Schmidt's game, badly approximable matrices and fractals

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Abstract

We prove that for every $M, N \in \mathbb{N}$, if $\tau$ is a Borel, finite, absolutely friendly measure supported on a compact subset $K$ of $\mathbb{R}^{MN}$, then $K \cap \text{BA}(M, N)$ is a winning set in Schmidt's game sense played on $K$, where $\text{BA}(M, N)$ is the set of badly approximable $M \times N$ matrices. As an immediate consequence we have the following application. If $K$ is the attractor of an irreducible finite family of contracting similarity maps of $\mathbb{R}^{MN}$ satisfying the open set condition (the Cantor's ternary set, Koch's curve and Sierpinski's gasket to name a few known examples), then

$$\dim K = \dim (K \cap \text{BA}(M, N)).$$

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0. Introduction

In his paper *Badly Approximable Systems of Linear Forms* [S2], W.M. Schmidt proved that the set of badly approximable $M \times N$ matrices in $\mathbb{R}^{MN}$ is uncountable and in fact of full Hausdorff dimension, i.e., $MN$. His proof is based on what is now referred to as Schmidt's game, first introduced by Schmidt in [S1]. More precisely, he proved that this set is $1\over 2$-winning, from which the conclusion regarding the Hausdorff dimension (and thus the cardinality of this set) is drawn. (See [S2] for a comprehensive review of partial results obtained prior to [S2].) In recent years similar questions have been posed regarding the intersection of badly approximable numbers and vectors with certain subsets of $\mathbb{R}^N$. For example, let $K\cap\text{BAD}(M,N)$ be any of the following sets: Cantor dust, Koch's curve, Sierpinski's gasket, or in general, an attractor of an irreducible finite family of contracting similarity maps of $\mathbb{R}^{MN}$ satisfying the open set condition. (This condition due to J.E. Hutchinson [H] is discussed in Section 4.) Denoting
by $\mathbf{BA}(M, N)$ the set of badly approximable $M \times N$ matrices (where the case $N = 1$ corresponds to badly approximable vectors) one may ask the following questions:

1. Is $K \cap \mathbf{BA}(M, N) \neq \emptyset$?
2. If $K \cap \mathbf{BA}(M, N) \neq \emptyset$, what is $\dim K \cap \mathbf{BA}(M, N)$?

Answers to both of these questions for $\mathbf{BA}(M, 1)$ have been independently given in [KW] and [KTV] and later strengthened by the author in [F] using Schmidt’s game, proving $\dim K \cap \mathbf{BA}(M, 1) = \dim K$ for the family of sets mentioned above, but for the general case where both $M$ and $N$ are strictly larger then one, the answer was, as far as we are aware of, unknown.

This paper’s main results, Theorem 1 and Corollary 3 generalize these results to the set of badly approximable matrices, hence proving an analogue to Schmidt’s result in $\mathbb{R}^{MN}$.

We emphasize that the major difference, and for all practical purposes the only difference, between our proof and that of Schmidt, is in Lemma 3.1 in our paper corresponding to Lemma 4 in [S2]. It is precisely in the proof of this lemma that player White has to specify his strategy. In Schmidt’s paper this is done by player White successively picking specific points in his opponent’s previous chosen balls as the centers for his balls. Unfortunately, we cannot follow this strategy simply by the fact that in any given ball centered on the support of our measure, we have no way of determining whether a specific point belongs to the support of the measure (apart of course from the center point). Thus we have to resort to measure theoretic reasoning postulating the existence of “good points”, i.e., points which could serve player White’s strategy as centers for his balls. This is done by utilizing results regarding absolutely friendly measures from D. Kleinbock, E. Lindenstrauss and B. Weiss, On fractal measures and Diophantine approximation [KLW]. Not originally intended for being a friendly environment for Schmidt’s game, it turns out that the support of these measures is indeed hospitable to this game.

Section 5 is dedicated to a short discussion regarding the winning dimension of a set. (See Section 5 for a formal definition.) We show that Schmidt’s optimal winning dimension result, $\text{Windim}(\mathbf{BA}(M, N) \cap \mathbb{R}^{MN}) = \frac{1}{2}$ cannot be reproduced when playing on the Cantor ternary set.

In the last section we raise a question regarding the measure of the intersection of $\mathbf{BA}(M, N)$ and the compact support of an absolutely friendly measure. We construct an example demonstrating the need for additional research on the necessary conditions for this measure of intersection to be 0.

1. Basic definitions, notations and formulation of main theorem

1.1. Badly approximable matrices

If $t \in \mathbb{R}$, let $\langle t \rangle$ denote the distance of $t$ from the nearest integer.

For $U \in \mathbb{R}^D$, $U = (u_1 \cdots u_D)$ we define

$$\text{dist}(U, \mathbb{Z}^D) = \|\langle u_1 \rangle, \ldots, \langle u_D \rangle\|_{\infty},$$

where for $V = (v_1, \ldots, v_D)$, $\|V\|_{\infty} = \max_{1 \leq i \leq D}\{|v_i|\}$.

Let $M, N \in \mathbb{N}$ and let $A$ be a real $M \times N$ matrix.

We say that $A$ is badly approximable if there exists a real constant $0 < C = C(A)$ such that for every $0 \neq x \in \mathbb{Z}^N$ we have

$$\text{dist}(Ax, \mathbb{Z}^M) > C\|x\|_{\infty}^{-\frac{N}{M}}. \quad (1.1)$$

Denote by $\mathbf{BA}(M, N)$ be the set of all $M \times N$ badly approximable matrices.

For the rest of this paper, if $U, V \in \mathbb{R}^D$, then $\|U\|$ is the usual vector length, i.e., $(\sum_{i=1}^D u_i^2)^{\frac{1}{2}}$ and $U \cdot V$ is the standard inner product.
1.2. Schmidt’s game

Let $(X,d)$ be a complete metric space and let $S \subseteq X$ be a given set (a target set). Schmidt’s game [S1] is played by two players White and Black, each equipped with parameters $\alpha$ and $\beta$ respectively, $0 < \alpha, \beta < 1$. The game starts with player Black choosing $y_0 \in X$ and $\rho > 0$ hence specifying a closed ball $U(0) = B(y_0, \rho)$. Player White may now choose any point $x_0 \in X$ provided that $W(0) = B(x_0, \alpha \rho) \subset U(0)$. Next, player Black chooses a point $y_1 \in X$ such that $U(1) = B(y_1, (\alpha \beta) \rho) \subset W(0)$. Continuing in the same manner we have a nested sequence of non-empty closed sets $U(0) \supset W(0) \supset U(1) \supset W(1) \supset \cdots \supset U(k) \supset W(k) \supset \cdots$ with diameters tending to zero as $k \to \infty$. As the game is played on a complete metric space, the intersection of these balls is a point $z \in X$. Call player White the winner if $z \in S$. Otherwise player Black is declared winner. A strategy consists of specifications for a player’s choices of centers for his balls as a consequence of his opponent’s previous moves. If for certain $\alpha$ and $\beta$ player White has a winning strategy, i.e., a strategy for winning the game regardless of how well player Black plays, we say that $S$ is an $(\alpha, \beta)$-winning set. If $S$ and $\alpha$ are such that $S$ is an $(\alpha, \beta)$-winning set for all possible $\beta$’s, we say that $S$ is an $\alpha$-winning set. Call a set winning if such an $\alpha$ exists.

We shall be considering a slight variant of Schmidt’s original definition of the game, say $K$-Schmidt’s game (and we thank B. Weiss for drawing our attention to this point).

