The Kakutani’s precompactness lemma

Jorge Mujica

IMECC-UNICAMP, Caixa Postal 6065, 13083-970 Campinas, SP, Brazil

Received 13 January 2004

Submitted by R.M. Aron

Dedicated to Professor John Horváth on the occasion of his 80th birthday

Abstract

In this paper we establish a theorem that extends and sharpens an old precompactness lemma due to Kakutani. We use this theorem to derive the classical Arzelà–Ascoli theorem and a theorem of Defant and Floret for families of linear operators. We also use this theorem to derive a theorem for composition operators which yields as immediate corollaries a theorem of Geue and a locally convex version of a theorem of Aron and Schottenloher.

© 2004 Elsevier Inc. All rights reserved.

1. The Kakutani’s precompactness lemma

If Z is a pseudometric space, then $B_Z(c; \varepsilon)$ denotes the open ball with center $c$ and radius $\varepsilon$, and $\overline{B}_Z(c; \varepsilon)$ denotes the corresponding closed ball. We recall that a set $K \subseteq Z$ is said to be precompact if for each $\varepsilon > 0$, $K$ can be covered by finitely many balls of radius $\varepsilon$. The following stronger notion of continuity will be useful.

Definition 1.1. Let $X$ be a topological space, and let $Z$ be a pseudometric space. A mapping $f : X \to Z$ is said to be $K$-continuous if for each $\varepsilon > 0$ we can write $X = U_1 \cup \cdots \cup U_m$, where each $U_j$ is open in $X$ and

$$d_Z(f(x), f(y)) < \varepsilon \quad \text{whenever } x, y \in U_j.$$

A family of mappings $f_i : X \to Z$ ($i \in I$) is said to be $K$-equicontinuous if for each $\varepsilon > 0$ we can write $X = U_1 \cup \cdots \cup U_m$, where each $U_j$ is open in $X$ and

$E-mail address: mujica@ime.unicamp.br.$

0022-247X/$ – see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2004.03.070
\[ d_Z(f_i(x), f_i(y)) < \varepsilon \quad \text{whenever} \quad i \in I \quad \text{and} \quad x, y \in U_j. \]

\( \mathcal{C}(X; Z) \) denotes the set of all continuous mappings from \( X \) into \( Z \), and \( \mathcal{C}_K(X; Z) \) denotes the subset of all \( K \)-continuous mappings from \( X \) into \( Z \).

Since uniform spaces can be described in terms of pseudometrics, we can define \( K \)-continuous mappings from topological spaces into uniform spaces in the obvious way. In the definition of \( K \)-continuous mappings, the letter \( K \) stands for compact or precompact, and is motivated by the next proposition.

**Proposition 1.2.** Let \( X \) be a topological space, and let \( Z \) be a pseudometric space. Then a mapping \( f : X \to Z \) is \( K \)-continuous if and only if \( f \) is continuous and \( f(X) \) is a precompact subset of \( Z \).

**Proof.** (\( \Rightarrow \)) If \( f \) is \( K \)-continuous, it is clearly continuous. To show that \( f(X) \) is precompact, let \( \varepsilon > 0 \) be given. By hypothesis, we can write \( X = U_1 \cup \cdots \cup U_m \), where each \( U_j \) is open in \( X \) and \( d_Z(f(x), f(y)) < \varepsilon \) whenever \( x, y \in U_j \). If we choose \( a_j \in U_j \) for each \( j \), then \( f(U_j) \subset B_Z(f(a_j); \varepsilon) \) for each \( j \), and it follows that

\[ f(X) = \bigcup_{j=1}^m f(U_j) \subset \bigcup_{j=1}^m B_Z(f(a_j); \varepsilon). \]

This shows that \( f(X) \) is precompact.

(\( \Leftarrow \)) \( f(X) \) being precompact, there are \( a_1, \ldots, a_m \in X \) such that \( f(X) \subset \bigcup_{j=1}^m B_Z(f(a_j); \varepsilon) \). Let \( U_j = f^{-1}(B_Z(f(a_j); \varepsilon)) \) for each \( j \). Since \( f \) is continuous, each \( U_j \) is open in \( X \). It follows that

\[ X = \bigcup_{j=1}^m f^{-1}(B_Z(f(a_j); \varepsilon)) = \bigcup_{j=1}^m U_j \]

and \( d_Z(f(x), f(y)) < 2\varepsilon \) whenever \( x, y \in U_j \). Hence \( f \) is \( K \)-continuous. \( \Box \)

We do not know if there is a similar characterization for \( K \)-equicontinuous families of mappings. In any case, we have the following weaker result.

**Proposition 1.3.** Let \( X \) be a compact topological space, and let \( Z \) be a pseudometric space. Then a family of mappings \( f_i : X \to Z \) (\( i \in I \)) is \( K \)-equicontinuous if and only if it is equicontinuous.

**Proof.** To prove the nontrivial implication, suppose that the family \( \{f_i : i \in I\} \) is equicontinuous. Then given \( a \in X \) and \( \varepsilon > 0 \), there is an open neighborhood \( U_a \) of \( a \) such that \( d_Z(f_i(x), f_i(a)) < \varepsilon \) whenever \( x \in U_a \) and \( i \in I \). Hence \( d_Z(f_i(x), f_i(y)) < 2\varepsilon \) whenever \( x, y \in U_a \) and \( i \in I \). The open sets \( U_a \), with \( a \in X \), cover \( X \). Since \( X \) is compact, there are \( a_1, \ldots, a_m \in X \) such that \( X = U_{a_1} \cup \cdots \cup U_{a_m} \), and thus \( \{f_i : i \in I\} \) is \( K \)-equicontinuous. \( \Box \)

\( K \)-equicontinuous nets have the following nice property.
**Proposition 1.4.** Let $(f_i)_{i \in I}$ be a $K$-equicontinuous net in $C_K(X; Z)$ which converges pointwise to an $f : X \to Z$. Then $f \in C_K(X; Z)$ and $(f_i)_{i \in I}$ converges to $f$ uniformly on $X$.

