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Sine kernel asymptotics for a class of singular measures

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Abstract

We construct a family of measures on $\mathbb{R}$ that are purely singular with respect to the Lebesgue measure, and yet exhibit universal sine kernel asymptotics in the bulk. The measures are best described via their Jacobi recursion coefficients: these are sparse perturbations of the recursion coefficients corresponding to Chebyshev polynomials of the second kind. We prove convergence of the renormalized Christoffel–Darboux kernel to the sine kernel for any sufficiently sparse decaying perturbation.

Keywords: Christoffel–Darboux kernel; Universality; Singular continuous measure

1. Introduction

Let $\mathrm{d}\mu(x) = w(x)\mathrm{d}x + \mathrm{d}\mu_{\mathrm{sing}}(x)$ be a compactly supported positive measure on $\mathbb{R}$, (where $\mathrm{d}\mu_{\mathrm{sing}}$ is the part of $\mathrm{d}\mu$ that is singular with respect to the Lebesgue measure), and let $\{p_n\}_{n=0}^{\infty}$ be the sequence of orthogonal polynomials associated with μ. Namely,

$$p_n = \gamma_n x^n + \text{lower order}$$

with $\gamma_n > 0$ and

$$\int_{\mathbb{R}} p_n(x) p_m(x) \mathrm{d}\mu(x) = \delta_{m,n}.$$

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The Christoffel–Darboux (CD) kernel,

$$K_n(x,y)=\sum_{j=0}^{n-1}p_j(x)p_j(y),$$

is the kernel of the projection onto the subspace of $L^2(\mathrm{d}\mu)$ of polynomials of degree less than n. It arises in various natural contexts and its properties have been the focus of many works (for reviews see [11,13]). A significant portion of these works study asymptotics of $K_n(x+\frac{a}{n},x+\frac{b}{n})$ as $n\to\infty$. There are two main motivations for studying these asymptotics. First, the CD kernel arises as the correlation kernel for the eigenvalues of the unitary ensembles of Hermitian matrices and so, its asymptotics describe the asymptotic distribution for these eigenvalues (see, e.g., [2]). Second, if y is a zero of p_n then x is a zero of p_n iff $K_n(x,y)=0$. Thus, the asymptotic properties of $K_n(x+\frac{a}{n},x+\frac{b}{n})$ are connected to the small scale behavior of the zeros of the p_n around x as $n\to\infty$ (see, e.g., [4,7,13]).

An important part in many works on the CD kernel, (including this one), is played by the Christoffel–Darboux formula:

$$K_n(x,y)=a_n\frac{p_n(x)p_{n-1}(y)-p_n(y)p_{n-1}(x)}{x-y}, \tag{1.1}$$

$$K_n(x,x)=a_n(p_n'(x)p_{n-1}(x)-p_n(x)p_{n-1}'(x)), \tag{1.2}$$

which can be derived from the recursion relations for orthogonal polynomials (see below).

Until recently, except for some classical cases, where the asymptotics of the p_n are well understood, the general methods for studying $K_n(x+\frac{a}{n},x+\frac{b}{n})$ required $\mathrm{d}\mu_{\mathrm{sing}}=0$ and some degree of smoothness from $w(x)$. For example, the very powerful Riemann–Hilbert methods require $w=\mathrm{e}^{-Q}$ with Q analytic and the $\overline{\partial}$-method allows one to relax this to Q'' satisfying a Lipschitz condition (for past results on universality, see, e.g., [8] and references therein). In these cases, it was shown that, for x_0 in the interior of the support of the measure,

$$\lim_{n\to\infty}\frac{K_n\left(x_0+\frac{a}{n},x_0+\frac{b}{n}\right)}{K_n(x_0,x_0)}=\frac{\sin(\pi\rho(x_0)(b-a))}{\pi\rho(x_0)(b-a)} \tag{1.3}$$

where $\rho(x)$ is the asymptotic density of the zeros of p_n at x. That is, $\rho(x)$ is the Radon–Nikodym derivative (with respect to the Lebesgue measure) of the weak limit $w-\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\delta_{x_j^n}$ where x_j^n, $j=1,2,\ldots,n$ are the *distinct* zeros of p_n. Clearly, this limit does not always exist, but in this paper we restrict our discussion to cases where it does.

