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An analytical theory of heated duct flows in supersonic combustors

Chenxi Wu,[a)] Fengquan Zhong, Jing Fan
State Key Laboratory of High Temperature Gas Dynamics, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100190, China

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Abstract One-dimensional analytical theory is developed for supersonic duct flow with variation of cross section, wall friction, heat addition, and relations between the inlet and outlet flow parameters are obtained. By introducing a self-similar parameter, effects of heat releasing, wall friction, and change in cross section area on the flow can be normalized and a self-similar solution of the flow equations can be found. Based on the result of self-similar solution, the sufficient and necessary condition for the occurrence of thermal choking is derived. A relation of the maximum heat addition leading to thermal choking of the duct flow is derived as functions of area ratio, wall friction, and mass addition, which is an extension of the classic Rayleigh flow theory, where the effects of wall friction and mass addition are not considered. The present work is expected to provide fundamentals for developing an integral analytical theory for ramjets and scramjets.
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Keywords duct flow with heating, self-similar solution, thermal choking, supersonic combustor

Supersonic combustor is one of the most important components of scramjets for hypersonic applications. The combustor is characterized by internal duct flow with effects of variations in cross section and heat releasing due to combustion, shock structures, and wall friction, etc. Therefore, analyses of supersonic combustor flow and optimizations are critical for improvement of scramjet performances.

There are many factors to determine the internal duct flow, among which except for shock wave, variation of cross section, heat releasing, and wall friction are the most important ones. The three factors are coupled with each other, causing significant difficulties for obtaining analytical relations between the inlet and outlet flow parameters such as density, velocity, pressure, and temperature. The previous study[1,2] focused on only one of the three factors and provided classic flow analyses such as friction flow without heating and area change ("Fanno flow"), isentropic flow with change in the cross section ("nozzle flow"), and heated flow with constant cross section and without friction effect ("Rayleigh flow"). However, if all these three effects (heating, friction, and variation of cross section) are taken into consideration, to the authors' knowledge, the available theoretical work can get flow properties only by solving one-dimensional differential equations of

[a)]Corresponding author. Email: mzw9802@163.com.

flow derived from conservation laws.[3]

Solving differential equations of flow with numerical methods can obtain distributions of pressure, velocity, and temperature along the flow direction with satisfactory agreements between numerical results and experiments.[4,5] Compared to analytical relations, it is not easy to identify in numerical solutions the effects of key factors on the flow and their interactions. Moreover, if aerodynamic properties of flight vehicle and scramjet performances are considered comprehensively, numerical solution of differential flow equations becomes even more unqualified to obtain variational rules of vehicle configuration, incoming flow conditions and fuel properties as well as interactions between these factors. In comparison, analytical relation of combustor flow parameters can reveal rules of the affecting factors more directly and clearly.

In this letter, analytical relations between inlet and outlet flow parameters of duct flow with effects of variation of cross section, heat releasing, wall friction, and heat exchange, addition of fuel mass and changes in gas properties (specific heat, gas constant, etc.) are obtained based on conservations of mass, momentum and energy of the flow and gas state equation. Self-similar solution of the flow is obtained with a self-similar parameter proposed in the present work. Based on the result of self-similar solution, a relation of the maximum heat addition for the occurrence of thermal chocking is derived with wall friction and change of cross section taken into account, thereby achieving an extension of the classic "Rayleigh flow" theory. The present work is expected to provide a key fundamental for establishing an overall theoretical framework for integral analyses of hypersonic air-breathing vehicles and scramjets.

As shown in Fig. 1, considering the flow between the duct inlet and outlet cross sections as a control volume, the mass, momentum and energy equations, and the gas state equation are given as

$$\dot{m}_k = \rho_k U_k A_k = \left(1+\alpha_{\mathrm{p}}\right)\rho_j U_j A_j, \tag{1}$$

$$p_k A_k + \rho_k A_k U_k^2 = p_j A_j + \rho_j A_j U_j^2 + p_j\left(A_k - A_j\right) - 0.5\rho_j U_j^2 C_{\mathrm{D},jk} A_{\mathrm{D},jk}, \tag{2}$$