**Proof.** Given $\varepsilon > 0$, we can write $X = U_1 \cup \cdots \cup U_m$, where each $U_j$ is open in $X$ and $d_Z(f_i(x), f_i(y)) < \varepsilon$ whenever $x, y \in U_j$ and $i \in I$. It follows that $d_Z(f(x), f(y)) \leq \varepsilon$ whenever $x, y \in U_j$, and therefore $f$ is $K$-continuous. To show uniform convergence, choose $a_j \in U_j$ for $1 \leq j \leq m$. There is $i_0 \in I$ such that $d_Z(f_i(a_j), f(a_j)) < \varepsilon$ whenever $i \geq i_0$ and $1 \leq j \leq m$. Each $x \in X$ belongs to some $U_j$. Hence for $i \geq i_0$ we have that

$$d_Z(f_i(x), f(x)) \leq d_Z(f_i(x), f_i(a_j)) + d_Z(f_i(a_j), f(a_j)) + d_Z(f(a_j), f(x)) < 3\varepsilon.$$ 

Thus $(f_i)_{i \in I}$ converges to $f$ uniformly on $X$. \qed

**Proposition 1.5.** Let $(f_n)_{n=1}^\infty$ be a sequence in $C_K(X; Z)$ which converges uniformly to an $f \in C_K(X; Z)$. Then $(f_n)_{n=1}^\infty$ is $K$-equicontinuous.

**Proof.** Given $\varepsilon > 0$, we can write $X = U_1 \cup \cdots \cup U_m$, where each $U_j$ is open in $X$ and $d_Z(f(x), f(y)) < \varepsilon$ whenever $x, y \in U_j$. There is $n_0 \in \mathbb{N}$ such that $d_Z(f_n(x), f_n(y)) < \varepsilon$ whenever $x, y \in U_j$ and $n > n_0$. It follows easily that $d_Z(f_n(x), f_n(y)) < 3\varepsilon$ whenever $x, y \in U_j$ and $n > n_0$. Since $f_1, \ldots, f_{n_0} \in C_K(X; Z)$, we can easily find open sets $V_1, \ldots, V_p$ such that $X = V_1 \cup \cdots \cup V_p$ and $d_Z(f_n(x), f_n(y)) < 3\varepsilon$ whenever $x, y \in V_j$ and $n \in \mathbb{N}$. \qed

We remark that the notion of $K$-equicontinuous family is closely connected with the notion of family with equal variation considered by Geue [6].

**Definition 1.6.** Let $X$ and $Y$ be arbitrary sets and let $Z$ be a pseudometric space. A mapping $f : X \times Y \to Z$ is said to be **separately precompact** if the set $f(X \times \{y\})$ is precompact in $Z$ for each $y \in Y$, and the set $f(X \times \{y\})$ is precompact in $Z$ for each $x \in X$.

The following theorem extends and sharpens a precompactness lemma due to Kakutani [10]. See also Bartle [2, Theorem 3.8].

**Theorem 1.7.** Let $X$ and $Y$ be arbitrary sets, let $Z$ be a pseudometric space, and let $f : X \times Y \to Z$ be a separately precompact mapping. Let $d_X : X \times X \to \mathbb{R}$ and $d_Y : Y \times Y \to \mathbb{R}$ be the pseudometrics defined by

$$d_X(x_1, x_2) = \sup_{y \in Y} d_Z(f(x_1, y), f(x_2, y)) \quad \text{and}$$

$$d_Y(y_1, y_2) = \sup_{x \in X} d_Z(f(x, y_1), f(x, y_2)).$$

Then the following conditions are equivalent:

1. The space $(X, d_X)$ is precompact.
(2) The space \((Y, d_Y)\) is precompact.

(3) For each \(\varepsilon > 0\) we can write \(X = X_1 \cup \cdots \cup X_m\), where

\[
d_Z(\{f(x_1, y), f(x_2, y)\}) < \varepsilon \quad \text{whenever } x_1, x_2 \in X_j \text{ and } y \in Y.
\]

(4) For each \(\varepsilon > 0\) we can write \(Y = Y_1 \cup \cdots \cup Y_n\), where

\[
d_Z(\{f(x, y_1), f(x, y_2)\}) < \varepsilon \quad \text{whenever } x \in X \text{ and } y_1, y_2 \in Y_k.
\]

(5) For each \(\varepsilon > 0\) we can write \(X = X_1 \cup \cdots \cup X_m\) and \(Y = Y_1 \cup \cdots \cup Y_n\), where

\[
d_Z(\{f(x_1, y_1), f(x_2, y_2)\}) < \varepsilon \quad \text{whenever } x_1, x_2 \in X_j \text{ and } y_1, y_2 \in Y_k.
\]

\[\text{If, in addition, } X \text{ (respectively } Y\text{) is a topological space, and all the partial mappings } f_j : X \to Z \text{ (respectively } f_i : Y \to Z\text{) are continuous, then the sets } X_j \text{ in (3) (respectively } Y_k \text{ in (4)) may be assumed to be open. In particular, the family of mappings } f_j : X \to Z \text{ (respectively } f_i : Y \to Z\text{) is } K\text{-equicontinuous.}\]

\[\text{If, in addition, } X \text{ and } Y \text{ are topological spaces, and all the partial mappings } f_j : X \to Z \text{ and } f_i : Y \to Z \text{ are continuous, then the sets } X_j \text{ and } Y_k \text{ in (5) may be assumed to be open. In particular, the mapping } f : X \times Y \to Z \text{ is } K\text{-continuous.}\]