The limit in (1.3) is known as the *universality limit in the bulk* since, apart from the normalizing factor of $\rho(x_0)$, the limiting kernel is independent of x_0 and the particular form of μ. Two new methods introduced by Lubinsky [9,10] enable the derivation of such a limit under much weaker requirements from the measure. In particular, in [9] it was shown that if μ is a regular measure on $(-2,2)$ which is absolutely continuous on a neighborhood of $x_0\in$ interior of supp(μ) and has a continuous and positive Radon–Nikodym derivative at x_0 then (1.3) holds uniformly for a,b in compact subsets of the complex plane. It is important to note that continuity of w at x_0 can be replaced by a Lebesgue point type condition. Moreover, there exist some extensions of this result to more general sets and less restrictive conditions on the derivative of the measure (see [1,3,14,13,15]). However, to the best of our knowledge, all existing methods for obtaining (1.3) require absolute continuity of the measure.

The purpose of this note is to present a class of purely singular measures for which (1.3) holds. We shall construct these measures through their Jacobi parameters—the parameters entering in the recursion relation of the p_n's:

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x) \quad n > 0, \tag{1.4}$$

$$xp_0(x) = a_1 p_1(x) + b_1 p_0(x). \tag{1.5}$$

It is a classical result that such a relation is satisfied by the set of orthogonal polynomials associated with any compactly supported, infinitely supported measure, with $a_n > 0$ and $b_n \in \mathbb{R}$ both bounded sequences (by 'infinitely supported' we mean that the support is not a finite set). On the other hand, any Jacobi matrix,

$$J(\{a_n, b_n\}_{n=1}^{\infty}) = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \ddots \\ 0 & a_2 & b_3 & a_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.6}$$

(with the a_n's positive and bounded and b_n's bounded), can be viewed as a bounded self-adjoint operator in ℓ^2 with $(1, 0, 0, 0, \dots)^T$ a cyclic vector. Thus, by the spectral theorem, J and $(1, 0, 0, 0, \dots)^T$ have a spectral measure associated with them. The mappings $J \mapsto \mu$ via the spectral theorem and $\mu \mapsto J$ via the orthogonal polynomial recursion relation, for bounded Jacobi matrices and compactly supported, infinitely supported probability measures, can be shown to be inverses of each other (see e.g. [2]), and so we obtain a 1–1 correspondence between these two families of objects.

Perhaps the simplest case is that of the (rescaled) Chebyshev polynomials of the second kind. In this case, $d\mu_0(x) = \frac{\sqrt{4-x^2}}{2\pi}\chi_{[-2,2]}(x)dx$, and the orthogonal polynomials (for $x = 2\cos(\theta)$) are $U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$. The asymptotic density of zeros is $\rho_0(x) = \pi^{-1}(\sqrt{4-x^2})^{-1}\chi_{[-2,2]}(x)$ and the corresponding Jacobi matrix is

$$J_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \ddots \\ 0 & 1 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.7}$$

We shall obtain our family of measures by adding a decaying sparse perturbation to the Jacobi matrix J_0. That is, we shall consider the Jacobi parameters

$$a_n \equiv 1 \qquad b_n = \begin{cases} v_j & n = N_j \\ 0 & \text{otherwise} \end{cases} \tag{1.8}$$

where $\{N_j\}_{j=1}^{\infty}$ is an increasing sequence of natural numbers satisfying

$$\frac{N_{j+1}}{N_j} \to \infty \tag{1.9}$$

and

$$v_j \to 0. \tag{1.10}$$

Such matrices are known as sparse Jacobi matrices and have served as the first explicit examples of discrete Schrödinger operators with singular continuous spectral measures. The review [6] contains a survey of some of the extensive research carried out in this context since Pearson's paper [12], where Schrödinger operators with sparse decaying potentials were introduced. Here we shall rely on Theorem 1.7 in [5].

Theorem 1.1 (*[5], Theorem 1.7*)**.** *Let J be a Jacobi matrix satisfying (1.8) so that (1.9) and (1.10) hold. Then:*

1. *If $\sum_{j=1}^{\infty} v_j^2 < \infty$ then the spectral measure associated with J is purely absolutely continuous on $(-2, 2)$.*
2. *If $\sum_{j=1}^{\infty} v_j^2 = \infty$ then the spectral measure associated with J is purely singular continuous on $(-2, 2)$.*

We shall prove the following

Theorem 1.2. *Let $\{v_j\}_{j=1}^{\infty}$ be a sequence of real numbers such that $v_j \to 0$ as $j \to \infty$. If the sequence $\{N_j\}_{j=1}^{\infty}$ is sufficiently sparse (see below) and μ is the measure corresponding to the Jacobi parameters given by (1.8), then for every $x \in (-2, 2)$ and any $a, b \in \mathbb{R}$*

$$\lim_{n\to\infty} \frac{K_n\left(x + \frac{a}{n}, x + \frac{b}{n}\right)}{K_n(x, x)} = \frac{\sin((\sqrt{4 - x^2})^{-1}(b - a))}{(\sqrt{4 - x^2})^{-1}(b - a)}, \tag{1.11}$$

where $K_n(x, y)$ is the corresponding CD kernel.