Specifically, the game is played on a compact subset $K$ of $\mathbb{R}^N$ and in his first move, player Black specifies a point $x_0$ in $K$ and a positive number $\rho$. These choices uniquely determine a standard Euclidean closed ball $B(x, \rho)$ centered at $x$ and of radius $\rho$. From this point on, center points picked by either players are in $K$ and the radii of their balls of choice, $\rho(B)$ (where the balls are considered as balls in $\mathbb{R}^N$) are well defined.

We emphasize that one could avoid using the notation $\rho(B)$ by referring to, for example, player Black balls’ radii as $(\alpha \beta)^k \rho$ for some $k \in \mathbb{N}$.

Finally we assume that the first ball $U(0)$ specified by player Black satisfies $\rho(U(0)) \leq \text{diam} K$. We remark that no loss of generality occurs by this assumption, as otherwise player’s White’s strategy is to play arbitrarily until the first integer $k$ is reached with $\rho(U_k) \leq \text{diam} K$.

1.3. Absolutely friendly measures

For our next definitions we assume $N \in \mathbb{N}$ and $P \subseteq \mathbb{R}^N$ is an affine subspace. We denote by $d_P(x)$ the Euclidean distance from $x \in \mathbb{R}^N$ to $P$.

Given $\epsilon > 0$, let

$$P^{(\epsilon)} = \{x \in \mathbb{R}^N : d_P(x) < \epsilon\}.$$

**Definition 1.** Let $\tau$ be a Borel, finite measure on $\mathbb{R}^N$. We say that $\tau$ is **absolutely friendly** if the following conditions are satisfied:

There exist constants $\rho_0, C, D$ and $a$ such that for every $0 < \rho \leq \rho_0$ and for every $x \in \text{supp}(\tau)$:

(i) for any $0 < \epsilon \leq \rho$, and any affine hyperplane $P$,

$$\tau(B(x, r) \cap P^{(\epsilon)}) < C \left( \frac{\epsilon}{\rho} \right)^a \tau(B(x, \rho)).$$

(ii) $\tau(B(x, \frac{1}{2} \rho)) > D \tau(B(x, \rho))$.

**Remark 1.** The second condition is usually referred to as the doubling or Federer property. The term “absolutely friendly” was first coined in [PV] where stronger assumptions regarding the definition of friendly measures (see [KLW]) were needed.
1.4. Main theorem

**Theorem 1.** For every $M, N \in \mathbb{N}$, if $\tau$ is a Borel, finite, absolutely friendly measure on $\mathbb{R}^{MN}$, supported on a compact subset $K$ of $\mathbb{R}^{MN}$, then $K \cap BA(M, N)$ is a winning set in Schmidt’s game sense, played on $K$.

2. Specific notations

Set

$$H = M \cdot N \quad \text{and} \quad L = M + N.$$ 

For the rest of the paper we shall assume $N \geq M$. This assumption will not imply any loss of generality since in fact “built in” the proof is the fact that if the set of badly approximable $M \times N$ matrices is winning in $K$-Schmidt’s game, so is the set of their transposes.

We shall be playing Schmidt’s game on $K$ as defined in Theorem 1, where we identify points in $\mathbb{R}^H$ with $M \times N$ real matrices.

For $k \in \mathbb{N}$ we denote the $k$th ball chosen by player White by $W(k)$ and respectively player Black’s balls by $U(k)$.

Let $\rho = \rho(U(0))$ be the first radius chosen by player Black.

Given a Borel, finite, absolutely friendly measure $\tau$ supported on a compact subset $K \subset \mathbb{R}^H$, define

$$\sigma = \sigma(\tau) = 3 \cdot \max \{ \|X\| : X \in K \}. \quad (2.2)$$

**Remark 2.** In Schmidt’s paper, $\sigma$ was defined as the maximal norm of a point in $U(0)$, and thus determined by $U(0)$. In our settings, as $\rho(U(0)) < \text{diam}(K)$ (see discussion regarding this assumption in Section 1.2), $\sigma$ is determined by $K = \text{supp}(\tau)$ and thus constants involving $\sigma$ in Schmidt’s paper are viewed as constants involving $\tau$.

We assign boldface lower case letters ($x, y$, etc.) to denote points in $\mathbb{R}^N$ and $\mathbb{R}^M$ while boldface upper case letters ($X, Y$, $B$, etc.) denote points in $\mathbb{R}^L$. Finally, upper case letters ($A, X, Y$, etc.) denote points in $\mathbb{R}^H$.

3. Proof of theorem

The proof will be presented in the following order. In the first subsection we shall begin by stating Lemmas 3.1, 3.2 and derive Corollary 2. We shall then proceed and ultimately prove our main theorem, Theorem 1. Once this result is established we shall prove Lemma 3.1 in the following section. (Theorem 3.2 could be proved in an identical way to Theorem 3.1.)

The rationale behind this way of presentation is the following. Lemmas 3.1 and 3.2 are, to quote Schmidt when referring to the analogous lemmas in [S2], difficult. One of the main difficulties is the need for seemingly obscure notations and definitions. Furthermore, in our case, we shall also need to utilize some deeper results concerning absolutely friendly measures. We hope that by demonstrating the relatively effortless way one derives the main theorem once these lemmas are proved will convince the reader of their necessity.

3.1. Proof of Theorem 1 assuming Lemma 3.1

3.1.1. Yet some more notation

We begin with some more notations and definitions.
For any
\[
A = \begin{pmatrix}
\gamma_{11} & \cdots & \gamma_{1N} \\
\vdots & \ddots & \vdots \\
\gamma_{M1} & \cdots & \gamma_{MN}
\end{pmatrix}
\]
let
\[
A_1 = (\gamma_{11}, \ldots, \gamma_{1N}, 1, 0, \ldots, 0), \quad \ldots, \quad A_M = (\gamma_{M1}, \ldots, \gamma_{MN}, 0, 0, \ldots, 1),
\]
\[
B_1 = (\gamma_{11}, \ldots, \gamma_{M1}, 1, 0, \ldots, 0), \quad \ldots, \quad B_N = (\gamma_{1N}, \ldots, \gamma_{MN}, 0, 0, \ldots, 1).
\]

Let \(X, Y \in \mathbb{Z}^L\) of the form
\[
X = (x_1, \ldots, x_N, \ldots, x_L): \ x = (x_1, \ldots, x_N) \neq (0, \ldots, 0),
\]
\[
Y = (y_1, \ldots, y_M, \ldots, y_L): \ y = (y_1, \ldots, y_M) \neq (0, \ldots, 0).
\]

Set
\[
A(X) = (|A_1 \cdot X|, \ldots, |A_M \cdot X|)
\]
and
\[
B(Y) = (|B_1 \cdot Y|, \ldots, |B_N \cdot Y|).
\]

We notice that a matrix \(A\) lies in \(BA(M, N)\) if and only if there exists a constant \(C\) such that for all \(X\) such as in (3.3)
\[
\|x\|_\infty^N \cdot \|A(X)\|_\infty^M > C.
\]

For a fixed \(N\) and \(v\), where \(1 \leq v \leq N\) and for any \(\mathcal{Y} = \{Y_1, \ldots, Y_N\}\), there are \(\binom{N}{v}^2\) matrices of the form
\[
(B_{i_1} \cdot Y_{j_1})
\]
where \(1 \leq i_1 < \cdots < i_v \leq N\) and \(1 \leq j_1 < \cdots < j_v \leq N\).

Define
\[
\tilde{M}_{v, \mathcal{Y}}(A) \in \mathbb{R}^{(\binom{N}{v})^2}
\]
as the vector whose components are the absolute value of the determinants of (3.8) arranged in some order.