\textbf{Proof.} (1) \(\Rightarrow\) (3). Given \(\varepsilon > 0\), there are \(a_1, \ldots, a_m \in X\) such that \(X = \bigcup_{j=1}^m B_X(a_j; \varepsilon)\). Given \(x_1, x_2 \in B_X(a_j; \varepsilon)\) and \(y \in Y\), we see that

\[
d_Z(f(x_1, y), f(x_2, y)) \leq d_X(x_1, x_2) < 2\varepsilon,
\]

and (3) follows.

(3) \(\Rightarrow\) (1). If we choose \(a_j \in X_j\) for every \(j\), we see that

\[
d_X(x, a_j) = \sup_{y \in Y} d_Z(f(x, y), f(a_j, y)) \leq \varepsilon
\]

for every \(x \in X_j\). Thus \(X_j \subseteq B_X(a_j; \varepsilon)\) and (1) follows.

Thus (1) \(\Leftrightarrow\) (3) and, by symmetry, (2) \(\Leftrightarrow\) (4). Clearly (5) \(\Rightarrow\) (3) and (5) \(\Rightarrow\) (4).

(3) \(\Rightarrow\) (5). Given \(\varepsilon > 0\), we can write \(X = X_1 \cup \cdots \cup X_m\), where

\[
d_Z(f(x_1, y), f(x_2, y)) < \varepsilon \quad \text{whenever } x_1, x_2 \in X_j \text{ and } y \in Y.
\]

Choose \(a_j \in X_j\) for every \(j\). Since \(f([a_1] \times Y)\) is precompact in \(Z\), there are \(c_{11}, \ldots, c_{1n_1} \in Z\) such that

\[
f([a_1] \times Y) \subseteq \bigcup_{k=1}^{n_1} B_Z(c_{1k}; \varepsilon).
\]

If we set

\[
Y_k = \{y \in Y : d_Z(f(a_1, y), c_{1k}) < \varepsilon\} = f_{1n_1}^{-1}(B_Z(c_{1k}; \varepsilon))
\]

for \(1 \leq k \leq n_1\), then \(Y = Y_{11} \cup \cdots \cup Y_{1n_1}\), and

\[
d_Z(f(a_1, y_1), f(a_1, y_2)) < 2\varepsilon \quad \text{whenever } y_1, y_2 \in Y_{1k}.
\]
By applying the same argument to the precompact set \( f([a_2] \times Y_{1k}) \), for each \( k \), we can write \( Y = Y_{21} \cup \cdots \cup Y_{2m_2} \), where
\[
d_Z(f(a_j, y_1), f(a_j, y_2)) < 2\varepsilon \quad \text{whenever } 1 \leq j \leq 2 \text{ and } y_1, y_2 \in Y_{2k}.
\]
Furthermore, the sets \( Y_{2k} \) are of the form
\[
Y_{2k} = Y_{1j} \cap f_{a_j}^{-1}(B_Z(c_{2k}; \varepsilon)), \quad \text{with } c_{2k} \in Z.
\]
After applying the same argument \( m \) times, we can write \( Y = Y_{m1} \cup \cdots \cup Y_{mn} \), where
\[
d_Z(f(a_j, y_1), f(a_j, y_2)) < 2\varepsilon \quad \text{whenever } 1 \leq j \leq m \text{ and } y_1, y_2 \in Y_{mk}.
\]
Furthermore, the sets \( Y_{mk} \) are of the form
\[
Y_{mk} = Y_{m-1,j} \cap f_{a_j}^{-1}(B_Z(c_{mk}; \varepsilon)), \quad \text{with } c_{mk} \in Z.
\]
Thus, given \( x_1, x_2 \in X_j \) and \( y_1, y_2 \in Y_{mk} \), it follows that
\[
d_Z(f(x_1, y_1), f(x_2, y_2)) \leq d_Z(f(x_1, y_1), f(a_j, y_1)) + d_Z(f(a_j, y_1), f(a_j, y_2)) + d_Z(f(a_j, y_2), f(x_2, y_2))
\]
\[
\leq \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon.
\]
Thus (3) \( \Rightarrow \) (5) and, by symmetry, (4) \( \Rightarrow \) (5).

Next assume that \( Y \) is a topological space, and all the partial mappings \( f_k : Y \rightarrow Z \) are continuous. In the proof of the implication (3) \( \Rightarrow \) (5) we showed that we can write \( X = X_1 \cup \cdots \cup X_m \) and \( Y = Y_{m1} \cup \cdots \cup Y_{mn} \), where
\[
d_Z(f(x_1, y_1), f(x_2, y_2)) < 4\varepsilon \quad \text{whenever } x_1, x_2 \in X_j \text{ and } y_1, y_2 \in Y_{mk}.
\]
In particular, it follows that
\[
d_Z(f_k(y_1), f_k(y_2)) < 4\varepsilon \quad \text{whenever } x \in X \text{ and } y_1, y_2 \in Y_{mk}.
\]
Furthermore, since the partial mappings \( f_{a_j} : Y \rightarrow Z \) are continuous, the sets \( Y_{mk} \) are open by construction. Thus the family of mappings \( f_k : Y \rightarrow Z \) is \( K \)-equicontinuous.