$$\dot{m}_k\left[\gamma_k R_{\mathrm{g},k}\left(T_k - T_j\right)/\left(\gamma_k - 1\right) + 0.5\left(U_k^2 - U_j^2\right)\right] = \dot{Q}_{\mathrm{C}} + \dot{Q}_{\mathrm{T}}, \tag{3}$$

$$p_j = \rho_j R_{\mathrm{g},j} T_j, \quad p_k = \rho_k R_{\mathrm{g},k} T_k, \tag{4}$$

where subscripts j and k denote the inlet and outlet, p, ρ, T, U represent the averaged pressure, density, temperature, velocity on the cross section, respectively, $\dot{m}$, A, γ, R_{g} are the mass flow rate, the confined core area (i.e. area of non-separated region in the center part of the duct flow),[3] ratio of specific heats, gas constant, α_{p} is the ratio of fuel mass flow rate to air mass flow rate, $A_{\mathrm{D},jk}$ and $C_{\mathrm{D},jk}$ are the area of the duct wall and the averaged friction coefficient, respectively, and $\dot{Q}_{\mathrm{C}}$, $\dot{Q}_{\mathrm{T}}$ are the heat rate released from the combustion, the heat exchange rate through the duct wall, respectively.

Equations (1)–(4) are complete for the four unknown parameters of p_k, ρ_k, T_k, U_k, but they are hard to be solved directly. It is found that if choosing the density ratio of ρ_k/ρ_j as a parameter and using definition of Mach number ($Ma \equiv U/\sqrt{\gamma R_{\mathrm{g}} T}$), a 2nd-order equation for ρ_k/ρ_j can be derived as

$$a\left(\rho_k/\rho_j\right)^2 + b\left(\rho_k/\rho_j\right) + c = 0, \tag{5}$$

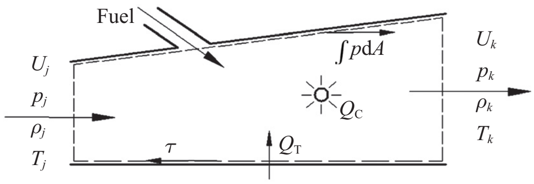

Fig. 1. Schematic diagram of a duct flow.

$$a = 1 + (\gamma_k - 1)\,\gamma_j R_{\mathrm{g},j} Ma_j^2/(2\gamma_k R_{\mathrm{g},k}) + (\gamma_k - 1)\left(\dot{Q}_{\mathrm{C}} + \dot{Q}_{\mathrm{T}}\right)/(\dot{m}_k \gamma_k R_{\mathrm{g},k} T_j), \tag{6a}$$

$$b = (R_{\mathrm{g},j}/R_{\mathrm{g},k})\left\{1 + \gamma_j Ma_j^2\left[A_j/A_k - C_{\mathrm{D},jk}A_{\mathrm{D},jk}/(2A_k)\right]\right\}, \tag{6b}$$

$$c = (R_{\mathrm{g},j}/R_{\mathrm{g},k})(\gamma_k + 1)\,\gamma_j(2\gamma_k)^{-1} Ma_j^2\left[(1+\alpha_{\mathrm{p}})\,A_j/A_k\right]^2. \tag{6c}$$

The relation between ρ_k/ρ_j and released heat due to combustion, area change, wall friction, etc., can be obtained from Eq. (5). By inserting density solution of Eq. (5) into Eqs. (1), (2), and (4), the ratio of outlet to inlet velocities, pressures, and temperatures can be obtained and they are not addressed here due to limit of pages.

It is known that Mach number is the most important parameter for combustion flows. Hence, using results of U_k/U_j and T_k/T_j, an analytical relation of the outlet Mach number is derived and given as

$$Ma_k = \left\{\gamma_k\left(1+\alpha_{\mathrm{p}}\right)^{-2}(A_k/A_j)^2\left[1/(\gamma_j Ma_j^2) + A_j/A_k - C_{\mathrm{D},jk}A_{\mathrm{D},jk}/(2A_k)\right](\rho_k/\rho_j) - \gamma_k\right\}^{-0.5}. \tag{7}$$

Figure 2 gives the result of outlet Mach number as functions of ratio of the outlet to the inlet areas A_k/A_j and ratio of the total temperatures T_{0k}/T_{0j} (representing the effect of heat releasing). The result is obtained via Eq. (7) with $Ma_j = 2.5$, $\gamma_j = 1.32$, $\gamma_k = 1.2$, $R_{\mathrm{g},k}/R_{\mathrm{g},j} = 1.0$, $C_{\mathrm{D},jk} = 0$, and $\alpha_{\mathrm{p}} = 0$. As shown in the figure, curves of outlet Mach number versus T_{0k}/T_{0j} for different A_k/A_j are somewhat similar. It indicates that the effect of heat releasing, wall friction, and area change on the outlet Mach number can be normalized and there exists a single self-similar parameter.

