ELSEVIER
Stochastic Processes and their Applications 102 (2002) 103–116
stochastic processes and their applications
www.elsevier.com/locate/spa

Regularization of differential equations by fractional noise

David Nualart[a,*,1], Youssef Ouknine[b,2]

[a] Facultat de Matemàtiques, Universitat de Barcelona, Gran Via, 585, 08007 Barcelona, Spain
[b] Faculté des Sciences Semlalia, Département de Mathématiques, Université Cadi Ayyad, BP 2390, Marrakech, Maroc

Received 2 January 2002; received in revised form 26 April 2002; accepted 29 April 2002

Abstract

Let $\{B_t^H, t \in [0,T]\}$ be a fractional Brownian motion with Hurst parameter H. We prove the existence and uniqueness of a strong solution for a stochastic differential equation of the form $X_t = x + B_t^H + \int_0^t b(s, X_s)\,\mathrm{d}s$, where $b(s,x)$ is a bounded Borel function with linear growth in x (case $H \leqslant \frac{1}{2}$) or a Hölder continuous function of order strictly larger than $1 - 1/2H$ in x and than $H - \frac{1}{2}$ in time (case $H > \frac{1}{2}$). © 2002 Elsevier Science B.V. All rights reserved.

MSC: 60H10; 60G18

Keywords: Fractional Brownian motion; Stochastic integrals; Malliavin calculus

1. Introduction

Let $B^H = \{B_t^H,\ t \in [0,T]\}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. That is, B^H is a centered Gaussian process with covariance

$$R_H(t,s) = E(B_t^H B_s^H) = \tfrac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}.$$

If $H = \frac{1}{2}$ the process B^H is a standard Brownian motion. Consider the following stochastic differential equation

$$X_t = x + B_t^H + \int_0^t b(s, X_s)\,\mathrm{d}s, \tag{1}$$

* Corresponding author. Tel.: +34-3-402-1656; fax: +34-3-402-1601.
E-mail address: nualart@mat.ub.es (D. Nualart).

[1] Supported by the DGES grant BFM2000-0598.
[2] Supported by Moroccan Program PARS MI 37.

PII: S0304-4149(02)00155-2

where $b:[0,T] \times \mathbb{R}$ is a Borel function. The purpose of this paper is to prove the existence and uniqueness of a strong solution to this equation under the following weak regularity assumptions on the coefficient $b(t,x)$:

(i) If $H \leqslant \frac{1}{2}$ (*singular case*), we assume the linear growth condition

$$|b(t,x)| \leqslant C(1+|x|). \tag{2}$$

(ii) If $H > \frac{1}{2}$ (*regular case*), we assume that b is Hölder continuous of order $1 > \alpha > 1 - 1/2H$ in x and of order $\gamma > H - \frac{1}{2}$ in time:

$$|b(t,x) - b(s,y)| \leqslant C(|x-y|^{\alpha} + |t-s|^{\gamma}). \tag{3}$$

In the case $H = \frac{1}{2}$, the process B^H is an ordinary Brownian motion. In this case the existence of a strong solution is well known by the results of Zvonkin (1974) and Veretennikov (1981). See also the work by Nakao (1972) and its generalization by Ouknine (1988). In these papers the equation may contain a nonconstant diffusion coefficient which is supposed to be bounded below by a positive constant and of bounded variation on any compact interval.

As we shall see, in the case of Eq. (1) driven by the fractional Brownian motion, the weak existence and uniqueness are established using a suitable version of Girsanov theorem established by Decreusefond and Üstunel (1999). Girsanov theorem for the fractional Brownian motion has also been used in the works by Norros et al. (1999) and Moret and Nualart (2002). Notice that in the regular case $H > \frac{1}{2}$ the coefficient $b(t,x)$ is supposed to be Hölder continuous, and this condition implies also the existence of a pathwise solution. In the singular case $H < \frac{1}{2}$, the existence of a strong solution could be deduced from an extension of Yamada–Watanabe's theorem to this context. We have used another argument to construct a strong solution in the case $H < \frac{1}{2}$ which uses a comparison theorem and a Krylov-type estimate. This method has also been used to handle one-dimensional heat equations with additive space–time white noise in Gyöngy and Pardoux (1993).

The paper is organized as follows. In Section 2 we give some preliminaries on fractional calculus and fractional Brownian motion. In Section 3 we formulate a Girsanov theorem and show the existence of a weak solution to Eq. (1). As a consequence we deduce the uniqueness in law and the pathwise uniqueness. Finally, Section 4 discusses the existence of a strong solution.

2. Preliminaries

2.1. Fractional calculus

An exhaustive survey on classical fractional calculus can be found in Samko et al. (1993). We recall some basic definitions and results.

For $f \in L^1([a,b])$ and $\alpha > 0$ the *left fractional Riemann–Liouville integral* of f *of order* α on (a,b) is given at almost all x by

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\, \mathrm{d}y,$$

where Γ denotes the Euler function.