The corresponding assertion concerning the space \( X \) and the partial mappings \( f_{a_j} : X \rightarrow Z \) is true by symmetry.

Finally, if \( X \) and \( Y \) are topological spaces, and the sets \( X_j \) in (3) and \( Y_{mk} \) in (4) are open, then it is clear that the sets \( X_j \) and \( Y_{mk} \) in (5) may be assumed to be open too. \( \square \)

The proof of Theorem 1.7 is a refinement of the elementary proof of the Kakutani’s precompactness lemma given in [11].

**Corollary 1.8.** Let \( X \) and \( Y \) be topological spaces, and let \( Z \) be a pseudometric space. Let \( f : X \times Y \rightarrow Z \) be a mapping such that \( f_{a_j} : X \rightarrow Z \) is \( K \)-continuous for every \( y \in Y \) and \( f_{a_j} : Y \rightarrow Z \) is \( K \)-continuous for every \( x \in X \). Then \( f : X \times Y \rightarrow Z \) is \( K \)-continuous.

**Corollary 1.9.** Let \( X \) and \( Y \) be compact topological spaces, and let \( Z \) be a pseudometric space. Then every separately continuous mapping \( f : X \times Y \rightarrow Z \) is continuous.

**Corollary 1.10.** Let \( X \) and \( Y \) be topological spaces, and let \( Z \) be a pseudometric space. If \( X \times Y \) is a \( k \)-space, then every separately continuous mapping \( f : X \times Y \rightarrow Z \) is continuous.
2. A theorem of Arzelà–Ascoli type for continuous mappings with precompact range

Let $X$ be a topological space and let $Z$ be a pseudometric space. The uniform topology on $C_K(X; Z)$, is the topology $\tau_a$ defined by the pseudometric

$$d(f, g) = \sup_{x \in X} d_Z(f(x), g(x)).$$

The compact-open topology on $C(X; Z)$, is the topology $\tau_c$ defined by the pseudometrics

$$d_K(f, g) = \sup_{x \in K} d_Z(f(x), g(x)),$$

where $K$ varies among the compact subsets of $X$.

If $X$ is compact, it is clear that $C(X; Z) = C_K(X; Z)$ and $\tau_c = \tau_a$ on $C(X; Z) = C_K(X; Z)$.

Now we can prove the following version of the Arzelà–Ascoli theorem for continuous mappings with precompact range.

**Theorem 2.1.** Let $X$ be a topological space, let $Z$ be a pseudometric space, and let $\{f_i: i \in I\} \subset C_K(X; Z)$. Then the following conditions are equivalent:

1. $\{f_i: i \in I\}$ is a precompact subset of $(C_K(X; Z), \tau_a)$.
2. $\{f_i: i \in I\}$ is $K$-equicontinuous and $\{f_i(x): i \in I\}$ is precompact in $Z$ for each $x \in X$.
3. $\{f_i: i \in I\}$ is $K$-equicontinuous and $\bigcup_{i \in I} f_i(X)$ is precompact in $Z$.

**Proof.** We will apply Theorem 1.7 to the mapping $f : I \times X \to Z$ defined by $f(i, x) = f_i(x)$ for all $i \in I$ and $x \in X$. By Proposition 1.2, $f_i(X)$ is precompact in $Z$ for each $i \in I$.

(1) $\Rightarrow$ (2). Since $d(f, g) \geq d_Z(f(x), g(x))$ for all $f, g \in C_K(X; Z)$ and $x \in X$, it is clear that since $\{f_i: i \in I\}$ is precompact in $C_K(X; Z)$, then $\{f_i(x): i \in I\}$ is precompact in $Z$ for each $x \in X$. Thus the mapping $f$ is separately precompact and Theorem 1.7 applies.

Since

$$d(f_i, f_j) = \sup_{x \in X} d_Z(f_i(x), f_j(x)) = d_I(i, j),$$

$\{f_i: i \in I\}$ being precompact in $C_K(X; Z)$ means that the space $(I, d_I)$ is precompact. By Theorem 1.7, the family $\{f_i: i \in I\}$ is $K$-equicontinuous.

(2) $\Rightarrow$ (1). Since $\{f_i(x): i \in I\}$ is precompact in $Z$ for each $x \in X$, the mapping $f$ is separately precompact, and Theorem 1.7 applies.

Since the family $\{f_i: i \in I\}$ is $K$-equicontinuous, the space $(I, d_I)$ is precompact, by Theorem 1.7. But we already know that $(I, d_I)$ precompact means that $\{f_i: i \in I\}$ is a precompact subset of $C_K(X; Z)$.

(2) $\Rightarrow$ (3). Since the family $\{f_i: i \in I\}$ is $K$-equicontinuous, for each $\varepsilon > 0$ we can write $X = U_1 \cup \cdots \cup U_m$, where each $U_j$ is open in $X$ and $d_Z(f_i(x), f_i(y)) < \varepsilon$ whenever $i \in I$ and $x, y \in U_j$. If we choose $a_j \in U_j$ for each $j$, then

$$f_i(X) = \bigcup_{j=1}^m f_i(U_j) \subset \bigcup_{j=1}^m B_Z(f_i(a_j); \varepsilon)$$
for every $i \in I$.