Remark. By '$\{N_j\}_{j=1}^{\infty}$ is sufficiently sparse' we mean that N_{k+1} has to be chosen sufficiently large as a function of $\{N_1, N_2, \ldots, N_k\}$. In other words, for any $k \geq 1$ there exists a function $\widetilde{N}_k(N_1, N_2, \ldots, N_k)$ such that $N_{k+1} \geq \widetilde{N}_k(N_1, N_2, \ldots, N_k)$. The sequence of functions $\widetilde{N}_k$ depends on $\{v_j\}_{j=1}^{\infty}$.

Corollary 1.3. *There exist purely singular measures such that (1.11) holds for every $x \in (-2, 2)$.*

Proof. As remarked above, Theorem 1.7 in [5] says that if $\sum_{j=1}^{\infty} v_j^2 = \infty$ and $\frac{N_{j+1}}{N_j} \to \infty$ then the measure is purely singular. Thus, by picking such sequences that satisfy the hypothesis of Theorem 1.2, we get a purely singular measure satisfying (1.11). □

The idea of the proof of Theorem 1.2 is quite simple. For any finite rank perturbation of J_0, universality holds. Thus, having chosen $\{N_j\}_{j=1}^{K}$, one may place N_{K+1} only after the renormalized CD kernel is very close to its sine kernel limit. The heart of the proof lies in showing that, since v_{K+1} is small, the perturbation at N_{K+1} is weak and does not produce a substantial change in the renormalized CD kernel.

After obtaining some preliminary results in Section 2, we prove Theorem 1.2 in Section 3.

2. Preliminaries

As our analysis is a perturbative analysis, we begin by sketching a proof of universality for the unperturbed model, namely, the second kind Chebyshev polynomials which correspond to the matrix J_0. As noted in the Introduction, a useful device is the Christoffel–Darboux formula:

$$K_n(x,y) = a_n \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{x-y}, \tag{2.1}$$

$$K_n(x,x) = a_n(p_n'(x)p_{n-1}(x) - p_n(x)p_{n-1}'(x)). \tag{2.2}$$

Using this formula, one can show directly that, for $\mathrm{d}\mu(x) = \frac{\sqrt{4-x^2}}{2\pi}\chi_{[-2,2]}(x)\mathrm{d}x$, $x \in (-2,2)$, and $a, b \in \mathbb{C}$, with $a \neq b$

$$\begin{aligned}\lim_{n\to\infty} \frac{K_n\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n} &= \lim_{n\to\infty} \frac{U_n\left(x+\frac{a}{n}\right)U_{n-1}\left(x+\frac{b}{n}\right) - U_n\left(x+\frac{b}{n}\right)U_{n-1}\left(x+\frac{a}{n}\right)}{b-a} \\ &= \frac{2\sin((\sqrt{4-x^2})^{-1}(b-a))}{\sqrt{4-x^2}(b-a)}\end{aligned} \tag{2.3}$$

and the convergence is uniform in $|a|, |b| < C$ and $|b-a| > \delta$ for every $\delta, C > 0$. In fact, it is not hard to see directly that the restriction $|a-b| > \delta$ is unnecessary, but we want to use an argument which will play an important role in what follows. Note that for fixed $a \in \mathbb{C}$, $K_n(x+\frac{a}{n}, x+\frac{b}{n})$ is analytic as a function of b. The limit function in (2.3) is analytic as well. By the uniform convergence in each annulus around a, it follows from Cauchy's integral formula that convergence holds also for $b = a$ and in fact is uniform in $|a|, |b| < C$ (where we interpret $\frac{\sin(0)}{0} = 1$).