This integral extends the usual n-order iterated integrals of f for $\alpha = n \in \mathbb{N}$. We have the first composition formula

$$I^{\alpha}_{a^+}(I^{\beta}_{a^+} f) = I^{\alpha+\beta}_{a^+} f.$$

The fractional derivative can be introduced as inverse operation. We assume $0 < \alpha < 1$ and $p > 1$. We denote by $I^{\alpha}_{a^+}(L^p)$ the image of $L^p([a,b])$ by the operator $I^{\alpha}_{a^+}$. If $f \in I^{\alpha}_{a^+}(L^p)$, the function ϕ such that $f = I^{\alpha}_{a^+}\phi$ is unique in L^p and it agrees with the *left-sided Riemann–Liouville derivative of f of order α* defined by

$$D^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \frac{f(y)}{(x-y)^{\alpha}} \, \mathrm{d}y.$$

The derivative of f has the following Weil representation:

$$D^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} \, \mathrm{d}y \right) \mathbf{1}_{(a,b)}(x), \tag{4}$$

where the convergence of the integrals at the singularity $x = y$ holds in L^p-sense.

When $\alpha p > 1$ any function in $I^{\alpha}_{a^+}(L^p)$ is $(\alpha - 1/p)$-Hölder continuous. On the other hand, any Hölder continuous function of order $\beta > \alpha$ has fractional derivative of order α. That is, $C^{\beta}([a,b]) \subset I^{\alpha}_{a^+}(L^p)$ for all $p > 1$.

Recall that by construction for $f \in I^{\alpha}_{a^+}(L^p)$,

$$I^{\alpha}_{a^+}(D^{\alpha}_{a^+} f) = f$$

and for general $f \in L^1([a,b])$ we have

$$D^{\alpha}_{a^+}(I^{\alpha}_{a^+} f) = f.$$

If $f \in I^{\alpha+\beta}_{a^+}(L^1)$, $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha + \beta \leqslant 1$ we have the second composition formula

$$D^{\alpha}_{a^+}(D^{\beta}_{a^+} f) = D^{\alpha+\beta}_{a^+} f.$$

2.2. Fractional Brownian motion

Let $B^H = \{B^H_t,\ t \in [0,T]\}$ be a fractional Brownian motion with Hurst parameter $0 < H < 1$ defined on the probability space $(\Omega, \mathscr{F}, P)$. For each $t \in [0,T]$ we denote by $\mathscr{F}^{B^H}_t$ the σ-field generated by the random variables B^H_s, $s \in [0,t]$ and the sets of probability zero.

We denote by $\mathscr{E}$ the set of step functions on $[0,T]$. Let $\mathscr{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathscr{H}} = R_H(t,s).$$

The mapping $\mathbf{1}_{[0,t]} \to B^H_t$ can be extended to an isometry between $\mathscr{H}$ and the Gaussian space $H_1(B^H)$ associated with B^H. We will denote this isometry by $\varphi \to B^H(\varphi)$.

The covariance kernel $R_H(t,s)$ can be written as

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,r)K_H(s,r)\,\mathrm{d}r,$$

where K_H is a square integrable kernel given by (see Decreusefond and Üstunel, 1999):

$$K_H(t,s) = \Gamma(H+\tfrac{1}{2})^{-1}(t-s)^{H-1/2}F\left(H-\tfrac{1}{2},\tfrac{1}{2}-H,H+\tfrac{1}{2},1-\frac{t}{s}\right),$$

$F(a,b,c,z)$ being the Gauss hypergeometric function. Consider the linear operator K_H^* from $\mathscr{E}$ to $L^2([0,T])$ defined by

$$(K_H^*\varphi)(s) = K_H(T,s)\varphi(s) + \int_s^T (\varphi(r)-\varphi(s))\frac{\partial K_H}{\partial r}(r,s)\,\mathrm{d}r.$$

For any pair of step functions φ and ψ in $\mathscr{E}$ we have (see Alòs et al., 2001)

$$\langle K_H^*\varphi, K_H^*\psi\rangle_{L^2([0,T])} = \langle\varphi,\psi\rangle_{\mathscr{H}}.$$

As a consequence, the operator K_H^* provides an isometry between the Hilbert spaces $\mathscr{H}$ and $L^2([0,T])$. Hence, the process $W=\{W_t,\ t\in[0,T]\}$ defined by

$$W_t = B^H((K_H^*)^{-1}(\mathbf{1}_{[0,t]})) \tag{5}$$

is a Wiener process, and the process B^H has an integral representation of the form

$$B_t^H = \int_0^t K_H(t,s)\,\mathrm{d}W_s, \tag{6}$$

because $(K_H^*\mathbf{1}_{[0,t]})(s) = K_H(t,s)$.

On the other hand, the operator K_H on $L^2([0,T])$ associated with the kernel K_H is an isomorphism from $L^2([0,T])$ onto $I_{0^+}^{H+1/2}(L^2([0,T]))$ and it can be expressed in terms of fractional integrals as follows (see Decreusefond and Üstunel, 1999):

$$(K_Hh)(s) = I_{0^+}^{2H}s^{1/2-H}I_{0^+}^{1/2-H}s^{H-1/2}h, \quad \text{if } H\leqslant 1/2, \tag{7}$$

$$(K_Hh)(s) = I_{0^+}^{1}s^{H-1/2}I_{0^+}^{H-1/2}s^{1/2-H}h, \quad \text{if } H\geqslant 1/2, \tag{8}$$

where $h\in L^2([0,T])$.