The self-similar parameter is found to be

$$\zeta = 1 - 4ac/b^2, \tag{8}$$

where a, b, c are given by Eq. (6).

Combining Eqs. (5) and (6), Eq. (7) can be rewritten in terms of ζ as

$$Ma_k = \sqrt{(1 \pm \zeta^{0.5})/(1 \mp \gamma_k \zeta^{0.5})}. \tag{9}$$

As for Eq. (9), if the outlet flow is supersonic, the sign "+" is for the numerator and "−" for the denominator; if the outlet flow is subsonic, the sign "−" is for the numerator and "+" for the denominator.

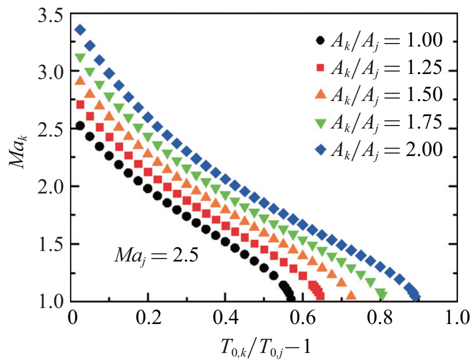

Fig. 2. Outlet Mach number of heated duct flow as functions of ratio of cross section areas and ratio of total temperatures.

Applying Eq. (9) to an element control volume of duct flow with individual variation in cross section, wall friction, heating or mass addition, the same formula of the outlet Mach number as those in Ref. 7 can be derived and given as

(1) isentropic flow

$$Ma_k = Ma_j\left\{1+\left[1+0.5\left(\gamma-1\right)Ma_j^2\right]\left(A_k-A_j\right)/\left[\left(Ma_j^2-1\right)A_j\right]\right\},$$

(2) flow in constant-area ducts with wall friction

$$Ma_k = Ma_j\left\{1-\left\{\gamma Ma_j^2\left[1+0.5\left(\gamma-1\right)Ma_j^2\right]C_{\mathrm{D},jk}A_{\mathrm{D},jk}\right\}/\left[2\left(Ma_j^2-1\right)A_k\right]\right\},$$

(3) flow in constant-area ducts with heating

$$Ma_k = Ma_j\left\{1-\left\{\left(1+\gamma Ma_j^2\right)\left[1+0.5\left(\gamma-1\right)Ma_j^2\right]\left(T_{0,k}-T_{0,j}\right)\right\}/\left[2\left(Ma_j^2-1\right)T_{0,j}\right]\right\}.$$

From the self-similar solution of Eq. (9), it is clear that $\zeta = 0$ leads to the outlet Mach number of 1, vice versa. Therefore, $\zeta = 0$ is the sufficient and necessary condition for the occurrence of thermal choking (i.e., the averaged Mach number is one). Curran et al.[6] proposed a relation of area change with increase in total temperature at thermal throat. It is, however, only a necessary condition for thermal choking.

There exists a maximum heating amount for supersonic duct flow that causes the Mach number at the duct outlet equal one (i.e., thermal choking). In the previous work, for example the work by Tong et al.,[7] the maximum heating for compressible duct flow with constant cross section ("Rayleigh flow") has been addressed. However, for a more generalized heated flow with wall friction, mass addition, and cross section variation, so far, no equation has been reported for the determination of the maximum heating.