Similarly for a fixed \(M\) and \(v\), where \(1 \leq v \leq M\) and for any \(\mathcal{Y} = \{Y_1, \ldots, Y_M\}\), there are \(\binom{M}{v}^2\) matrices of the form
\[
(A_{i_1} \cdot Y_{j_1})
\]
where \(1 \leq i_1 < \cdots < i_v \leq M\) and \(1 \leq j_1 < \cdots < j_v \leq M\).
Define

\[ \vec{M}_{v, Y'}(A) \in \mathbb{R}^{(H)^2} \]

as the vector whose components are the absolute value of the determinants of (3.9) arranged in some order.

As we shall consequently see, we shall only be assuming that the elements of the sets \( Y \) and \( Y' \) are orthonormal, but the proofs DO NOT depend on a specific \( Y \) or \( Y' \). (This is perhaps the most important part of our main theorem’s proof.) Thus from this point on, for any fixed \( Y \), we shall write \( \vec{M}_v(A) \) to mean \( \vec{M}_{v, Y}(A) \).

Define \( \vec{M}_0(A) \) (similarly \( \vec{M}'_0(A) \)) and \( \vec{M}_{-1}(A) \) (similarly \( \vec{M}'_{-1}(A) \)) as the one-dimensional vector \( [1] \).

For a closed ball \( B \subset \mathbb{R}^H \), let

\[ M_v(B) = \max_{A \in B} |\vec{M}_v(A)| \quad \text{and} \quad M'_v(B) = \max_{A \in B} |\vec{M}'_v(A)|. \]

3.1.2. More on absolutely friendly measures

For a ball \( B \subset \mathbb{R}^N \) and a real valued function \( f \) on \( \mathbb{R}^N \), let

\[ \|f\|_B = \sup_{x \in B} |f(x)|. \]

As an immediate consequence of Proposition 7.3 in [KLW] one has the following corollary.

**Corollary 1.** Let \( \tau \) be a Borel, finite, absolutely friendly measure on \( \mathbb{R}^H \). Then for every \( k \) there exist \( K = K(k) \) and \( \delta = \delta(k) \) such that if \( f \) is a real polynomial function on \( \mathbb{R}^H \) of a bounded total degree \( k \), then for any ball \( B \subset \mathbb{R}^H \) centered on \( \text{supp}(\tau) \) and any \( \epsilon > 0 \),

\[ \tau \left( \{ x \in B : |f(x)| < \epsilon \} \right) \leq K \left( \frac{\epsilon}{\|f\|_B} \right)^{\delta} \tau(B). \]

Thus given a Borel, finite, absolutely friendly measure \( \tau \) on \( \mathbb{R}^H \) and a polynomial function of total bounded degree \( L \) with associated constants \( K = K(L) \) and \( \delta = \delta(L) \) as in Corollary 1, let

\[ 0 < \epsilon_0 = \epsilon_0(\tau, M, N) \]

be small enough as to satisfy

\[ K(\epsilon_0)^{\delta} < \frac{1}{2}. \]

3.1.3. Statement of main lemmas

**Lemma 3.1.** Given \( \tau \), a Borel, finite absolutely friendly measure with \( \text{supp}(\tau) = K \), where \( K \) is a compact subset of \( \mathbb{R}^H \), we play Schmidt’s game on \( K \) such that all balls chosen by the two players are centered on \( K \). Let \( \epsilon_0 \) be as defined in (3.11). Then for any \( \psi > 0 \), there exists

\[ 0 < \alpha_1 = \alpha_1(M, N, \psi, \tau), \]

and for any \( 0 < \beta < 1, 0 \leq \nu \leq N \), there exists

\[ \mu_{\nu} = \mu_{\nu}(M, N, \alpha_1, \beta, \psi, \tau) \]
such that for any $Y_1, \ldots, Y_N$ orthonormal vectors in $\mathbb{R}^L$, if a ball $U \subset \mathbb{R}^L$ satisfying $\rho(U) = \rho_0 < 1$ is reached by player Black at some stage of the $(\alpha_1, \beta)$ game, then player White has a strategy enforcing the first of player Black’s ball $U(i_\nu)$ with

$$\rho(U(i_\nu)) < \rho_0 \mu \nu$$

to satisfy for every $A \in U(i_\nu)$

$$|\tilde{M}_\nu(A)| > \left(\frac{\epsilon_0}{2}\right)^{\nu} \psi \rho_0 \mu \nu M_{\nu - 1}(U(i_\nu)).$$

(3.12)

The following lemma can be proved almost exactly as Lemma 3.1, substituting $M$ for $N$ in the appropriate places.

**Lemma 3.2.** Given $\tau$, a Borel, finite absolutely friendly measure with $\text{supp}(\tau) = K$, where $K$ is a compact subset of $\mathbb{R}^H$, we play Schmidt’s game on $K$ such that all balls chosen by the two players are centered on $K$. Let $\epsilon_0$ be as defined in (3.11). Then for any $\psi > 0$, there exists

$$0 < \alpha_2 = \alpha_2(M, N, \psi, \tau),$$

and for any $0 < \beta < 1$, there exists

$$\mu_\nu = \mu_\nu(M, N, \alpha_2, \beta, \psi, \tau)$$

such that for any $Y_1, \ldots, Y_M$ orthonormal vectors in $\mathbb{R}^L$, if a ball $U \subset \mathbb{R}^L$ satisfying $\rho(U) = \rho_0 < 1$ is reached by player Black at some stage of the $(\alpha_2, \beta)$ game, then player White has a strategy enforcing the first of player Black’s ball $U(i_\nu)$ with

$$\rho(U(i_\nu)) < \rho_0 \mu \nu$$

to satisfy for every $A \in U(i_\nu)$

$$|\tilde{M}_\nu(A)| > \left(\frac{\epsilon_0}{2}\right)^{\nu} \psi \rho_0 \mu \nu M_{\nu - 1}(U(i_\nu)).$$

(3.13)

3.1.4. Immediate corollary

**Corollary 2.** Given $\tau$, a Borel, finite absolutely friendly measure with $\text{supp}(\tau) = K$, where $K$ is a compact subset of $\mathbb{R}^H$, we play Schmidt’s game on $K$ such that all balls chosen by the two players are centered on $K$. There exists

$$0 < \alpha = \alpha(M, N, \tau),$$

and given any $0 < \beta < 1$, there exists

$$\mu = \mu(M, N, \alpha, \beta, \tau)$$

such that for any $0 < \mu' \leq \mu$ and for any $Y_1, \ldots, Y_N$ orthonormal vectors in $\mathbb{R}^L$, if a ball $U \subset \mathbb{R}^L$ centered on $K$ satisfying $\rho(U) < 1$ is reached by player Black at some stage of the game, then player White has a strategy enforcing the first of player Black’s ball $U(i)$ with
\[ \rho(U(l)) < \rho(U)\mu' \]
to satisfy for every \( A \in U(l) \)

\[ |\tilde{M}_N(A)| > L\sqrt{L}\rho(U)\mu'M_{N-1}(U(l)). \]  
(3.14)

Alternatively under the same assumptions on \( U \), for any \( Y_1, \ldots, Y_M \) orthonormal vectors in \( \mathbb{R}^L \) player White has a strategy enforcing the first of player Black’s balls \( U(l) \) with

\[ \rho(U(l')) < \rho(U)\mu' \]
to satisfy for every \( A \in U(l') \)

\[ |\tilde{M}_M'(A)| > L\sqrt{L}\rho(U)\mu'M_{M-1}'(U(l')). \]  
(3.15)

**Proof.** Replace \( \psi \) in Lemmas 3.1 and 3.2 with \( L\sqrt{L}(\frac{2}{\epsilon_0})^L \).

Set

\[ \alpha = \min\{\alpha_1, \alpha_2\} \quad \text{and} \quad \mu = \min\{\mu_N, \mu'_M\}. \]

Notice that by Lemma 3.1 (Lemma 3.2), if

\[ 0 < \mu' \leq \mu_N \quad \text{similarly} \quad 0 < \mu' \leq \mu_M \]

and

\[ \rho(U(i_N)) < \mu'\rho_0 \quad \text{similarly} \quad \rho(U(i_M)) < \mu'\rho_0, \]

then obviously every \( A \in U(i_N) \) (\( A \in U(i_M) \)) will satisfy (3.14). \( \square \)

**3.1.5. Two geometric lemmas**

For what follows we shall need to use Lemmas 1 and 2 in \([S2]\]. Let \( X, Y, A(X) \) and \( B(Y) \) be as in (3.3), (3.4), (3.5) and (3.6).