We will make use of the following definition of $\mathscr{F}_t$-fractional Brownian motion.

Definition 1. Let $\{\mathscr{F}_t,\ t\in[0,T]\}$ be a right-continuous increasing family of σ-fields on $(\Omega,\mathscr{F},P)$ such that $\mathscr{F}_0$ contains the sets of probability zero. A fractional Brownian motion $B^H=\{B^H,\ t\in[0,T]\}$ is called an $\mathscr{F}_t$-fractional Brownian motion if the process W defined in (5) is an $\mathscr{F}_t$-Wiener process.

3. Existence of a weak solution, and pathwise uniqueness property

3.1. Girsanov transform

As in the previous section, let B^H be a fractional Brownian motion with Hurst parameter $0<H<1$ and denote by $\{\mathscr{F}_t^{B^H},\ t\in[0,T]\}$ its natural filtration.

Given an adapted process with integrable trajectories $u = \{u_t,\ t \in [0,T]\}$ consider the transformation

$$\tilde{B}_t^H = B_t^H + \int_0^t u_s \, \mathrm{d}s. \tag{9}$$

We can write

$$\tilde{B}_t^H = B_t^H + \int_0^t u_s \, \mathrm{d}s = \int_0^t K_H(t,s) \, \mathrm{d}W_s + \int_0^t u_s \, \mathrm{d}s$$

$$= \int_0^t K_H(t,s) \, \mathrm{d}\tilde{W}_s,$$

where

$$\tilde{W}_t = W_t + \int_0^t \left(K_H^{-1} \left(\int_0^{\cdot} u_s \, \mathrm{d}s \right)(r) \right) \mathrm{d}r. \tag{10}$$

Notice that $K_H^{-1}(\int_0^{\cdot} u_s \, \mathrm{d}s)$ belongs to $L^2([0,T])$ almost surely if and only if $\int_0^{\cdot} u_s \, \mathrm{d}s \in I_{0^+}^{H+1/2}(L^2([0,T]))$. As a consequence we deduce the following version of the Girsanov theorem for the fractional Brownian motion, which has been obtained in (Decreusefond and Üstunel, 1999, Theorem 4.9):

Theorem 2. *Consider the shifted process* (9) *defined by a process* $u = \{u_t,\ t \in [0,T]\}$ *with integrable trajectories. Assume that*

(i) $\int_0^{\cdot} u_s \, \mathrm{d}s \in I_{0^+}^{H+1/2}(L^2([0,T]))$, *almost surely*.
(ii) $E(\xi_T) = 1$, *where*

$$\xi_T = \exp\left(-\int_0^T \left(K_H^{-1} \int_0^{\cdot} u_s \, \mathrm{d}s \right)(s) \, \mathrm{d}W_s - \frac{1}{2} \int_0^T \left(K_H^{-1} \int_0^{\cdot} u_s \, \mathrm{d}s \right)^2 (s) \, \mathrm{d}s \right).$$

Then the shifted process $\tilde{B}^H$ *is an* $\mathscr{F}_t^{B^H}$*-fractional Brownian motion with Hurst parameter* H *under the new probability* $\tilde{P}$ *defined by* $\mathrm{d}\tilde{P}/\mathrm{d}P = \xi_T$.

Proof. By the standard Girsanov theorem applied to the adapted and square integrable process $K_H^{-1}(\int_0^{\cdot} u_s \, \mathrm{d}s)$ we obtain that the process $\tilde{W}$ defined in (10) is an $\mathscr{F}_t^{B^H}$— Brownian motion under the probability $\tilde{P}$. Hence, the result follows. □

From (7) and (8) the inverse operator K_H^{-1} is given by

$$K_H^{-1} h = s^{H-1/2} D_{0^+}^{H-1/2} s^{1/2-H} h', \quad \text{if } H > 1/2, \tag{11}$$

$$K_H^{-1} h = s^{1/2-H} D_{0^+}^{1/2-H} s^{H-1/2} D_{0^+}^{2H} h \quad \text{if } H < 1/2 \tag{12}$$

for all $h \in I_{0+}^{H+1/2}(L^2([0,T]))$. If h is absolutely continuous, we can write for $H < 1/2$

$$K_H^{-1}h = s^{H-1/2} I_{0+}^{1/2-H} s^{1/2-H} h'. \tag{13}$$

In order to show (13) let us compute

$$\begin{aligned} K_H s^{H-1/2} I_{0+}^{1/2-H} s^{1/2-H} h' &= I_{0+}^{2H} s^{1/2-H} I_{0+}^{1/2-H} s^{2H-1} I_{0+}^{1/2-H} s^{1/2-H} h' \\ &= \alpha_H \int_0^s (s-u)^{2H-1} u^{1/2-H} \int_0^u (u-w)^{-1/2-H} w^{2H-1} \\ &\quad \times \int_0^w r^{1/2-H} (w-r)^{-1/2-H} h'_r \, \mathrm{d}r \, \mathrm{d}w \, \mathrm{d}u \\ &= \alpha_H \int_0^s h'_r r^{1/2-H} \int_r^s u^{1/2-H} (s-u)^{2H-1} \\ &\quad \times \int_r^u (u-w)^{-1/2-H} w^{2H-1} (w-r)^{-1/2-H} \, \mathrm{d}w \, \mathrm{d}u \, \mathrm{d}r, \end{aligned}$$

where $\alpha_H = 1/(\Gamma(\frac{1}{2}-H)^2 \Gamma(2H))$. Making the change of variable $z = (u(w-r))/(w(u-r))$ the integral in $\mathrm{d}w$ equals to $B(\frac{1}{2}-H, \frac{1}{2}-H) u^{H-1/2} r^{H-1/2} (u-r)^{-2H}$, and we obtain

$$K_H s^{H-1/2} I_{0+}^{1/2-H} s^{1/2-H} h' = \int_0^s h'_r \, \mathrm{d}r,$$

which implies (13).