Using the conclusion that $\zeta = 0$ corresponds to the unity of Mach number and Eq. (8), the maximum heat addition as functions of area ratio A_k/A_j, friction coefficient, inlet flow conditions can be derived as

$$\frac{(\gamma_k-1)\dot{Q}_\mathrm{C}^{\max}}{\dot{m}_k\gamma_k R_{\mathrm{g},k}T_j}=\frac{(R_{\mathrm{g},j}/R_{\mathrm{g},k})\{A_k/A_j+\gamma_j Ma_j^2[1-C_{\mathrm{D},jk}A_{\mathrm{D},jk}/(2A_j)]\}^2}{2(1+\alpha_\mathrm{p})^2\left[(\gamma_k+1)\gamma_j Ma_j^2/\gamma_k\right]}-\left[1+(\gamma_k-1)\gamma_j R_{\mathrm{g},j}Ma_j^2/(2\gamma_k R_{\mathrm{g},k})\right]-\frac{(\gamma_k-1)\dot{Q}_\mathrm{T}}{\dot{m}_k\gamma_k R_{\mathrm{g},k}T_j}. \tag{10}$$

The above equation for maximum heating is an extension of the classic Rayleigh flow theory, which gives the relation of maximum heating with constant cross section area.[7]

With Eq. (10) and the definition of Mach number and ignoring changes in gas constant and specific heat, the self-similar parameter ζ may be written as

$$\zeta=1-\frac{1+(\gamma_k-1)\gamma_j R_{\mathrm{g},j}Ma_j^2/(2\gamma_k R_{\mathrm{g},k})+(\gamma_k-1)(\dot{Q}_\mathrm{C}+\dot{Q}_\mathrm{T})/(\dot{m}_k\gamma_k R_{\mathrm{g},k}T_j)}{1+(\gamma_k-1)\gamma_j R_{\mathrm{g},j}Ma_j^2/(2\gamma_k R_{\mathrm{g},k})+(\gamma_k-1)(\dot{Q}_\mathrm{C}^{\max}+\dot{Q}_\mathrm{T})/(\dot{m}_k\gamma_k R_{\mathrm{g},k}T_j)}=1-\frac{T_{0,k}/T_j}{T_{0,k}^{\max}/T_j}=\frac{T_{0,k}^{\max}-T_{0,k}}{T_{0,k}^{\max}}, \tag{11}$$

where $T_{0,k}^{\max}$ is the maximum total temperature at the outlet, corresponding to the maximum heat addition that leads to thermal choking.

From Eq. (11), it is clearly seen that the self-similar parameter is a ratio of the actual total temperature to the possible maximum total temperature at the duct outlet.

In this paper, an analytical theory for supersonic duct flow with variation of cross section, wall friction and heat addition is developed, and relations between the inlet and the outlet flow parameters are obtained. By introducing a self-similar parameter, effects of heat releasing, wall friction, and change in cross section area on the flow can be normalized, and a self-similar solution of the flow equations can be found. Based on the result of self-similar solution, the sufficient and necessary condition for the occurrence of thermal choking is derived. At the same time, the relation of maximum heat addition leading to thermal choking of duct flow is derived as functions of area ratio, wall friction and mass addition, which is an extension of the classic Rayleigh flow theory, where the effects of wall friction and mass addition are not considered.

The present theory is applicable to a compressible flow through ducts with varied cross section, heating, wall friction, mass addition, but no shock discontinuity. The present work provides fundamentals for developing an overall theoretical framewotk for integral analyses of hypersonic air-breathing vehicles and scramjets.

1. J. D. Anderson. Hypersonic and High Temperature Gas Dynamics. AIAA, USA (1989).
2. A. H. Shapiro. The Dynamics and Thermodynamics of Compressible Fluid Flow. John Wiley & Sons, New York (1953).
3. W. H. Heiser, D. T. Pratt. Hypersonic Airbreathing Propulsion. AIAA, USA (1994).
4. S. M. Torrez, N. A. Scholten, D. J. Micka. A scramjet engine model including effects of precombustion shocks and dissociation. *AIAA Paper 4619* (2008).
5. X. Wang, F. Q. Zhong, L. H. Chen, et al. A coupled heat transfer analysis with effects of thermal/catalytic cracking of kerosene for actively cooled supersonic combustor. *Journal of Propulsion Technology* **34**, 45–51 (2013).
6. E. T. Curran, W. H. Heiser, D. T. Pratt. Fluid phenomena in scramjet combustion systems. *Annual Review of Fluid Mechanics* **28**, 323–360 (1996).
7. B. G. Tong, X. Y. Kong, G. H. Deng. Gas Dynamics. Higher Education Press, Beijing (1990) (in Chinese).