Set

\[ \lambda = N/L. \]  
(3.16)

Given \( 1 < R \in \mathbb{R} \), let

\[ \delta = R^{-NL^2}, \quad \delta^T = R^{-ML^2}. \]  
(3.17)

**Lemma 3.3.** There exists a constant \( R_1 = R_1(M, N, \sigma) \) such that for every \( i \in \mathbb{N} \) and \( R \geq R_1 \), if a ball \( B \) satisfies

\[ \rho(B) < R^{-(\lambda+i)}, \]

with the system
having no solution $X$ for all $A_i$ associated with points in $B$, then the number of linearly independent vectors $Y$ satisfying the system

$$0 < \|y\|_\infty < \delta R^{M(1+i)},$$
$$\|B(Y)\|_\infty < \delta R^{M(1+i) - N}$$

for all $B_i$ associated with points in $B$ is at most $N$.

**Lemma 3.4.** There exists a constant $R_2 = R_2(M, N, \sigma)$ such that for every $j \in \mathbb{N}$ and $R \geq R_2$, if a ball $B$ satisfies

$$\rho(B) < R^{-L(1+j)}$$

with the system

$$0 < \|y\|_\infty < \delta R^{N(1+i)},$$
$$\|B(Y)\|_\infty < \delta R^{M(1+i) - N}$$

having no integer solution $Y$ for all $B_i$ associated with points in $B$, then the number of linearly independent vectors $X$ satisfying the system

$$0 < \|x\|_\infty < \delta R^{M(\lambda+i)},$$
$$\|A(X)\|_\infty < \delta R^{-N(\lambda+i) - M}$$

for $i = j + 1$ for all $A_i$ associated with points in $B$ is at most $M$.

### 3.1.6. One last lemma

We remind that $\rho = \rho(U(0))$ is the first radius chosen by player Black, and we assume $N \geq M$.

**Lemma 3.5.** Set $\alpha$ as in Corollary 2, and given $0 < \beta < 1$, let $\mu$ be as in Corollary 2 and $\lambda$ as in (3.16). Then there exists $R = R(M, N, \alpha, \beta, \rho, \tau)$ such that player White can direct the game in such a way that for every $i, k \in \mathbb{N}$, if $U(k)$ of the game satisfies

$$\rho(U(k)) < R^{-L(\lambda+i)},$$

then for all $A \in U(k)$ the system

$$0 < \|x\|_\infty < \delta R^{M(\lambda+i)},$$
$$\|A(X)\|_\infty < \delta R^{-N(\lambda+i) - M}$$

has no solution $X$ as in (3.3), where $\delta$ is as in (3.17).

He can also direct the game such that for every $i, h \in \mathbb{N}$ if $U(h)$ satisfies

$$\rho(U(h)) < R^{-L(1+i)}$$

(3.21)
then for all $A \in U(h)$ the system

\[
0 < \|y\|_\infty < \delta^T R^{N(1+i)}, \\
\|B(y)\|_\infty < \delta^T R^{-M(1+i)-N}
\]

(3.22)

(3.23)

has no solution $Y$ as in (3.4), where $\delta^T$ is as in (3.17).

**Proof.** In order for Lemmas 3.3 and 3.4 to be applicable, we first demand that

\[
R > \max\{R_1, R_2\},
\]

where $R_1$ and $R_2$ are as defined in Lemmas 3.3 and 3.4.

Next, let

\[
R > \max\{\rho^{-\frac{1}{\mu}}, (\alpha \beta)^{-\frac{1}{\mu}}\}.
\]

(3.24)

Condition (3.24) ensures that $\rho(U(k_0)) < \rho$, and the sequence

\[
U(k_0) \supset U(h_0) \supset U(k_1) \supset U(h_1) \supset \cdots
\]

(3.25)

is strictly decreasing.

Finally we demand that

\[
R > (\alpha \beta \mu)^{-\frac{1}{\mu}}.
\]

(3.26)

Let $U(k_i)$ be the first ball of the game with (3.18), and $U(h_i)$ for the first ball with (3.21).

We shall prove the lemma by induction on $i$.

1. **Base of the induction.**

   We notice that for $i = 0$

   \[
   \|x\| \geq 1 > \delta R^{M\lambda}.
   \]

   Therefore (3.19) and (3.20) have no solution $X$ if $A \in U(k_0)$.

2. **The induction hypothesis.**

   We assume

   \[
   U(0) \supset W(0) \supset \cdots \supset U(k_0) \supset \cdots \supset U(k_i)
   \]

   have been already chosen such that for every $0 \leq j \leq i$, (3.19) and (3.20) have no solution for $A \in U(k_j)$, and dually we assume that

   \[
   U(0) \supset W(0) \supset \cdots \supset U(k_0) \supset \cdots \supset U(h_j)
   \]

   \[
   U(0) \supset W(0) \supset \cdots \supset U(k_0) \supset \cdots \supset U(h_i)
   \]

   have been already chosen such that for every $0 \leq j \leq i$, (3.22) and (3.23) have no solution for $A \in U(h_j)$.

   Thus it remains to prove that if

   \[
   U(0) \supset W(0) \supset \cdots \supset U(k_0) \supset \cdots \supset U(k_i)
   \]
have been already chosen such that for every \(0 \leq j \leq i\), (3.19) and (3.20) have no solution for \(A \in U(k_j)\), player White can enforce that (3.22) and (3.23) have no solution if \(A \in U(h_i)\).

Suppose that there are solutions \(Y\) of (3.22) and (3.23) with vectors \(B_1, \ldots, B_N\) associated with a point \(A\) in \(U(k_i)\). By our assumptions it is sufficient to consider points \(Y\) satisfying

\[
\delta^T R^{N(1+i-1)} \leq \|Y\|_\infty < \delta^T R^{N(1+i)}.
\]

Thus in particular

\[
\delta^T R^{N(1+i-1)} \leq \|Y\|_\infty.
\]

By Lemma 3.3, the vectors \(Y\) will be contained in an \(N\)-dimensional subspace of \(\mathbb{R}^L\). Let \(Y_1, \ldots, Y_N\) be an orthonormal basis of this subspace and suppose that the integer point \(Y = t_1Y_1 + \cdots + t_NY_N\) satisfies (3.23) and (3.27).

We have that

\[
\delta^T R^{N(1+i-1)} \leq \|Y\|_\infty \leq |Y| = \sqrt{t_1^2 + \cdots + t_N^2} \leq \sqrt{N} \max(|t_1|, \ldots, |t_N|).
\]

And so,

\[
\frac{1}{\sqrt{N}} R^{N(1+i-1)} \leq \max(|t_1|, \ldots, |t_N|), \tag{3.28}
\]

and

\[
|t_1(B_1 \cdot Y_1) + \cdots + t_N(B_N \cdot Y_N)| \leq \delta^T R^{-M(1+i)-N},
\]

\[
\vdots
\]

\[
|t_1(B_N \cdot Y_1) + \cdots + t_N(B_N \cdot Y_N)| \leq \delta^T R^{-M(1+i)-N}.
\]

Let \(D\) be the determinant of \((B_u \cdot Y_v)_{1 \leq u, v \leq N}\), and let \(D_{uv}\) be the cofactor of \(B_u \cdot Y_v\) in this determinant.