From (13) it follows that in the case $H \leqslant \frac{1}{2}$ a sufficient condition for (i) is $\int_0^T u_s^2 \, \mathrm{d}s < \infty$. On the other hand, from (11) we get that if $H > \frac{1}{2}$ we need $u \in I_{0+}^{H-1/2}(L^2([0,T]))$, and a sufficient condition is the fact that the trajectories of u are Hölder continuous of order $H - \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$.

3.2. Existence of a weak solution

Consider the stochastic differential equation

$$X_t = x + B_t^H + \int_0^t b(s, X_s) \, \mathrm{d}s, \quad 0 \leqslant t \leqslant T, \tag{14}$$

where b is a Borel function on $[0,T] \times \mathbb{R}$.

By a *weak solution* to Eq. (14) we mean a couple of adapted continuous processes (B^H, X) on a filtered probability space $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t, \ t \in [0,T]\})$, such that:

(i) B^H is an $\mathscr{F}_t$—fractional Brownian motion in the sense of Definition 1.
(ii) X and B^H satisfy (14).

Theorem 3. *Suppose that $b(t,x)$ satisfies the linear growth condition* (2) *if $H < \frac{1}{2}$ or the Hölder continuity condition* (3) *if $H > \frac{1}{2}$. Then Eq.* (14) *has a weak solution.*

Proof. Set $\tilde{B}_t^H = B_t^H - \int_0^t b(s, B_s^H + x)\,\mathrm{d}s$. We claim that the process $u_s = -b(B_s^H + x)$ satisfies conditions (i) and (ii) of Theorem 2. If this claim is true, under the probability measure $\tilde{P}$, $\tilde{B}^H$ is an $\mathscr{F}_t^{B^H}$—fractional Brownian motion, and $(B^H, \tilde{B}^H)$ is a weak solution of (14) on the filtered probability space $(\Omega, \mathscr{F}, \tilde{P}, \{\mathscr{F}_t^{B^H}, t \in [0,T]\})$.

Set

$$v_s = -K_H^{-1}\left(\int_0^{\cdot} b(r, B_r^H + x)\,\mathrm{d}r\right)(s).$$

In order to show that the process v satisfies conditions (i) and (ii) of Theorem 2 we distinguish the two cases $H < \frac{1}{2}$ and $H > \frac{1}{2}$. Along the proof c_H will denote a generic constant depending only on H.

Case $H < \frac{1}{2}$: From (13) and the linear growth property of b we obtain

$$\begin{aligned} |v_s| &= |s^{H-1/2} I_{0^+}^{1/2-H} s^{1/2-H} b(s, B_s^H + x)| \\ &= c_H s^{H-1/2}\left|\int_0^s (s-r)^{-1/2-H} r^{1/2-H} b(r, B_r^H + x)\,\mathrm{d}r\right| \\ &\leqslant c_H C(1 + |x| + \|B^H\|_\infty). \end{aligned} \tag{15}$$

From (13) it follows that the operator K_H^{-1} preserves the adaptability property. Hence, the process v is adapted and then condition (ii) can be proved using Novikov criterion. Indeed (see, for instance, Theorem 1.1 in Friedman (1975)) it suffices to show that there exists a constant $\lambda > 0$ such that

$$\sup_{0 \leqslant s \leqslant T} E(\exp(\lambda v_s^2)) < \infty, \tag{16}$$

which is an immediate consequence of (15) and the exponential integrability of the square of a seminorm of a Gaussian process (see Fernique, 1974).

Case $H > \frac{1}{2}$: Again (11) implies that the process v is adapted. From (11) we obtain

$$\begin{aligned} v_s &= -s^{H-1/2} D_{0^+}^{H-1/2} s^{1/2-H} b(s, B_s^H + x) \\ &:= -c_H(\alpha(s) + \beta(s)), \end{aligned}$$

where

$$\begin{aligned} \alpha(s) = {}& b(s, B_s^H + x) s^{1/2-H} \\ &+ \left(H - \frac{1}{2}\right) s^{H-1/2} b(s, B_s^H + x) \int_0^s \frac{s^{1/2-H} - r^{1/2-H}}{(s-r)^{1/2+H}}\,\mathrm{d}r \\ &+ \left(H - \frac{1}{2}\right) s^{H-1/2} \int_0^s \frac{b(s, B_s^H + x) - b(r, B_s^H + x)}{(s-r)^{1/2+H}} r^{1/2-H}\,\mathrm{d}r \end{aligned}$$

and

$$\beta(s) = \left(H - \frac{1}{2}\right) s^{H-1/2} \int_0^s \frac{b(r, B_s^H + x) - b(r, B_r^H + x)}{(s-r)^{1/2+H}} r^{1/2-H}\,\mathrm{d}r.$$