On the other hand, the set $\{f_i(\alpha_j): i \in I\}$ is precompact in $Z$ for each $j$, and therefore the set $\{f_i(\alpha_j): j \in I, 1 \leq j \leq m\}$ is also precompact in $Z$. Thus there are $c_1, \ldots, c_n \in Z$ such that

$$\{f_i(\alpha_j): i \in I, 1 \leq j \leq m\} \subset \bigcup_{k=1}^{n} B_Z(c_k; \varepsilon).$$

It follows that

$$\bigcup_{i \in I} f_i(X) \subset \bigcup_{k=1}^{n} B_Z(c_k; 2\varepsilon).$$

Thus $\bigcup_{i \in I} f_i(X)$ is a precompact subset of $Z$.

Since the implication $(2) \Rightarrow (1)$ is obvious, the proof of the theorem is complete. □

**Corollary 2.2.** Let $X$ be a compact topological space, let $Z$ be a pseudometric space, and let $\{f_i: i \in I\} \subset C(X; Z)$. Then the following conditions are equivalent:

1. $\{f_i: i \in I\}$ is a precompact subset of $(C(X; Z), \tau_u)$.
2. $\{f_i: i \in I\}$ is equicontinuous and $\{f_i(x): i \in I\}$ is precompact in $Z$ for each $x \in X$.
3. $\{f_i: i \in I\}$ is equicontinuous and $\bigcup_{i \in I} f_i(X)$ is precompact in $Z$.

**Corollary 2.3.** Let $X$ be a $k$-space, let $Z$ be a pseudometric space, and let $\{f_i: i \in I\} \subset C(X; Z)$. Then the following conditions are equivalent:

1. $\{f_i: i \in I\}$ is a precompact subset of $(C(X; Z), \tau_u)$.
2. $\{f_i: i \in I\}$ is equicontinuous and $\{f_i(x): i \in I\}$ is precompact in $Z$ for each $x \in X$.
3. $\{f_i: i \in I\}$ is equicontinuous and $\bigcup_{i \in I} f_i(K)$ is precompact in $Z$ for each compact set $K \subset X$.

Floret [5] used the Kakutani’s precompactness lemma [10] to prove the implication $(2) \Rightarrow (1)$ in the scalar-valued version of Corollary 2.2. For a variant of the Arzelà–Ascoli theorem see Geue [6, Theorem 2.1].

### 3. A precompactness theorem of Defant and Floret for families of linear operators

Let $C$ be an absolutely convex subset of a real or complex vector space $E$. Let $p_C$ denote the Minkowski functional of $C$, that is

$$p_C(x) = \inf\{\lambda > 0: x \in \lambda C\},$$

and let $[C]$ denote the seminormed space $(\operatorname{span} C, p_C)$. We say that a set $A$ is $C$-precompact if $A$ is a precompact subset of $[C]$.

Given two dual systems $(E_1, E_2)$ and $(F_1, F_2)$, let $\mathcal{L}((E_1, E_2), (F_1, F_2))$ denote the space of all linear mappings $T : E_1 \rightarrow F_1$ which are $\sigma(E_1, E_2)$–$\sigma(F_1, F_2)$-continuous.
For each $T \in \mathcal{L}((E_1, E_2), (F_1, F_2))$, the dual mapping $T' \in \mathcal{L}((F_2, F_1), (E_2, E_1))$ is defined by

$$\langle Tx, y' \rangle = \langle x, T'y' \rangle$$

for all $x \in E_1, y' \in F_2$.

We refer to Grothendieck [8] or Horváth [9] for the terminology from the theory of topological vector spaces.

As pointed out by Floret [5], by applying the Kakutani’s precompactness lemma to the mapping $f: A \times B \to K$ defined by $f(x, y') = \langle Tx, y' \rangle = \langle x, T'y' \rangle$, one immediately obtains the following results of Grothendieck [7,8].

**Corollary 3.1.** Let $\langle E_1, E_2 \rangle$ and $\langle F_1, F_2 \rangle$ be two dual systems, let $T \in \mathcal{L}((E_1, E_2), (F_1, F_2))$, and let $A \subset E_1$ and $B \subset F_2$. Then $T(A)$ is $B^\circ$-precompact if and only if $T(B)$ is $A^\circ$-precompact.

**Corollary 3.2.** Let $\langle E_1, E_2 \rangle$ be a dual system, and let $A \subset E_1$ and $B \subset E_2$. Then $A$ is $B^\circ$-precompact if and only if $B$ is $A^\circ$-precompact.

Kakutani [10] used his precompactness lemma to give a proof of the classical Schauder theorem (which follows at once from Corollary 3.1).

Given $A \subset E_1$ and $V \subset F_2$, we set

$$N(A, V) = \{ T \in \mathcal{L}((E_1, E_2), (F_1, F_2)): T(A) \subset V \}.$$

Then we can prove the following result, a slight improvement of a theorem of Defant and Floret [3].

**Theorem 3.3.** Let $\langle E_1, E_2 \rangle$ and $\langle F_1, F_2 \rangle$ be two dual systems, let $\{ T_i : i \in I \} \subset \mathcal{L}((E_1, E_2), (F_1, F_2))$, and let $A \subset E_1$ and $B \subset F_2$. Then the conditions (1)–(4) below are equivalent:

1. $\{ T_i : i \in I \}$ is $N(A, B^\circ)$-precompact.
2. $T_i(A)$ is $B^\circ$-precompact for each $i \in I$.
3. $\{ T_i : i \in I \}$ is $N(B, A^\circ)$-precompact.
4. $T_i(B)$ is $A^\circ$-precompact for each $i \in I$.

