \frac{1}{S(\lambda)} \sum_j \frac{1}{\lambda_j^2 + \lambda^2} \left[\lambda g_j - \lambda_j^2 f_j\right] w_j(1) \tag{3.19}$$

with

$$S(\lambda) = 1 + \lambda k \sum_m \frac{w_m(1) w_m'(1)}{\lambda_m^2 + \lambda^2}. \tag{3.20}$$

We further need an identity for $S(\lambda)$ given by (3.20),

$$S(\lambda) = 1 + \lambda k \sum_m \frac{w_m(1) w_m'(1)}{\lambda_m^2 + \lambda^2} = 1 + ik \frac{\sin\sqrt{-i\lambda}\sinh\sqrt{-i\lambda}}{1 + \cos\sqrt{-i\lambda}\cosh\sqrt{-i\lambda}}. \tag{3.21}$$

Indeed, both functions are meromorphic and the direct calculation shows that their residues at $\lambda = \pm i\lambda_n$ coincide. Further, it is easy to construct a system of extended contours such that the RHS of (3.21) is uniformly bounded on it. All of the above proves identity (3.21) by referring to the standard results on the theory of meromorphic functions. Comment. The eigenfunctions of the original problem (see (3.6) with $f = g = 0$) have the form

$$u_n(x) = \sin\sqrt{-i\mu_n}x - \sinh\sqrt{-i\mu_n}x - T_n(\cos\sqrt{-i\mu_n}x - \cosh\sqrt{-i\mu_n}x)$$

with $T_n = (\cos\sqrt{-i\mu_n} + \cosh\sqrt{-i\mu_n})/(\sinh\sqrt{-i\mu_n} - \sin\sqrt{-i\mu_n})$ and the eigenvalues μ_n satisfying

$$1 + \cos\sqrt{-i\mu_n}\cosh\sqrt{-i\mu_n} + ik\sin\sqrt{-i\mu_n}\sinh\sqrt{-i\mu_n} = 0, \tag{3.22}$$

which is, by (3.21), equivalent to $S(\mu) = 0$. Location of the eigenvalues μ_n satisfying (3.22) is described by the following

Lemma 3.2. *Let $0 < k \neq 1$. Then the only point of accumulation of $\{\mu_n\}$ is ∞. There is no more than finite number of eigenvalues on the bisectrix $\Im\mu = \pm\Re\mu$. For large n, the eigenvalues have the asymptotic form*

$$\mu_n \asymp -2\pi nT + i\left((\pi n)^2 - T^2\right), \tag{3.23}$$

$$\text{where } T \equiv \tanh^{-1}\frac{1}{k} = \frac{1}{2}\left(\ln\left|\frac{k+1}{k-1}\right| + i\arg\frac{k+1}{k-1}\right).$$

Hence, the spectrum is asymptotically located on the parabola. For $k > 1$, it has the form:

$$\Im\mu \asymp \frac{(\Re\mu)^2}{4T^2} - T^2.$$

According to (3.16)–(3.19), the desired inequality (3.7) is equivalent to the following (we may assume τ to be large and positive):

$$\begin{aligned}
&\sum_n \frac{\lambda_n^2}{(\lambda_n^2 - \tau^2)^2}\left|g_n + i\tau f_n - \frac{kw_n'(1)}{S(i\tau)}\sum_j \frac{w_j(1)}{\lambda_j^2 - \tau^2}\left(i\tau g_j - \lambda_j^2 f_j\right)\right|^2 \\
&\quad + \sum_n \frac{1}{(\lambda_n^2 - \tau^2)^2}\left|i\tau g_n - \lambda_n^2 f_n - i\tau\frac{kw_n'(1)}{S(i\tau)}\sum_j \frac{w_j(1)}{\lambda_j^2 - \tau^2}\left(i\tau g_j - \lambda_j^2 f_j\right)\right|^2 \\
&\leqslant \frac{C_3}{\tau}\left(\sum_n \lambda_n^2|f_n|^2 + |g_n|^2\right).
\end{aligned} \tag{3.24}$$

Substitution $\lambda_n f_n \mapsto f_n$ and $\tau = s^2$, $s > 0$, yield a more convenient form of the last inequality:

$$\sum_n \frac{1}{(\lambda_n^2 - s^4)^2}\left(\left|\lambda_n g_n + is^2 f_n - \frac{k\lambda_n w_n'(1)}{S(is^2)}\sum_j \frac{w_j(1)}{\lambda_j^2 - s^4}\big(is^2 g_j - \lambda_j f_j\big)\right|^2 + \left|is^2 g_n - \lambda_n f_n - is^2\frac{kw_n'(1)}{S(is^2)}\sum_j \frac{w_j(1)}{\lambda_j^2 - s^4}\big(is^2 g_j - \lambda_j f_j\big)\right|^2\right) \leqslant \frac{C_3}{s^2}\left(\sum_n |f_n|^2 + |g_n|^2\right). \tag{3.25}$$