Using the estimate

$$|b(s, B_s^H + x)| \leqslant |b(0,x)| + C(|s|^\gamma + |B_s^H|^\alpha)$$

and the equality

$$\int_0^s \frac{r^{1/2-H} - s^{1/2-H}}{(s-r)^{1/2+H}} \, \mathrm{d}r = c_H s^{1-2H}$$

we obtain

$$|\alpha(s)| \leqslant c_H (s^{1/2-H}[|b(0,x)| + C(|s|^\gamma + |B_s^H|^\alpha)] + C s^{\gamma+1/2-H})$$

$$\leqslant c_H s^{1/2-H} (C \|B^H\|_\infty^\alpha + C s^\gamma + |b(0,x)|).$$

As a consequence, taking into account that $\alpha < 1$, we have for any $\lambda > 1$

$$E\left(\exp\left(\lambda \int_0^T \alpha(s)^2 \, \mathrm{d}s\right)\right) < \infty. \tag{17}$$

In order to estimate the term $\beta(s)$, we apply the Hölder continuity condition (3) and we get

$$|\beta(s)| \leqslant c_H s^{H-1/2} \int_0^s \frac{|B_s^H - B_r^H|^\alpha}{(s-r)^{H+1/2}} r^{1/2-H} \, \mathrm{d}r$$

$$\leqslant c_H s^{1/2-H+\alpha(H-\varepsilon)} G^\alpha,$$

where we have fixed $\varepsilon < H - \frac{1}{\alpha}(H - \frac{1}{2})$ and we denote

$$G = \sup_{0 \leqslant s < r \leqslant T} \frac{|B_s^H - B_r^H|}{|s-r|^{H-\varepsilon}}.$$

By Fernique's Theorem, taking into account that $\alpha < 1$, for any $\lambda > 1$ we have

$$E\left(\exp\left(\lambda \int_0^T \beta(s)^2 \, \mathrm{d}s\right)\right) < \infty,$$

and we deduce condition (ii) of Theorem 2 by means of Novikov criterion. $\square$

3.3. Uniqueness in law and pathwise uniqueness

Let (X, B^H) be a weak solution of the stochastic differential equation (14) defined in the filtered probability space $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t, \ t \in [0,T]\})$. Define

$$u_s = \left(K_H^{-1} \int_0^\cdot b(r, X_r) \, \mathrm{d}r\right)(s).$$

Let $\tilde{P}$ defined by

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}P} = \exp\left(-\int_0^T u_s \, \mathrm{d}W_s - \frac{1}{2}\int_0^T u_s^2 \, \mathrm{d}s\right). \tag{18}$$

We claim that the process u_s satisfies conditions (i) and (ii) of Theorem 2. In fact, u_s is an adapted process and taking into account that X_t has the same regularity properties as the fBm we deduce that $\int_0^T u_s^2\,\mathrm{d}s < \infty$ almost surely. Finally, we can apply again Novikov theorem in order to show that $E(\mathrm{d}\tilde{P}/\mathrm{d}P) = 1$, because by Gronwall's lemma

$$\|X\|_\infty \leqslant (x + \|B^H\|_\infty + C_1 T)\mathrm{e}^{C_2 T}$$

and

$$|X_t - X_s| \leqslant |B_t^H - B_s^H| + C_3|t-s|(1 + \|X\|_\infty)$$

for some constants $C_i,\ i = 1,2,3$.

By the classical Girsanov theorem the process

$$\tilde{W}_t = W_t + \int_0^t u_s\,\mathrm{d}r$$

is an $\mathscr{F}_t$-Brownian motion under the probability $\tilde{P}$. In terms of the process $\tilde{W}_t$ we can write

$$X_t = x + \int_0^t K_H(t,s)\,\mathrm{d}\tilde{W}_s.$$

Hence, $X - x$ is an $\mathscr{F}_t$-fractional Brownian motion with respect to the probability $\tilde{P}$ with Hurst parameter equal to H. As a consequence, the processes $X - x$ and $\tilde{B}_t^H$ have the same distribution under the probability P. In fact, if Ψ is a bounded measurable functional on $C([0,T])$, we have

$$\begin{aligned}
E_P(\Psi(X-x)) &= \int_\Omega \Psi(\xi - x)\frac{\mathrm{d}P}{\mathrm{d}\tilde{P}}(\xi)\,\mathrm{d}\tilde{P} \\
&= E_{\tilde{P}}\left(\Psi(X-x)\exp\left(\int_0^T \left(K_H^{-1}\int_0^\cdot b(r,X_r)\,\mathrm{d}r\right)(s)\,\mathrm{d}W_s\right.\right. \\
&\quad \left.\left. + \frac{1}{2}\int_0^T \left(K_H^{-1}\int_0^\cdot b(r,X_r)\,\mathrm{d}r\right)^2(s)\,\mathrm{d}s\right)\right) \\
&= E_{\tilde{P}}\left(\Psi(X-x)\left(\exp\int_0^T \left(K_H^{-1}\int_0^\cdot b(r,X_r)\,\mathrm{d}r\right)(s)\,\mathrm{d}\tilde{W}_s\right.\right. \\
&\quad \left.\left. - \frac{1}{2}\int_0^T \left(K_H^{-1}\int_0^\cdot b(r,X_r)\,\mathrm{d}r\right)^2(s)\,\mathrm{d}s\right)\right) \\
&= E_P\left(\Psi(B^H)\left(\exp\int_0^T \left(K_H^{-1}\int_0^\cdot b(r,B_r^H + x)\,\mathrm{d}r\right)(s)\,\mathrm{d}W_s\right)\right) \\
&\quad \left.\left. - \frac{1}{2}\int_0^T \left(K_H^{-1}\int_0^\cdot b(r,B_r^H + x)\,\mathrm{d}r\right)^2(s)\,\mathrm{d}s\right)\right) \\
&= E_P(\Psi(\tilde{B}^H)).
\end{aligned}$$