If, in addition, $A$ is a topological space and each restriction $T_i|A : A \to [B^\circ]$ is continuous, then the conditions (1)–(4) are equivalent also to the condition (5) below:

1. The family of restrictions $T_i|A : A \to [B^\circ]$ ($i \in I$) is $K$-equicontinuous.
2. $\{ T_i(x) : i \in I \}$ is $B^\circ$-precompact for each $x \in A$.
3. $T_i(A)$ is $B^\circ$-precompact for each $i \in I$. 

4. $\mathcal{T}_i$ is $K$-equicontinuous.
Proof. Defant and Floret [3] derived their theorem from the Grothendieck Corollary 3.2, together with a vector-valued version of the Arzelà–Ascoli theorem. We will derive Theorem 3.3 directly from Theorem 1.7.

Defant and Floret [3] observed that the equivalence (1) ⇔ (2) follows from the identity

\[ p_{N(A,B^\ast)}(T) = \sup_{x \in A} \sup_{y' \in B} \|Tx, y'\| = \sup_{x \in A} \sup_{y' \in B} \|x, T'y'\| = p_{N(B,A^\ast)}(T). \]

Clearly (3a) ⇒ (4b), and Defant and Floret [3] used Corollary 3.2 to prove that (3) ⇒ (4a). Thus (3) ⇒ (4) and, by symmetry, (4) ⇒ (3).

Since

\[ p_{N(A,B^\ast)}(T) = \sup_{x \in A} \sup_{y' \in B} \|Tx, y'\| = \sup_{x \in A} p_{B^\ast}(Tx), \]

it follows that (1) ⇒ (3a) ⇒ (4b) and, by symmetry, (2) ⇒ (4a) ⇒ (3b). Thus (1) ⇔ (2) ⇒ (3) ⇔ (4).

To show that (3) ⇔ (4) ⇒ (1) ⇔ (2), we consider the mapping \( f : I \times A \to [B^\ast] \) defined by \( f(i,x) = T_i x \) for all \( i \in I \) and \( x \in A \). Then \( f(I \times \{x\}) = \{T_i x : i \in I\} \) is \( B^\circ \)-precompact, by (4b), and \( f((i) \times A) = T_i(A) \) is \( B^\circ \)-precompact, by (3a). Thus \( f \) is separately precompact, and Theorem 1.7 applies. Observe that

\[ d_f(i, j) = \sup_{x \in A} \sup_{y' \in B} \|T_i x - T_j x, y'\| = p_{N(A,B^\ast)}(T_i - T_j) \quad (\ast) \]

and

\[ d_A(x_1 - x_2) = \sup_{i \in I} \sup_{y' \in B} \|T_i x_1 - T_i x_2, y'\| = \sup_{i \in I} \sup_{y' \in B} \|x_1 - x_2, T_i'y'\| \]

\[ = p_{\bigcup_{i \in I} T_i(B)^\ast}(x_1 - x_2). \quad (\ast\ast) \]

By (4a), \( \bigcup_{i \in I} T_i'(B) \) is \( A^\ast \)-precompact. By Corollary 3.2, \( A \) is \( (\bigcup_{i \in I} T_i'(B))^{\ast\ast} \)-precompact. By \( (\ast\ast), (A, d_A) \) is precompact. By Theorem 1.7, \( (I, d_I) \) is precompact. By \( (\ast) \{T_i : i \in I\} \) is \( N(A, B^\ast) \)-precompact, proving (1a). Since (1b) is an obvious consequence of (3a), the proof of the implication (3) ⇐ (4) ⇒ (1) ⇐ (2) is complete.

Conditions (5b) and (5c) are direct consequences of (4b) and (1b) and guarantee that the mapping \( f : I \times A \to [B^\ast] \) from the proof of the implication (3) ⇐ (4) ⇒ (1) ⇐ (2) is separately precompact. By Theorem 1.7, (5a) means that \( (I, d_I) \) is precompact, and we already know that is equivalent to (1a). This completes the proof of the theorem. \( \square \)

By symmetry we can state a condition similar to (5) for the restrictions \( T'_i[B : B \to [A^\circ]] \).

By letting \( A \) and \( B \) vary over suitable subsets of \( E_1 \) and \( F_2 \), we can recover the precompactness theorems for families of linear operators obtained by several authors. See Palmer [12], Geue [6], Ruess [13], and Defant and Floret [3], and the references in those papers.
4. A precompactness theorem for composition operators

Let $E$ and $F$ be Hausdorff locally convex spaces, and let $\mathcal{P}_b(mE; F)$ be the space of all continuous $m$-homogeneous polynomials from $E$ into $F$, with the topology of uniform convergence on the bounded subsets of $E$. The sets

$$N(A, V) = \{ P \in \mathcal{P}(mE; F): P(A) \subset V \},$$

where $A$ varies among the bounded subsets of $E$, and $V$ varies among the 0-neighborhoods in $F$, form a 0-neighborhood base in $\mathcal{P}_b(mE; F)$. Observe that if $V$ is closed and absolutely convex, then the Minkowski functional of $N(A, V)$ is given by

$$p_{N(A,V)}(P) = \sup_{x \in A} p_V(P(x)).$$

If $m = 1$, then $\mathcal{P}_b(mE; F)$ coincides with the space of continuous linear mappings $L_b(E; F)$. We refer to the book of Dineen [4] for background information on the theory of polynomials between locally convex spaces.

We now use Theorem 1.7 to prove the following precompactness theorem for composition operators.