By considering the limit for $a = b = 0$, this immediately implies

$$\begin{aligned}\lim_{n\to\infty} \frac{K_n\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{K_n(x,x)} &= \lim_{n\to\infty} \frac{K_n\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n} \lim_{n\to\infty} \frac{n}{K_n(x,x)} \\ &= \frac{\sin((\sqrt{4-x^2})^{-1}(b-a))}{(\sqrt{4-x^2})^{-1}(b-a)}\end{aligned} \tag{2.4}$$

which is precisely (1.3) (recall $\rho_0(x) = \pi^{-1}(\sqrt{4-x^2})^{-1}\chi_{[-2,2]}(x)$).

Now fix sequences $\{v_j\}_{j=1}^{\infty}$ and $\{N_j\}_{j=1}^{\infty}$ with $N_{j+1}/N_j \to \infty$, and let μ be the spectral measure corresponding to the Jacobi parameters given by (1.8). Let $\{p_n(x)\}_{n=0}^{\infty}$ be the orthogonal polynomials associated with μ.

We use variation of parameters. We consider (1.8) as a perturbation on J_0. Fix $x \in (-2,2)$ and let $\psi_n^1(x)$ and $\psi_n^2(x)$ be the two solutions of the difference equation $x\psi_n(x) = \psi_{n+1}(x) + \psi_{n-1}(x)$ for $n \geq 0$, satisfying the boundary conditions

$$\psi_0^1(x) = 1, \qquad \psi_{-1}^1(x) = 0 \qquad \psi_0^2(x) = 0, \qquad \psi_{-1}^2(x) = 1. \tag{2.5}$$

Explicitly, it is easy to see that if $x = 2\cos(\theta)$ for $\theta \in (0,\pi)$, then

$$\psi_n^1(x) = \frac{\sin(n+1)\theta}{\sin(\theta)} = U_n(x) \tag{2.6}$$

(the second kind Chebyshev polynomials) and

$$\psi_n^2(x) = \frac{-\sin(n\theta)}{\sin(\theta)}. \tag{2.7}$$

We note that the CD formula, as well as universality holds for $\psi_n^1(x) = U_n(x)$ (as shown above) and also for $\psi_n^2(x) \equiv -\psi_{n-1}^1(x)$.

Now, define $A_n(x) \in \mathbb{C}^2$ $(n \geq 1)$ by

$$\begin{aligned} p_n(x) &= A_{n,1}(x)\psi_n^1(x) + A_{n,2}(x)\psi_n^2(x) \\ p_{n-1}(x) &= A_{n,1}(x)\psi_{n-1}^1(x) + A_{n,2}(x)\psi_{n-1}^2(x), \end{aligned} \tag{2.8}$$

or, in matrix form,

$$\begin{pmatrix} p_n(x) \\ p_{n-1}(x) \end{pmatrix} = \begin{pmatrix} \psi_n^1(x) & \psi_n^2(x) \\ \psi_{n-1}^1(x) & \psi_{n-1}^2(x) \end{pmatrix} \begin{pmatrix} A_{n,1}(x) \\ A_{n,2}(x) \end{pmatrix}. \tag{2.9}$$

We denote

$$T_n(x) = \begin{pmatrix} \psi_n^1(x) & \psi_n^2(x) \\ \psi_{n-1}^1(x) & \psi_{n-1}^2(x) \end{pmatrix} = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}^n, \tag{2.10}$$

where the right equality holds since the recursion relations (1.4), (1.5) for J_0 can be written in matrix form as

$$\begin{pmatrix} p_n(x) \\ p_{n-1}(x) \end{pmatrix} = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_{n-1}(x) \\ p_{n-2}(x) \end{pmatrix}.$$

Note that $\det T_n(x) = 1$. Moreover, for any closed interval $I \subseteq (-2, 2)$ there exists $M_I > 0$ such that for any n and any $x \in I$,

$$\|T_n(x)\| \leq M_I. \tag{2.11}$$

In fact, it will be crucial later on, to be able to extend this bound slightly to the complex plane.