In conclusion we have proved the following result

Theorem 4. *Suppose that $b(t,x)$ satisfies the assumptions of Theorem* 3. *Then two weak solutions must have the same distribution.*

As a corollary we deduce the pathwise uniqueness of the solution to Eq. (14):

Theorem 5. *Suppose that $b(t,x)$ satisfies the assumptions of Theorem* 3. *Then two weak solutions defined on the same filtered probability space must coincide almost surely.*

Proof. Let X^1 and X^2 be two weak solutions defined on the same filtered probability space $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t,\ t \in [0,T]\})$ with respect to the same fractional Brownian motion. It is easy to see that $\sup(X^1, X^2)$ and $\inf(X^1, X^2)$ are also solutions, then they have the same laws which implies that $X^1 = X^2$. □

4. Existence of strong solutions

Since b is continuous in the case $H > 1/2$, we have existence of a solution. In particular, if b satisfies the Hölder continuity assumption (3) then we have existence and pathwise uniqueness result and this is better than the corresponding result for ordinary differential equations because the uniqueness fails. (Take, for instance, $b(x) = \sqrt{|x|}$.)

Moreover, in the case $H > 1/2$ we can establish the uniqueness and existence of a strong solution for the equation

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}B_s^H + \int_0^t b(s, X_s)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant T, \tag{19}$$

where σ is a Hölder continuous function of order $\delta > 1/H - 1$ such that $|\sigma(z)| \geqslant c > 0$. By definition a solution to Eq. (19) is an adapted process whose trajectories are Hölder continuous of order $H - \varepsilon$ for all $\varepsilon > 0$. Under these assumptions, the stochastic integral that appears in Eq. (19) exists pathwise. We refer to Zähle (1998) for the definition of this pathwise integral using fractional calculus.

Set

$$F(x) = \int_0^x \frac{1}{\sigma(z)}\,\mathrm{d}z.$$

Then, using the change-of-variables formula for the fractional Brownian motion (see, for instance, Zähle, 1998, Theorem 4.3.1) we obtain that a process X is a solution to Eq. (19) if and only if the process $Y_t = F(X_t)$ is a solution of

$$Y_t = F(x) + B_t^H + \int_0^t \frac{b(s, F^{-1}(Y_s))}{\sigma(F^{-1}(Y_s))}\,\mathrm{d}s.$$

We conclude that if $\delta > 1 - 1/2H$ then there is a unique strong solution to Eq. (19).

Let us now prove the existence of a strong solution in the singular case $H < \frac{1}{2}$. Let us first establish a Krylov-type inequality that will play an essential role in the sequel.

Proposition 6. *Suppose now that b is uniformly bounded. Let X denote a weak solution to Eq.* (14). *Fix $\rho > H + 1$. There exists a constant K depending on T, $\|b\|_\infty$ and ρ such that for any measurable nonnegative function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ we have*

$$E \int_0^T g(t,X_t)\,\mathrm{d}t \leqslant K \left(\int_0^T \int_{\mathbb{R}} g(t,x)^\rho \,\mathrm{d}x\,\mathrm{d}t \right)^{1/\rho}. \tag{20}$$

Proof. Let $Z = \mathrm{d}\tilde{P}/\mathrm{d}P$ be the Radon–Nikodym density given by (18). By Hölder's inequality with $1/\alpha + 1/\beta = 1$

$$\begin{aligned} E \int_0^T g(t,X_t)\,\mathrm{d}t &= \tilde{E}Z^{-1} \int_0^T g(t,X_t)\,\mathrm{d}t \\ &= K_T(\tilde{E}Z^{-\alpha})^{1/\alpha} \left(\tilde{E} \int_0^T g(t,X_t)^\beta \,\mathrm{d}t \right)^{1/\beta}. \end{aligned}$$

The expectation $\tilde{E}Z^{-\alpha}$ is uniformly bounded for any $\alpha > 1$ because, by the arguments used in the proof of Theorem 3 we obtain

$$\begin{aligned} \tilde{E}Z^{-\alpha} &= \tilde{E} \exp\left(\alpha \int_0^T u_s \,\mathrm{d}W_s + \frac{\alpha}{2} \int_0^T u_s^2 \,\mathrm{d}s \right) \\ &= \tilde{E} \exp\left(\alpha \int_0^T u_s \,\mathrm{d}\tilde{W}_s - \frac{\alpha}{2} \int_0^T u_s^2 \,\mathrm{d}s \right) < \infty. \end{aligned}$$