According to (3.21),

$$S\big(is^2\big) = 1 + ik\frac{\sin s \sinh s}{1 + \cos s \cosh s} \quad \Rightarrow \quad \big|S\big(is^2\big)\big| \geqslant 1.$$

Note, we may leave only a "tail" of the series in the LHS and RHS because any finite sum in the LHS may be estimated as

$$O\big(s^{-4}\big)\sum_{n=1}^{N}\big(|f_n|^2 + |g_n|^2\big) \leqslant \frac{C_4}{s^2}\sum_{n=1}^{N}\big(|f_n|^2 + |g_n|^2\big).$$

Hence, we may substitute λ_n by its asymptotic representation (3.13), $\lambda_n \asymp N_n^2$, $N_n \equiv n + \frac{1}{2}$, where we remove π by scaling s. The expressions inside absolute values in the LHS of (3.24) and (3.25) are equal to zero as $s \to N_n$, and hence, by their analyticity, the LHS do not have any discontinuity at $s = N_n$. Yet, we prove the estimate (3.25) separately for $s \neq N_n$ and $s = N_n$. Let first s be uniformly separated from all N_n. We find an estimate for the LHS of (3.25):

$$|\mathrm{LHS}| \leqslant C_5 \sum_n \frac{1}{(N_n + s)^2(N_n^2 + s^2)^2} \times \Bigg(N_n^4|g_n|^2 + s^4|f_n|^2 + C_6 N_n^6 \sum_j \frac{1}{(N_j + s)^2(N_j^2 + s^2)^2}\big(s^4|g_j|^2 + N_j^4|f_j|^2\big) + s^4|g_n|^2 + N_n^4|f_n|^2 + C_7 s^4 N_n^2 \sum_j \frac{1}{(N_j + s)^2(N_j^2 + s^2)^2}\big(s^4|g_j|^2 + N_j^4|f_j|^2\big)\Bigg). \tag{3.26}$$

The following asymptotic estimates hold:

$$\sup_n \frac{N_n^{2m}}{(N_n + s)^2(N_n^2 + s^2)^2} = O\big(s^{-6+2m}\big), \quad m = 0, 1, 2, 3. \tag{3.27}$$

Using (3.27) we continue the estimate (3.26) as follows:

$$|\mathrm{LHS}| \leqslant \frac{C_8}{s^2}\sum_n \big(|f_n|^2 + |g_n|^2\big) = \frac{C_8}{\tau}\sum_n \big(|f_n|^2 + |g_n|^2\big), \tag{3.28}$$

which proves the desired inequality (3.7) (see also (3.24)) but only for τ that is uniformly separated from $\forall\lambda_n$. The nbh of $\forall\lambda_n$, $\tau \in [\lambda_n - \delta, \lambda_n + \delta]$ (or $s \in [N_n - \delta, N_n + \delta]$) is considered now.

Let $s \in [\sqrt{\lambda_m}-\delta, \sqrt{\lambda_m}+\delta]$ for a particular m. We may disregard all terms of the series over n in (3.25) with $n \neq m$ because an estimate above is applicable. Hence, only the following estimate is required:

$$\frac{1}{(\lambda_m^2 - s^4)^2}\left(\left|\lambda_m g_m + is^2 f_m - \frac{k\lambda_m w_m'(1)}{S(is^2)} \sum_j \frac{w_j(1)}{\lambda_j^2 - s^4}\big(is^2 g_j - \lambda_j f_j\big)\right|^2 + \left|is^2 g_m - \lambda_m f_m - is^2 \frac{kw_m'(1)}{S(is^2)} \sum_j \frac{w_j(1)}{\lambda_j^2 - s^4}\big(is^2 g_j - \lambda_j f_j\big)\right|^2\right)$$
$$\leqslant \frac{C_9}{s^2}\left(\sum_n |f_n|^2 + |g_n|^2\right) \quad \forall s \in [\sqrt{\lambda_m}-\delta, \sqrt{\lambda_m}+\delta]. \tag{3.29}$$