**Theorem 4.1.** Let $E$, $F$, $G$, and $H$ be Hausdorff locally convex spaces. Let $P \in \mathcal{P}(mE; F)$ and $T \in L(G; H)$, with $P \neq 0$ and $T \neq 0$, and let $\Phi$ be the continuous linear mapping defined by

$$\Phi : S \in L_b(F; G) \to T \circ S \circ P \in \mathcal{P}_b(mE; H).$$

Then $\Phi$ maps equicontinuous sets onto precompact sets if and only if both $P$ and $T$ map bounded sets onto precompact sets.

**Proof.** We can readily verify that $\Phi$ is linear and continuous. Indeed, if $A$ is a bounded subset of $E$, and $W$ is a 0-neighborhood in $H$, then it is clear that

$$\Phi(N(P(A), T^{-1}(W))) \subset N(A, W).$$

(a) We first show that if both $P$ and $T$ map bounded sets onto precompact sets, then $\Phi$ maps equicontinuous sets onto precompact sets.

Let $S$ be an equicontinuous subset of $L(F; G)$, let $A$ be a bounded subset of $E$, and let $W$ be a closed, absolutely convex 0-neighborhood in $H$. Since $S$ is equicontinuous, there is a closed, absolutely convex 0-neighborhood $V$ in $F$ such that $S(V) \subset T^{-1}(W)$, and therefore $T \circ S(V) \subset W$ for every $S \in S$.

Let $f : P(A) \times S \to (H, p_W)$ be defined by $f(y, S) = T \circ S(y)$ for every $y \in P(A)$ and $S \in S$. Then $f(P(A) \times [S]) = T \circ S \circ P(A)$ is precompact in $H$, and therefore in $(H, p_W)$ for every $S \in S$, and $f([y] \times S) = [T \circ S(y)]: S \in S$ is precompact in $H$, and therefore in $(H, p_W)$, for every $y \in P(A)$. Thus $f$ is separately precompact and Theorem 1.7 applies.

We claim that

$$d_{P(A)}(y_1, y_2) \leq p_V(y_1 - y_2) \quad (*)$$

and

$$d_S(S_1, S_2) = p_{N(A,W)}(\Phi(S_1) - \Phi(S_2)). \quad (**)$$
On the one hand, since $T \circ S(V) \subset W$ for every $S \in \mathcal{S}$, it follows that $p_W(T \circ S(y)) \leq p_V(y)$ for every $y \in F$, and therefore

$$d_{P(A)}(y_1, y_2) = \sup_{S \in \mathcal{S}} p_W(T \circ S(y_1 - y_2)) \leq p_V(y_1 - y_2),$$

thus proving (*). On the other hand,

$$d_S(S_1, S_2) = \sup_{y \in P(A)} p_W(T \circ S_1(y) - T \circ S_2(y))$$

$$= \sup_{x \in A} \Phi(S_1)(x) - \Phi(S_2)(x) = p_{N(A, W)}(\Phi(S_1) - \Phi(S_2)),$$

thus proving (**).

From (*) and (**) and Theorem 1.7 we see that since $P(A)$ is $p_V$-precompact, then $(P(A), d_{P(A)})$ is precompact, hence $(\mathcal{S}, d_S)$ is precompact, and therefore $\Phi(\mathcal{S})$ is $N(A, W)$-precompact. Thus $\Phi(\mathcal{S})$ is precompact in $\mathcal{P}_b^m(E; H)$ for every equicontinuous set $S \subset \mathcal{L}(F; G)$. This proves (a).

(b) We next show that if $\Phi$ maps equicontinuous sets onto precompact sets, then $T$ maps bounded sets onto precompact sets.

Let $B$ be a bounded subset of $G$. Since $P \neq 0$, there is $x_0 \in E$ such that $P(x_0) \neq 0$. By the Hahn–Banach theorem, there is $y'_0 \in F'$ such that $y'_0 \circ P(x_0) = 1$. For each $z \in G$ let $S_z \in \mathcal{L}(F; G)$ be defined by $S_z(y) = y'_0(y)z$ for every $y \in F$. Then

$$\Phi(S_z)(x_0) = T \circ S_z \circ P(x_0) = T \circ z.$$

We claim that the set $\{S_z : z \in B\}$ is equicontinuous in $\mathcal{L}(F; G)$. Indeed let $W$ be an absolutely convex 0-neighborhood in $G$, and let $\delta > 0$ such that $\delta B \subset W$. Let $V$ be a 0-neighborhood in $F$ such that $|y'_0(y)| < \delta$ for every $y \in V$. It follows that $S_z(V) \subset \delta B \subset W$ for every $z \in B$, and therefore $\{S_z : z \in B\}$ is equicontinuous in $\mathcal{L}(F; G)$. Thus $\{\Phi(S_z) : z \in B\}$ is precompact in $\mathcal{P}_b^m(E; H)$, and

$$T(B) = \{Tz : z \in B\} = \{\Phi(S_z)(x_0) : z \in B\}$$

is precompact in $H$. This proves (b).

(c) We finally show that if $\Phi$ maps equicontinuous sets onto precompact sets, then $P$ maps bounded sets onto precompact sets.