By Cramer’s rule we get for every \(1 \leq v \leq N\)

\[
|t_v D| \leq N\delta^T R^{-M(1+i)-N} \max(|D_{1v}|, \ldots, |D_{Nv}|) \tag{3.29}
\]

and in conjunction with (3.28) we get

\[
|D| \leq N\sqrt{N} R^{-L(1+i)} \max(|D_{11}|, |D_{12}|, \ldots, |D_{NN}|). \tag{3.30}
\]

Player White’s strategy is to play in such a way such that (3.30) is not satisfied by any \(B_1, \ldots, B_N\) associated with a point \(A \in U(h_i)\).

Set \(\rho_0 = \rho(U(k_i))\) and let \(0 < \mu'\) be chosen to satisfy

\[
\mu' \rho_0 = R^{-L(1+i)}. \tag{3.31}
\]

Notice that by definition,

\[
\alpha \beta R^{-L(\lambda+i)} \leq \rho_0 < R^{-L(\lambda+i)},
\]
and it follows by condition (3.26) that
\[
\mu' < (\alpha \beta)^{-1} R^{L(i+1) - L(1+i)} = (\alpha \beta)^{-1} R^{-M} < \mu.
\]  
(3.32)

Applying Corollary 2, player White can enforce the first ball \( U(i_N) = U(h_i) \) with
\[
\rho(U(h_i)) < \rho_0 \mu' = R^{-L(1+i)}
\]
to satisfy for every \( A \in U(i_N) \)
\[
|\tilde{M}_N(A)| > L \sqrt{L} \rho_0 \mu' M_{N-1} U(i_N).
\]

Thus for every \( A \in U(h_i) \)
\[
|D| = |\tilde{M}_N(A)| > L \sqrt{L} R^{-L(1+i)} M_{N-1} U(h_i)
\]
\[
> N \sqrt{N} R^{-L(1+i)} \max(|D_{11}|, |D_{12}|, \ldots, |D_{NN}|),
\]
and so (3.30) is not satisfied by any \( B_1, \ldots, B_N \) associated with a point \( A \in U(h_i) \).

One can show in almost the same way that if \( U(h_i) \) has already chosen such that (3.22) and (3.23) have no solution for \( A \in U(h_i) \), player White can enforce \( U(k_{i+1}) \) to satisfy that for no \( A \in U(k_{i+1}) \) the system (3.19) and (3.20) has no solution.  \(\Box\)

3.1.7. Proof of Theorem 1

**Proof of Theorem 1.** Let \( \alpha \) be as defined in Lemma 3.5 and let \( 0 < \beta < 1 \). Once player Black chooses his initial radius \( \rho \) for his first ball \( U(0) \), \( R \) as defined in Lemma 3.5 could be chosen by player White.

Let \( X \) be as defined in (3.3), i.e. \( X \in \mathbb{Z}^L \) and
\[
X = (x_1, \ldots, x_N, \ldots, x_L): x = (x_1, \ldots, x_N) \neq (0, \ldots, 0).
\]

Then for some \( i \in \mathbb{N} \),
\[
\delta R^{M(\lambda+i-1)} \leq \|x\|_{\infty} < \delta R^{M(\lambda+i)}.
\]

By Lemma 3.5, player White can direct the game in such a way that if \( U(k_i) \) of the game satisfies
\[
\rho(U(k_i)) < R^{-L(\lambda+i)},
\]
then for all \( A \in U(k_i) \)
\[
\|A(X)\|_{\infty} \geq \delta R^{-N(\lambda+i) - M}.
\]

Successively applying Lemma 3.5 to ever increasing \( i \), player White can direct the game such that \( A = \bigcap_{i=0}^{\infty} U(k_i) \) will satisfy for every \( X \) as defined in (3.3)
\[
\left(\|x\|_{\infty}\right)^N \left(\|A(X)\|_{\infty}\right)^M \geq \delta^L R^{-NM - M^2}.
\]

Recalling (3.7) we are done, letting
\[
0 < C < \delta^L R^{-NM - M^2}.  \quad \Box
\]
3.2. Proof of Lemma 3.1

3.2.1. Preliminaries

For the rest of this subsection, we shall need the following notation.

Let $\sigma, \psi, \mu > 0$ and suppose that $\nu \in \mathbb{N}$ with $0 \leq \nu \leq N$. Let $U \subset \mathbb{R}^H$ be a closed ball and we denote $\rho(U) = \rho_0$.

We say that $(U, B, \sigma, M, N, \psi, \mu, \nu)$ satisfy $(\ast)$ if

1. $\rho_0 < 1$.  
2. For every $A \in U$, $|A| \leq \sigma$.  
3. $B$ is a closed ball such that $B \subset U$.  
4. $\rho(B) < \mu \rho_0$.  
5. For any given $Y_1, \ldots, Y_N$ orthonormal vectors in $\mathbb{R}^L$ we have for every $A \in B$,

$$|\tilde{M}_{\nu-1}(A)| > \psi \rho_0 \mu M_{\nu-2}(B).$$

The next three propositions are proved by Schmidt. See Lemma 5, Corollaries 1, 2 and Lemma 6 in [S2].

**Proposition 1.** Suppose $(U, B, \sigma, M, N, \psi, \mu, \nu)$ satisfy $(\ast)$. There exists $C_1 = C_1(M, N)$ such that for any $\epsilon > 0$ if $U'$ is a ball contained in $B$ satisfying

$$\rho(U') < \epsilon C_1 \rho(B),$$

then for any $A'$ and $A''$ in $U'$

$$|\tilde{M}_{\nu-1}(A') - \tilde{M}_{\nu-1}(A'')| < \epsilon \rho_0 \mu M_{\nu-2}(B).$$

Furthermore, if

$$\rho(U') < \frac{1}{2} \psi C_1 \rho(B),$$

then for every $A \in U'$,

$$|\tilde{M}_{\nu-1}(A)| \geq \frac{1}{2} M_{\nu-1}(U').$$

Before formulating the next two propositions we need the following notation. Given $Y_1, \ldots, Y_N$ orthonormal vectors in $\mathbb{R}^L$ and $A \in \mathbb{R}^H$ let

$$D_v(A) = D(B_1, \ldots, B_v) = \det \begin{pmatrix} B_1 \cdot Y_1 & \cdots & B_1 \cdot Y_v \\ \vdots & \ddots & \vdots \\ B_v \cdot Y_1 & \cdots & B_v \cdot Y_v \end{pmatrix}.$$

Thus for every $0 \leq \nu \leq N$, $D_v$ is a real polynomial function, $D_v : \mathbb{R}^H \to \mathbb{R}$ of bounded total degree less than or equal to $N$, and in particular, less than $L$.