On the other hand, applying again Hölder's inequality with $1/\gamma' + 1/\gamma = 1$ and $\gamma > H+1$ yields

$$\begin{aligned} \tilde{E} \int_0^T g(t,X_t)^\beta \,\mathrm{d}t &= \int_0^T \frac{1}{\sqrt{2\pi} t^H} \int_{\mathbb{R}} g(t,y)^\beta \mathrm{e}^{-(y-x)^2/2t^{2H}} \,\mathrm{d}y\,\mathrm{d}t \\ &\leqslant \frac{1}{\sqrt{2\pi}} \left(\int_0^T \int_{\mathbb{R}} g(t,y)^{\beta\gamma} \,\mathrm{d}y\,\mathrm{d}t \right)^{1/\gamma} \\ &\quad \times \left(\int_0^T \int_{\mathbb{R}} t^{-H\gamma'} \mathrm{e}^{-\delta(y-x)^2/2t^{2H}} \,\mathrm{d}y\,\mathrm{d}t \right)^{1/\gamma'} \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_0^T \int_{\mathbb{R}} g(t,y)^{\beta\gamma} \,\mathrm{d}y\,\mathrm{d}t \right)^{1/\gamma} \left(\int_0^T t^{(1-\gamma')H} \,\mathrm{d}t \right)^{1/\gamma'} \\ &\leqslant \frac{c_\gamma, T}{\sqrt{2\pi}} \left(\int_0^T \int_{\mathbb{R}} g(t,y)^{\beta\gamma} \,\mathrm{d}y\,\mathrm{d}t \right)^{1/\gamma}. \end{aligned}$$

Proposition 7. *Consider a sequence $b_n(t,x)$ of measurable functions uniformly bounded by C, such that*

$$\lim_{n\to\infty} b_n(t,x) = b(t,x)$$

for almost all $(t,x)\in[0,T]\times\mathbb{R}$. Suppose also that the corresponding solutions $X_t^{(n)}$ of the equations

$$X_t^{(n)} = x + B_t^H + \int_0^t b_n(s, X_s^{(n)})\,\mathrm{d}s, \quad 0 \leqslant t \leqslant T,$$

converge a.s. to some process X_t for all $t\in[0,T]$. Then the process X_t is a solution of Eq. (14).

Proof. It suffices to show that

$$\lim_{n\to\infty} E\int_0^T |b_n(s,X_s^{(n)}) - b(s,X_s)|\,\mathrm{d}s = 0.$$

We can write

$$J(n) := E\int_0^T |b_n(s,X_s^{(n)}) - b(s,X_s)|\,\mathrm{d}s \leqslant J_1(n) + J_2(n),$$

where

$$J_1(n) := \sup_k E\int_0^T |b_k(s,X_s^{(n)}) - b_k(s,X_s)|\,\mathrm{d}s,$$

$$J_2(n) := E\int_0^T |b_n(s,X_s) - b(s,X_s)|\,\mathrm{d}s.$$

Let $\kappa:\mathbb{R}\to\mathbb{R}$ be a smooth function such that $0\leqslant\kappa(z)\leqslant 1$ for every z, $\kappa(z)=0$ for $|z|\geqslant 1$ and $\kappa(0)=1$. Fix $\varepsilon>0$ and choose $R>0$ such that

$$E\int_0^T |1-\kappa(X_t/R)|\,\mathrm{d}t < \varepsilon.$$

The sequence of functions b_k is relatively compact in $L^2([0,T]\times[-R,R])$. Hence, we can find finitely many bounded smooth functions $H_1,\dots,H_N$ such that for every k

$$\int_0^T\int_{-R}^R |b_k(t,x) - H_i(t,x)|^2\,\mathrm{d}r\,\mathrm{d}t < \varepsilon^2$$

for some H_i. We have

$$E\int_0^T |b_k(t,X_t^{(n)}) - b_k(t,X_t)|\,\mathrm{d}t \leqslant E\int_0^T |b_k(t,X_t^{(n)}) - H_i(t,X_t^{(n)})|\,\mathrm{d}t$$

$$+\sum_{j=1}^N E\int_0^T |H_j(t,X_t^{(n)}) - H_j(t,X_t)|\,\mathrm{d}t$$

$$+E\int_0^T |b_k(t,X_t)-H_i(t,X_t)|\,\mathrm{d}t$$

$$:= I_1(n,k)+I_2(n)+I_3(k).$$

By Proposition 6

$$I_1(n,k)=E\int_0^T \kappa(X_t^{(n)}/R)|b_k(t,X_t^{(n)})-H_i(t,X_t^{(n)})|\,\mathrm{d}t$$

$$+E\int_0^T [1-\kappa(X_t^{(n)}/R)]|b_k(t,X_t^{(n)})-H_i(t,X_t^{(n)})|\,\mathrm{d}t$$

$$\leqslant K\left(\int_0^T\int_{-R}^R |b_k(t,x)-H_i(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}+CE\int_0^T [1-\kappa(X_t^{(n)}/R)]\,\mathrm{d}t$$

for some constant C depending on $\|b\|_\infty$ and $\sup_i \|H_i\|_\infty$. Hence,

$$\lim_{n\to\infty}\sup_k I_1(n,k)\leqslant K\varepsilon+CE\int_0^T [1-\kappa(X_t/R)]\,\mathrm{d}t\leqslant (K+C)\varepsilon.$$