We further split the series over j in (3.29) in $j = m$ and $j \neq m$ and prove the corresponding estimates separately. For $j \neq m$, we find

$$\frac{1}{(\lambda_m^2 - s^4)^2}\left(\left|\frac{k\lambda_m w_m'(1)}{S(is^2)} \sum_{j\neq m} \frac{w_j(1)}{\lambda_j^2 - s^4}\big(is^2 g_j - \lambda_j f_j\big)\right|^2 + \left|is^2 \frac{kw_m'(1)}{S(is^2)} \sum_{j\neq m} \frac{w_j(1)}{\lambda_j^2 - s^4}\big(is^2 g_j - \lambda_j f_j\big)\right|^2\right)$$
$$\leqslant \frac{C_{10}}{s^2}\sum_n \big(|f_n|^2 + |g_n|^2\big) \tag{3.30}$$

or equivalently

$$\frac{1}{(\lambda_m^2 - s^4)^2}\left|\frac{kw_m'(1)}{S(is^2)}\right|^2 \big(\lambda_m^2 + s^4\big)\left|\sum_{j\neq m} \frac{w_j(1)}{\lambda_j^2 - s^4}\big(is^2 g_j - \lambda_j f_j\big)\right|^2$$
$$\leqslant \frac{C_{11}}{s^2}\sum_n \big(|f_n|^2 + |g_n|^2\big) \quad \forall s \in [\sqrt{\lambda_m}-\delta, \sqrt{\lambda_m}+\delta], \tag{3.31}$$

so that $|\sqrt{\lambda_j} - s| \geqslant C_{12}$, $j \neq m$.

We now use (3.15) to estimate the sum over $j \neq m$ in (3.31):

$$\left|\sum_{j\neq m} \frac{w_j(1)}{\lambda_j^2 - s^4}\big(is^2 g_j - \lambda_j f_j\big)\right|^2 \leqslant C_{13}\sum_j \frac{s^4|g_j|^2 + \lambda_j^2|f_j|^2}{\lambda_j^2 + s^4} \cdot \frac{1}{(\sqrt{\lambda_j} + s)^2}$$
$$\leqslant \frac{C_{13}}{s^2}\sum_j \big(|f_j|^2 + |g_j|^2\big).$$

To prove (3.31) we need to prove that

$$\frac{1}{(\lambda_m^2 - s^4)^2}\left|\frac{kw_m'(1)}{S(is^2)}\right|^2 \big(\lambda_m^2 + s^4\big) \leqslant C_{14}$$

or equivalently (see (3.15))

$$\big|(\sqrt{\lambda_m} - s)S\big(is^2\big)\big| \geqslant C_{15}, \quad s \in [\sqrt{\lambda_m}-\delta, \sqrt{\lambda_m}+\delta]. \tag{3.32}$$

According to (3.20),

$$\lim_{s\to\sqrt{\lambda_m}}(\sqrt{\lambda_m}-s)S(is^2)=\frac{ik\lambda_m w_m(1)w_m'(1)}{4\lambda_m\sqrt{\lambda_m}}=O(1)\quad\text{as } m\to\infty, \tag{3.33}$$

which proves (3.32). We finally consider the case $j=m$ in (3.29). We need to prove:

$$\begin{aligned}&\frac{1}{(\lambda_m^2-s^4)^2}\left(\left|\lambda_m g_m+is^2 f_m-\frac{k\lambda_m w_m'(1)}{S(is^2)}\frac{w_m(1)}{\lambda_m^2-s^4}\big(is^2 g_m-\lambda_m f_m\big)\right|^2\right.\\&\quad\left.+\left|is^2 g_m-\lambda_m f_m-is^2\frac{kw_m'(1)}{S(is^2)}\frac{w_m(1)}{\lambda_m^2-s^4}\big(is^2 g_m-\lambda_m f_m\big)\right|^2\right)\\&\quad\leqslant\frac{C_{16}}{s^2}\left(\sum_n|f_n|^2+|g_n|^2\right),\qquad s\in[\sqrt{\lambda_m}-\delta,\sqrt{\lambda_m}+\delta].\end{aligned} \tag{3.34}$$

Using the limit in (3.33) yields an equivalent form of (3.34):

$$\frac{1}{\lambda_m^2+s^4}\left(|g_m-if_m|^2+\left|\frac{is^2}{\lambda_m}g_m-f_m\right|^2\right)\leqslant\frac{C_{16}}{s^2}\left(\sum_n|f_n|^2+|g_n|^2\right),$$

which holds due to $s\in[\sqrt{\lambda_m}-\delta,\sqrt{\lambda_m}+\delta]$. All of the above proves the desired estimates for the resolvent. □

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