**Proposition 2.** Suppose $(U, B, \sigma, M, N, \psi, \mu, \nu)$ satisfy $(\ast)$. There exists $C_2 = C_2(M, N)$ such that for any $\epsilon > 0$, if $U'$ satisfies (3.33), then for every $A'$ and $A''$ in $U'$

$$|\nabla D_v(A') - \nabla D_v(A'')| < C_2 \epsilon \rho_0 \mu M_{\nu-2}(B).$$
**Proposition 3.** Suppose \((U, B, \sigma, M, N, \psi, \mu, \nu)\) satisfy \((*)\). There exist \(C_3 = C_3(N, \psi)\) and \(C_4 = C_4(M, N, \sigma)\) such that if \(U'\) is a ball contained in \(B\) satisfying (3.34) and \(A \in U'\) with
\[
\left| \tilde{M}_v(A) \right| < C_3 \psi M_{v-1}(U')
\] (3.35)
and \(D_{v-1}(A)\) has the largest absolute value among the coordinates of \(\tilde{M}_{v-1}(A)\), then
\[
\left| \nabla D_v(A) \right| > C_4 M_{v-1}(U')
\] (3.36)

### 3.2.2. **Proof of Lemma 3.1**

**Proof of Lemma 3.1.** Given \(\psi > 0\) define for every \(0 \leq \nu \leq N\)
\[
\psi_{\nu} = \left( \frac{\epsilon_0}{2} \right)^{\nu} \psi.
\] (3.37)
Assuming \(\psi = \psi_{\nu}\) for \(0 \leq \nu \leq N\) and noticing that in our setup \(\sigma = \sigma(\tau)\), let
\[
C_{\nu}^3 = C_{\nu}^3(N, \psi) \quad \text{and} \quad C_4 = C_4(M, N, \sigma(\tau))
\]
be as in Proposition 3.
Define
\[
C_{3}^{\text{Min}} = C_{3}^{\text{Min}}(N, \psi, \tau) = \min_{1 \leq \nu \leq N} C_{\nu}^3
\] (3.38)
Given \(\psi > 0\) let \(\psi_N\) be as defined in (3.37) and let \(\alpha_1\) be so small as to satisfy
\[
(\alpha_1)^{\frac{1}{2}} < \min \left\{ \frac{1}{2}, \frac{1}{4}, \frac{\psi_N \epsilon_0}{N \max_{0 \leq \nu \leq N} \left( \psi_{\nu} \right)}, \frac{15}{32} \frac{C_3^{\text{Min}}}{\psi} \right\}
\] (3.39)

Our initial setup is a closed ball \(U \subset \mathbb{R}^H\) with \(\rho(U) = \rho_0 < 1\). For every \(A \in U\), \(|A| \leq \sigma\) for some positive \(\sigma = \sigma(\tau)\) and \(0 < \beta < 1\) is given. We shall prove the lemma by induction on \(\nu\).

**1. Base of the induction.**
For \(\nu = 0\), \(\psi_0 = \psi\). Let \(\mu_0 < \frac{1}{\psi}\) and let \(Y_1, \ldots, Y_N\) be any set of orthonormal vectors in \(\mathbb{R}^L\). By definition we have for any ball \(V \subset \mathbb{R}^H\) and any \(A \in V\),
\[
\tilde{M}_0(A) = 1 > \psi_0 \mu_0 \rho_0 = \psi_0 \mu_0 \rho_0 M_{-1}(V).
\] (3.40)

**2. The induction hypothesis.**
We assume the validity of the lemma for \(\nu - 1\) (\(\nu \geq 1\)), i.e., there exists \(\mu_{\nu-1}\) such that player White can play in such a way that the first of player Black's balls \(U(i_{\nu-1}) \subset U\) to satisfy
\[
\rho(U(i_{\nu-1})) < \mu_{\nu-1} \rho_0
\]
satisfies for every \(A \in U(i_{\nu-1})\)
\[
\left| \tilde{M}_{\nu-1}(A) \right| > \psi_{\nu-1} \rho_0 \mu_{\nu-1} M_{\nu-2}(U(i_{\nu-1})).
\] (3.41)
We assume that \( U(i_{\nu-1}) \) with this property is given and thus
\[
(U, U(i_{\nu-1}), \sigma(\tau), M, N, \psi_{\nu-1}, \mu_{\nu-1}, \nu-1)
\]
satisfy \((\ast)\) by the induction hypothesis and our initial conditions. We shall define \( \mu_{\nu} \) and show how player White can play in such a way that \( U(i_{\nu}) \) satisfies (3.14).

Let \( j_{\nu} \) be the first integer exceeding \( i_{\nu-1} \) satisfying
\[
\rho(U(j_{\nu})) < \frac{1}{2} C_1 \rho(U(i_{\nu-1})) \cdot \min\left\{ \psi_{\nu}, \frac{1}{8} \psi N, \frac{C_4}{C_2} \right\}.
\]
(3.42) (As we shall soon see, \( U(j_{\nu}) \) will play the part of \( U' \) in Propositions 1–3.)

By definition
\[
\rho(U(i_{\nu-1})) \geq \alpha_1 \beta \rho_0 \mu_{\nu-1},
\]
and thus there exists
\[
c_{\nu-1} = c_{\nu-1}(M, N, \alpha_1, \beta, \psi, \tau)
\]
(3.43) such that
\[
\rho(U(j_{\nu})) \geq c_{\nu-1} \rho_0.
\]

Set
\[
\mu_{\nu} = \mu_{\nu}(M, N, \alpha_1, \beta, \psi, \tau) = \alpha_1^2 c_{\nu-1}
\]
(3.44) and
\[
K_{\nu-1} = K_{\nu-1}(M, N, \alpha_1, \beta, \psi, \tau, \rho_0) = \frac{\rho(U(j_{\nu}))}{\rho_0}.
\]
(3.45)

For later use we observe that trivially \( K_{\nu-1} \geq c_{\nu-1} \).

When \( U(i_{\nu-1}) \) is given, player White plays in an arbitrary way until \( U(j_{\nu}) \) is reached.

The trivial case.

If it so happens that for every \( A \in U(j_{\nu}) \)
\[
|\tilde{M}_{\nu}(A)| > \psi_{\nu} \rho_0 \mu_{\nu} M_{\nu-1}(U(j_{\nu})),
\]
player White's strategy is to play in an arbitrary way until the first ball \( U(i_{\nu}) \) to satisfy
\[
\rho(U(i_{\nu})) < \mu_{\nu} \rho_0
\]
is reached, and every \( A \in U(i_{\nu}) \) will trivially satisfy (3.14).

The non-trivial case.

Suppose that there exists \( A' \in U(j_{\nu}) \) such that
\[
|\tilde{M}_{\nu}(A')| \leq \psi_{\nu} \rho_0 \mu_{\nu} M_{\nu-1}(U(j_{\nu})).
\]
(3.46)
Set
\[ \epsilon = \frac{1}{16} \psi_N \frac{C_4}{C_2} \left( \leq \frac{1}{16} \psi_{v-1} \frac{C_4}{C_2} \right). \]

We notice that since \( U(j_v) \subset U(i_{v-1}) \) we have by (3.41)
\[ M_{v-1}(U(j_v)) > \psi_{v-1} \rho_0 \mu_{v-1} M_{v-2}(U(i_{v-1})). \] (3.47)

By Proposition 2, for any \( A' \) and \( A'' \) in \( U(j_v) \)
\[ \left| \nabla D_v(A') - \nabla D_v(A'') \right| < C_2 \epsilon \rho_0 \mu_{v-1} M_{v-2}(U(i_{v-1})) < \frac{1}{16} C_4 M_{v-1}(U(j_v)). \] (3.48)

By (3.42), \( U(j_v) \) satisfies (3.34), and since \( \rho_0 < 1 \) and \( \mu_v < C_3^{-1} \) by (3.39) and (3.44), the point \( A' \) satisfies (3.35).

As no special assumptions were made neither on the \( B_v \)'s nor the \( Y_v \)'s, we may assume \( D_{v-1}(A') \)
has the largest absolute value among the coordinates of \( M_{v-1}(A') \). By Proposition 3,
\[ \left| \nabla D_v(A') \right| > C_4 M_{v-1}(U(j_v)). \] (3.49)

Let
\[ D' = \nabla D_v(A'). \] (3.50)

Denote the center of \( U(j_v) \) by \( A(j_v) \).