Let $A$ be a bounded subset of $E$, and let $V$ be a closed, absolutely convex 0-neighborhood in $F$. Since $T \neq 0$, there is $z_0 \in G$ such that $Tz_0 \neq 0$. Let $W$ be a closed, absolutely convex 0-neighborhood in $H$ such that $p_W(Tz_0) = 1$. For each $y' \in V^\circ$ let $S_{y'} \in \mathcal{L}(F; G)$ be defined by $S_{y'}(y) = y'(y)z_0$ for every $y \in F$. We claim that the set $S = \{S_{y'} : y' \in V^\circ\}$ is equicontinuous in $\mathcal{L}(F; G)$. Indeed, let $N$ be an absolutely convex 0-neighborhood in $G$, and let $\delta > 0$ such that $\delta z_0 \in N$. It follows that $\delta S_{y'}(V) \subset N$ for every $y' \in V^\circ$, and therefore $S$ is equicontinuous. Since

$$p_W(T \circ S_{y'}(y)) = |y'(y)| \leq 1 \quad \text{for every } y \in V, y' \in V^\circ,$$

it follows that

$$T \circ S_{y'}(V) \subset W \quad \text{for every } y' \in V^\circ.$$
Let \( f : P(A) \times S \to (H, p_W) \) be defined as before by \( f(y, S) = T \circ S(y) \) for every \( y \in P(A) \) and \( S \in S \). Since \( \Phi \) maps equicontinuous sets onto precompact sets, \( T \) maps bounded sets onto precompact sets, and it follows as before that \( f \) is separately precompact, and Theorem 1.7 applies.

We claim that
\[
d_{P(A)}(y_1, y_2) = p_V(y_1 - y_2).
\]

Since \( T \circ S(y')(V) \subset W \) for every \( y' \in V^o \), it follows as before that
\[
d_{P(A)}(y_1, y_2) = \sup_{y' \in V^o} p_W(T \circ S(y_1 - y_2)) \leq p_V(y_1 - y_2).
\]

To show equality we fix \( y_1, y_2 \in P(A) \). By the Hahn–Banach theorem there is \( y' \in F' \) such that \( y'(y_1 - y_2) = p_V(y_1 - y_2) \) and \( |y'(y)| \leq p_V(y) \) for every \( y \in F \). Hence \( y' \in V^o \) and
\[
p_W(T \circ S(y_1 - y_2)) = |y'(y_1 - y_2)| = p_V(y_1 - y_2).
\]

This shows (**). Since the identity
\[
d_S(S_1, S_2) = p_N(A, W)(\Phi(S_1) - \Phi(S_2))
\]
is true as before, another application of Theorem 1.7 shows that, since \( \Phi \) is \( N(A, W) \)-precompact, then \( (S, d_S) \) is precompact, hence \( (P(A), d_{P(A)}) \) is precompact, and therefore \( P(A) \) is \( p_V \)-precompact. Thus \( P(A) \) is precompact in \( F \) for every bounded set \( A \subset E \). This shows (c) and completes the proof of the theorem.

**Corollary 4.2.** Let \( E \) and \( F \) be Hausdorff locally convex spaces. Let \( P \in P(mE; F) \), \( P \neq 0 \), and let \( P^* \) be the continuous linear mapping defined by
\[
P^* : y' \in F_b' \mapsto y' \circ P \in P(mE).
\]

Then \( P \) maps bounded sets onto precompact sets if and only if \( P^* \) maps equicontinuous sets onto precompact sets.

**Proof.** It suffices to apply Theorem 4.1 with \( G = H = K \) and \( T = \text{identity} \).

When \( m = 1 \) Theorem 4.1 reduces to a precompactness theorem of Geue [6, Theorem 4.1]. Corollary 4.2 is a locally convex version of a result of Aron and Schottenloher [1, Proposition 3.2].

We end this paper with a holomorphic version of Theorem 4.1. Let \( E \) and \( F \) be complex, Hausdorff locally convex spaces, and let \( \mathcal{H}_b(E; F) \) be the space of all holomorphic mappings from \( E \) into \( F \) which map bounded sets onto bounded sets, with the topology of uniform convergence on the bounded subsets of \( E \). The sets
\[
N(A, V) = \{ f \in \mathcal{H}_b(E; F) : f(A) \subset V \},
\]
where \( A \) varies among the bounded subsets of \( E \), and \( V \) varies among the \( 0 \)-neighborhoods in \( F \), form a \( 0 \)-neighborhood base in \( \mathcal{H}_b(E; F) \). Observe that if \( V \) is closed and absolutely convex, then the Minkowski functional of \( N(A, V) \) is given by
\[
p_{N(A, V)}(f) = \sup_{x \in A} p_V(f(x)).
\]

We then have the following holomorphic version of Theorem 4.1.

**Theorem 4.3.** Let $E$, $F$, $G$, and $H$ be complex, Hausdorff locally convex spaces. Let $f \in \mathcal{H}_b(E; F)$ and $T \in \mathcal{L}(G; H)$, with $f \neq 0$ and $T \neq 0$, and let $\Phi$ be the continuous linear mapping defined by

$$\Phi : S \in \mathcal{L}_b(F; G) \rightarrow T \circ S \circ f \in \mathcal{H}_b(E; H).$$

Then $\Phi$ maps equicontinuous sets onto precompact sets if and only if both $f$ and $T$ map bounded sets onto precompact sets.

The proof of Theorem 4.3 is just a repetition of the proof of Theorem 4.1. We leave the details to the reader. We also have the following corollary.

**Corollary 4.4.** Let $E$ and $F$ be complex, Hausdorff locally convex spaces. Let $f \in \mathcal{H}_b(E; F)$, $f \neq 0$, and let $f^*$ be the continuous linear mapping defined by

$$f^* : y' \in F'_b \rightarrow y' \circ f \in \mathcal{H}_b(E).$$

Then $f$ maps bounded sets onto precompact sets if and only if $f^*$ maps equicontinuous sets onto precompact sets.

Corollary 4.4 may be regarded as a variant of another result of Aron and Schottenloher [1, Proposition 3.6].

**References**