Similarly,

$$\sup_k I_3(k)\leqslant (K+C)\varepsilon.$$

Consequently,

$$\lim_{n\to\infty}\sup_k I(n,k)\leqslant 2(K+C)\varepsilon,$$

and this implies that $\lim_{n\to\infty} J_1(n)=0$. For the term $J_2(n)$ we can write

$$J_2(n)=E\int_0^T \kappa(X_t/R)|b_n(t,X_t)-b(t,X_t)|\,\mathrm{d}t$$

$$+E\int_0^T [1-\kappa(X_t/R)]|b_n(t,X_t)-b(t,X_t)|\,\mathrm{d}t$$

$$\leqslant K\left(\int_0^T\int_{-R}^R |b_n(t,x)-b(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}+CE\int_0^T [1-\kappa(X_t/R)]\,\mathrm{d}t$$

and we use the same arguments as before. □

Theorem 8. *Assume that $b(t,x)$ satisfies the linear growth condition* (2). *Then, there exists a unique strong solution to Eq.* (14).

Proof. We already know that pathwise uniqueness holds by Theorem 5. For any $R>0$ define $b_R(t,x)=b(t,(x\wedge R)\vee(-R))$. The linear growth condition implies that b_R is a bounded measurable function. Let ρ be a smooth nonnegative with compact support in $\mathbb{R}$ such that $\int_{\mathbb{R}}\rho(z)\,\mathrm{d}z=1$. For $j\in\mathbb{N}$ define

$$b_{R,j}(t,x)=j\int_{\mathbb{R}} b_R(t,z)\rho(j(x-z))\,\mathrm{d}z.$$

Moreover, let for $n \leqslant k$

$$\tilde{b}_{R,n,k} = \bigwedge_{j=n}^{k} b_{R,j} \quad \text{and} \quad \tilde{b}_{R,n} = \bigwedge_{j=n}^{\infty} b_{R,j}.$$

Clearly, $\tilde{b}_{R,n,k}$ is Lipschtiz in the variable x uniformly with respect to t, and $\tilde{b}_{R,n,k} \downarrow \tilde{b}_{R,n}$ as $k \to \infty$, $\tilde{b}_{R,n} \uparrow b_R$ as $n \to \infty$, for almost all x, for any t. Eq($\tilde{b}_{R,n,k}$) has a unique solution $\tilde{X}_{R,n,k}$. By the comparison criterion for ordinary differential equations the sequence $\tilde{X}_{R,n,k}$ decreases with k, hence it has a limit $\tilde{X}_{R,n}$. Again by the comparison theorem $\tilde{X}_{R,n,k}$ (and hence $\tilde{X}_{R,n}$) is bounded from above (resp. from below) by the solution with the constant coefficient R (resp. $-R$). Moreover, $\tilde{X}_{R,n}$ increases as n increases. So again, $\tilde{X}_{R,n}$ converges and its limit, denoted by X_R.

Finally, we apply Proposition 7. □

Acknowledgements

The work was carried out during a stay of Youssef Ouknine at the Institut de Matemàtica de la Universitat de Barcelona (IMUB). He would like to thank the IMUB for hospitality and support.

References

Alòs, E., Mazet, O., Nualart, D., 2001. Stochastic calculus with respect to Gaussian processes. Ann. Probab. 29, 766–801.

Decreusefond, L., Üstunel, A.S., 1999. Stochastic analysis of the fractional Brownian motion. Potential Anal. 10, 177–214.

Fernique, X.M., 1974. Regularité des trajectories de fonctions aléatoires gaussiennes. In: Hennequin, P.L. (Ed.), École d'Eté de Saint-Flour IV, Lecture Notes in Mathematics, Vol. 480, Springer, Berlin, pp. 2–95.

Friedman, A., 1975. Stochastic Differential Equations and Applications. Academic Press, New York.

Gyöngy, I., Pardoux, E., 1993. On quasi-linear stochastic partial differential equations. Probab. Theory Related Fields 94, 413–425.

Moret, S., Nualart, D., 2002. Onsager–Machlup functional for the fractional Brownian motion. Probab. Theory Related Fields, to appear.

Nakao, S., 1972. On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations. Osaka J. Math. 9, 513–518.

Norros, I., Valkeila, E., Virtamo, J., 1999. An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motion. Bernoulli 5, 571–587.

Ouknine, Y., 1988. Généralisation d'un Lemme de S. Nakao et applications. Stochastics 23, 149–157.

Samko, S.G., Kilbas, A.A., Mariachev, O.I., 1993. Fractional Integrals and Derivatives. Gordon and Breach Science, London.

Veretennikov, A.Ju., 1981. On strong solutions and explicit formulas for solutions of stochastic integral equations. Math. USSR-Sb. 39, 387–403.

Zähle, M., 1998. Integration with respect to fractal functions and stochastic calculus I. Probab. Theory Related Fields 111, 333–374.

Zvonkin, A.K., 1974. A transformation of the phase space of a diffusion process that removes the drift. Math. USSR-Sb. 22, 129–149.