If \[ D_v(A(j_v)) \geq 0, \] (3.51)

let
\[ A_M = A(j_v) + (1 - \alpha_1) \frac{1}{|D'|} \rho(U(j_v)) D'. \]

Thus
\[ (A_M - A(j_v)) \cdot D' = (1 - \alpha_1) \rho(U(j_v)) |D'|. \]

Since \( \alpha_1 < \frac{1}{4} \) we have
\[ (A_M - A(j_v)) \cdot D' > \frac{3}{4} \rho(U(j_v)) |D'|. \] (3.52)

In view of (3.51), (3.48), (3.52) and (3.44)
\[ D_v(A_M) \geq D_v(A_M) - D_v(A(j_v)) \]

\[ = \int_0^1 (A_M - A_{j_v}) \cdot (\nabla D_v((1 - s)A(j_v) + sA_M)) \, ds \]

\[ = (A_M - A_{j_v}) \cdot D' + \int_0^1 (s A_M - A_{j_v}) \cdot (\nabla D_v((1 - s)A(j_v) + sA_M) - D')) \, ds \]

\[ \geq \frac{3}{4} \rho(U(j_v))|D'| - 2(1 - \alpha_1) \rho(U(j_v)) \frac{1}{16} C_4 M_{v-1}(U(j_v)) \]

\[ > \frac{15}{32} C_4 K^{v-1} \rho_0 M_{v-1}(U(j_v)) > \alpha_1^2 K^{v-1} \psi_{v-1}^{1/2} \rho_0 M_{v-1}(U(j_v)). \]

Thus

\[ D_v(A_M) > \alpha_1^2 K^{v-1} \psi_{v-1}^{1/2} \rho_0 M_{v-1}(U(j_v)). \]

In the case

\[ D_v(A(j_v)) < 0, \]

we let

\[ A_M = A(j_v) - (1 - \alpha_1) \frac{D'}{|D'|} \rho(U(j_v)), \]

and we get

\[ -D_v(A_M) < -\alpha_1^2 K^{v-1} \psi_{v-1}^{1/2} \rho_0 M_{v-1}(U(j_v)). \]

Combining we get

\[ |D_v(A_M)| > \alpha_1^2 K^{v-1} \psi_{v-1}^{1/2} \rho_0 M_{v-1}(U(j_v)). \] (3.53)

Let

\[ \Omega = B(A(j_v), (1 - \alpha_1) \rho(U(j_v))). \]

Since \( A_M \in \Omega \), we conclude by (3.53)

\[ \|D_v\|_\Omega > \alpha_1^2 K^{v-1} \psi_{v-1}^{1/2} \rho_0 M_{v-1}(U(j_v)). \] (3.54)

By (3.10) and (3.11) we have

\[ \tau\left( \{ A \in \Omega : |D_v(A)| < \epsilon_0 \alpha_1^2 K^{v-1} \psi_{v-1}^{1/2} \rho_0 M_{v-1}(U(j_v)) \} \right) \leq \frac{1}{2} \tau(\Omega). \]

Thus there exists \( A_0 \in \Omega \cap \text{supp}(\tau) \) such that
and player White chooses a ball $W(j_v) = B(A_0, \alpha_1 \rho(U(j_v)))$. Assume $A \in W(j_v)$.

Notice that every coordinate of $M_v(A)$ is a certain determinant of a $v \times v$ matrix depending on some $\gamma_j$. The absolute values of the partial derivatives of every such determinant are no greater then

$$|M_v(A)| \leq NM_{v-1}(U(j_v)).$$

Set $C = N \cdot \max_{0 \leq v \leq N} \binom{N}{v}$. By elementary calculus (mean value theorem),

$$|\tilde{M}_v(A_0) - \tilde{M}_v(A)| \leq \sqrt{\left( \max_{0 \leq v \leq N} \binom{N}{v} \right)^2} \cdot 2\alpha_1 \rho(U(j_v)) \cdot NM_{v-1}(U(j_v))$$

$$= 2C\alpha_1 \rho(U(j_v))M_{v-1}(U(j_v)) < 2C\psi N\epsilon_0 \alpha_1^2 K^{v-1} \rho_0 M_{v-1}(U(j_v))$$

$$\leq \frac{1}{2} \epsilon_0 \psi_{v-1} \alpha_1^2 K^{v-1} \rho_0 M_{v-1}(U(j_v)).$$

Combining with (3.55) we get for every $A \in W(j_v)$,

$$|\tilde{M}_v(A)| > \frac{1}{2} \epsilon_0 \psi_{v-1} \alpha_1^2 K^{v-1} \rho_0 M_{v-1}(U(j_v)) \geq \psi \nu \mu \nu \rho_0 M_{v-1}(U(j_v)).$$

We conclude that every $A \in U(j_v + 1)$ satisfies (3.14) and the first ball $U(i_v)$ satisfying

$$\rho(U(i_v)) < \mu \nu \rho_0,$$

will satisfy (3.14). Player White can play in an arbitrary way until such a ball is reached by player Black. □

4. Application to fractals

A map $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is a similarity if it can be written as

$$\phi(x) = \rho \Theta(x) + y,$$

where $\rho \in \mathbb{R}^+, \Theta \in O(N, \mathbb{R})$ and $y \in \mathbb{R}^N$. It is said to be contracting if $\rho < 1$. It is known (see [H] for a more general statement) that for any finite family $\phi_1, \ldots, \phi_m$ of contracting similarities there exists a unique non-empty compact set $\mathcal{K}$, called the attractor or limit set of the family, such that

$$\mathcal{K} = \bigcup_{i=1}^m \phi_i(\mathcal{K}).$$

Say that $\phi_1, \ldots, \phi_m$ as above satisfy the open set condition (first introduced in [H]) if there exists an open subset $U \subset \mathbb{R}^N$ such that

$$\phi_i(U) \subset U \quad \text{for all} \ i = 1, \ldots, m,$$

and

$$i \neq j \Rightarrow \phi_i(U) \cap \phi_j(U) = \emptyset.$$
The family \( \{ \phi_i \} \) is called **irreducible** if there is no finite collection of proper affine subspaces which is invariant under each \( \phi_i \). The well-known self-similar sets, like Cantor’s ternary set, Koch’s curve or Sierpinski’s gasket, are all examples of attractors of irreducible families of contracting similarities satisfying the open set condition.

As an immediate consequence of Theorem 1 we have

**Corollary 3.** Let \( \{ \phi_1, \ldots, \phi_k \} \) be a finite irreducible family of contracting similarity maps of \( \mathbb{R}^n \) satisfying the open set condition with \( K \) its attractor. Then

\[
\dim(\text{BA}(M, N) \cap K) = \dim K.
\]

**Proof.** Let \( \delta \) be the Hausdorff dimension of \( K \), and \( \tau \) the restriction of the \( \delta \)-dimensional Hausdorff measure to \( K \).

It is known that \( \tau \) is an absolutely friendly measure. (See [KLW, Theorem 2.3, Lemma 8.2 and 8.3].) Furthermore, it was proved in [F, Corollary 5.3] that for this particular case a winning set enjoys full dimension. \( \Box \)

5. **Windim**

Let \( M \) be a complete metric space. Define the winning dimension of \( S \subset M \), \( \text{Windim}(S) \), as follows. If \( S \) is \( \alpha \)-winning for no \( \alpha > 0 \), then \( \text{Windim}(S) = 0 \).

Otherwise \( \text{Windim}(S) \) is the least upper bound on all \( 0 < \alpha < 1 \) such that \( S \) is \( \alpha \)-winning.

In [S2] Schmidt was able to prove that the winning dimension of \( \text{BA}(M, N) \) in \( \mathbb{R}^H \) is \( \frac{1}{2} \). This is the best possible result for any proper subset of \( \mathbb{R}^H \). In this paper as well in [F], no upper bound on the winning dimension of \( \text{BA}(M, N) \cap K \) (where \( K \) is as defined in Corollary 3) is given and a natural question would be whether one could improve the proof leading to the optimal upper bound of \( \frac{1}{2} \) similar to the case in [S2]. In what follows we prove that in general one cannot.

Let \( C \) denote the usual middle third Cantor set and we remind that \( \text{BA}(1, 1) \) is the set of badly approximable numbers.

**Proposition 4.** \( \text{Windim}(C \cap \text{BA}(1, 1)) \leq \frac{1}{3} \).

**Proof.** For every \( k \in \mathbb{N} \) we denote by \( U(k) \) (respectively \( W(k) \)) player Black’s \( k \)th ball choice (respectively player White’s \( k \)th ball choice).

Given any \( \alpha > \frac{1}{2} \), we prove that player Black can always pick \( \beta = \beta(\alpha) \) and specify a strategy such that \( \bigcap_{k=0}^{\infty} W(k) = \bigcap_{k=0}^{\infty} W(k) \) is not a badly approximable number.

Given \( \alpha = \frac{1}{3} + \epsilon \) for some \( \epsilon > 0 \), there exists \( N \in \mathbb{N}^+ \) such that \( \frac{1}{3N} < \epsilon \).

Thus it is sufficient to prove the proposition for \( \alpha = \frac{1}{3} + \frac{1}{3N} \) for any given \( N \).

Given \( \alpha = \frac{1}{3} + \frac{1}{3N} \) let

\[
\beta = \frac{1}{3^{N-1}} + 1 \quad \text{and} \quad U(0) = B(0, 1).
\]

We have

\[
\rho(U(k)) = \frac{1}{3k} \quad \text{and} \quad \rho(W(k)) = \frac{1}{3k+1} + \frac{1}{3N(k+1)}.
\]

We prove by induction on \( k \) that player Black can always choose 0 as the center of his balls. Obviously for \( k = 0 \) the condition is satisfied.

Assume for \( k \), i.e., \( U(k) = B(0, \frac{1}{3N}) \). As \( \rho(W(k)) = \frac{1}{3N+1} + \frac{1}{3N(k+1)} \), the rightmost point player White can choose for his center is \( \frac{1}{3N+1} \). To see this notice that the next point bigger then \( \frac{1}{3N+1} \)
in C is \( \frac{2}{3Nk+1} \). But player White cannot choose this point on account that his radius is too large, i.e.,
\[ \rho(W(k)) > \frac{1}{3Nk+1}. \]

We notice that the closed interval \([-\frac{1}{3Nk+1}, \frac{1}{3Nk+1} + \frac{1}{3Nk+1}] \subset W(k) \) for any choice of player White's ball \( W(k) \). In particular,
\[
U(k+1) = B\left(0, \frac{1}{3N(k+1)}\right) \subset W(k) \quad \text{for any possible } W(k).
\]

6. Measure of intersection

The result regarding the intersection's dimension of \( \text{BA}(M, N) \) and the compact support of an absolutely friendly measure on \( \mathbb{R}^H \) (where \( H = M \cdot N \)) is similar to that of the dimension of \( \text{BA}(M, N) \), i.e., in both cases the result is a full dimension. It is well known that \( \lambda(\text{BA}(M, N)) = 0 \) where \( \lambda \) is the Lebesgue measure in \( \mathbb{R}^H \). It is therefore logical to ask the following:

**Question 1.** Let \( \tau \) be an absolutely friendly measure on \( \mathbb{R}^H \) supported on \( \mathcal{K} \), a compact subset of \( \mathbb{R}^H \). Is it true in general that
\[
\tau(\mathcal{K} \cap \text{BA}(M, N)) = 0.
\]

The answer is negative as the following example demonstrates, but one may pose the seemingly simpler question to which the answer, as far as the author is aware of, is unknown.

**Question 2.** What is the Hausdorff measure of intersection between the ternary Cantor set and the set of badly approximable numbers? (Where obviously the dimension of the measure is the Cantor set's dimension.) In other words, is it true that almost all points on the Cantor set are not badly approximable with respect to the appropriate Hausdorff measure?

**Example 1.** Consider continued fraction expansions \([0; a_1, a_2, \ldots, a_n, \ldots]\) such that for every \( i \in \mathbb{N} \), \( a_i \in \{1, 3\} \).

Let \( A_{a_1, a_2, \ldots, a_n} \) denote the interval containing all numbers with continued fraction expansion initial segment \([0; a_1, a_2, \ldots, a_n]\).

Thus for example \( A_1 = [\frac{1}{2}, 1] \), \( A_3 = [\frac{1}{3}, \frac{1}{3}] \) and \( A_{1,1,1,3} = [\frac{7}{14}, \frac{9}{14}] \).

Let \( I = [0, 1] \) be the 0th stage of the construction, \( A_1 \) and \( A_3 \) belong to the first stage of the construction \( E_1 \), \( A_{1,1} \), \( A_{1,3} \), \( A_{3,1} \), \( A_{3,3} \) to the second, \( E_2 \) and so on.

By definition, if there exists \( i \in \mathbb{N} \), \( 1 \leq i \leq n \), such that \( a_i \neq a'_i \), then
\[
A_{a_1, a_2, \ldots, a_i, \ldots, a_n} \cap A_{a_1, a_2, \ldots, a'_i, \ldots, a_n} = \emptyset.
\]

We recall that if we recursively define
\[
q_{-1} = 0, \quad q_0 = 1 \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2}, \quad (6.56)
\]
then
\[
l(A_{a_1, a_2, \ldots, a_n}) = \frac{1}{q_n(q_n + q_{n-1})},
\]
where \( l(A_{a_1, a_2, \ldots, a_n}) \) is the length of the interval \( A_{a_1, a_2, \ldots, a_n} \).
Claim 1. For any \( n \in \mathbb{N} \),

\[
\frac{1}{12} < \frac{l(A_{a_1, a_2, \ldots, a_n, a_{n+1}})}{l(A_{a_1, a_2, \ldots, a_n})} < \frac{1}{2}.
\]

Proof. By (6.56),

\[
l(A_{a_1, a_2, \ldots, a_n, 3}) < l(A_{a_1, a_2, \ldots, a_n, 1}).
\]

Thus,

\[
l(A_{a_1, a_2, \ldots, a_n, a_{n+1}}) = l(A_{a_1, a_2, \ldots, a_n}) = q_n(q_n + q_{n-1}) \leq \frac{q_n(q_n + q_{n-1})}{(q_n + q_{n-1})(2q_n + q_{n-1})} = \frac{q_n}{2q_n + q_{n-1}} < \frac{1}{2}.
\]

On the other hand,

\[
l(A_{a_1, a_2, \ldots, a_n, a_{n+1}}) = l(A_{a_1, a_2, \ldots, a_n}) = q_n(q_n + q_{n-1}) \geq \frac{q_n(q_n + q_{n-1})}{(3q_n + q_{n-1})(4q_n + q_{n-1})} > \frac{1}{12},
\]

by a simple calculation and using (6.56).

We define the measure \( \tau \) as follows:

\[
\tau(A_{a_1, a_2, \ldots, a_i, \ldots, a_n}) = \frac{1}{2^n}.
\]

Following W.A. Veech [V, Section 2, Proposition 2.5] and [KW] (Section 6, Remark 6.2 — in which a generalization of Veech’s definitions and results are discussed in relation to the friendly conditions), it is easily checked that \( \tau \) is an absolutely friendly measure and obviously,

\[
\tau(BA(1, 1) \cap \text{supp}(\tau)) = 1. \quad \square
\]

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