Lemma 2.1. *Let $I \subset (-2, 2)$ be a closed interval. There exists $M_I > 0$ such that for any $x \in I$, $t \in \mathbb{R}$ with $|t| \leq 1$,*

$$\left\| T_n\left(x + \frac{it}{n}\right) \right\| \leq M_I. \tag{2.12}$$

Proof. We shall show that we can uniformly bound $|\psi_n^{1,2}(x + \frac{it}{n})|$. Fix x, t and let $2\cos(\theta_0) = x$ and $2\cos(\theta_0 + \delta_n) = x + \frac{it}{n}$. By expanding to a Taylor series

$$\frac{it}{n} = 2\cos(\theta_0 + \delta_n) - 2\cos(\theta_0) = 2\delta_n \sin(\theta_0) + o(\delta_n)$$

as $\delta_n \to 0$ (and so as $n \to \infty$), we see that $\delta_n = O(\frac{1}{n})$ and the implicit constant depends on $|\sin(\theta_0)|^{-1}$ which is uniformly bounded on I. Write

$$\frac{\sin(n(\theta_0 + \delta_n))}{\sin(\theta_0 + \delta_n)} = \frac{\sin(n\theta_0)\cos(n\delta_n) + \cos(n\theta_0)\sin(n\delta_n)}{\sin(\theta_0 + \delta_n)}.$$

The denominator above is bounded from below on I, $\sin(n\theta_0)$ and $\cos(n\theta_0)$ are both uniformly bounded on I and $n\delta_n$ is uniformly bounded on I as well, by the discussion above. Therefore, $\psi_n^{1,2}(x + \frac{it}{n})$ are uniformly bounded on I and we are done. □

We assume, without loss of generality, that $M_I \geq 1$ for all I. For concreteness, we let $I_j = [-2+\frac{1}{j}, 2-\frac{1}{j}]$ for $j \geq 1$ and $M_j \equiv M_{I_j}$.

It is well known (and follows from (2.8)/ (2.9)) that $A_n(x)$ satisfies the recurrence relation:

$$A_{n+1}(x) = A_n(x) + \Phi_n(x) A_n(x) \tag{2.13}$$

where

$$\Phi_n(x) = -b_{n+1} \begin{pmatrix} \psi_n^1(x)\psi_n^2(x) & (\psi_n^2(x))^2 \\ -(\psi_n^1(x))^2 & -\psi_n^1(x)\psi_n^2(x) \end{pmatrix}. \tag{2.14}$$

By noting $(I + \Phi_n(x))^{-1} = I - \Phi_n(x)$ we also get that

$$A_n(x) = A_{n+1}(x) - \Phi_n(x) A_{n+1}(x). \tag{2.15}$$

By extending the definition of M_I, we also assume that $\|\Phi_n(x)\| \leq |b_{n+1}| M_I^2$ for any $n \in \mathbb{N}$, $x \in I$.

Thus, we immediately see that along stretches where $b_{n+1} = 0$, $A_n(x)$ is constant in n. Moreover, in this case, $A_n(x+\frac{a}{n}) - A_n(x)$ is small for large n, so we can approximate $p_n(x+\frac{a}{n})$ by $A_{n,1}(x)\psi_n^1(x+\frac{a}{n}) + A_{n,2}(x)\psi_n^2(x+\frac{a}{n})$.

For constant A we have

Lemma 2.2. *For any* $A = \binom{A_1}{A_2} \in \mathbb{C}^2$, *let*

$$\varphi_n^A(x) = A_1 \psi_n^1(x) + A_2 \psi_n^2(x) \tag{2.16}$$

and let

$$K_n^A(x, y) = \sum_{j=0}^{n-1} \varphi_j^A(x) \varphi_j^A(y).$$

Then, for any $x \in (-2, 2)$ *and any* $a, b \in \mathbb{C}$,

$$\lim_{n\to\infty} \frac{1}{(A_1^2 + A_2^2 - A_1 A_2 x)} \frac{K_n^A\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n} = \frac{2\sin((\sqrt{4-x^2})^{-1}(b-a))}{\sqrt{4-x^2}(b-a)}. \tag{2.17}$$

Moreover, for any $C > 0$ *and any closed interval* $I \subseteq (-2, 2)$, *the convergence is uniform in* $1/C < \|A\| < C$, $|a|, |b| < C$ *and* $x \in I$.

Proof. It is a simple calculation, using (2.6) and (2.7), to see that for $a \neq b$

$$\lim_{n\to\infty} \frac{K_n^A\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n} = \frac{2(A_1^2 + A_2^2 - A_1 A_2 x)\sin((\sqrt{4-x^2})^{-1}(b-a))}{\sqrt{4-x^2}(b-a)}$$

and that the convergence is uniform in $\|A\| < C$, $x \in I$, and $|a|, |b| < C$ with $|a-b| > \delta$ for any $\delta > 0$. The computation is essentially the same as the one leading to (2.3) where the A_j^2 terms come from the terms of the form $\psi_n^j(x+\frac{a}{n})\psi_{n-1}^j(x+\frac{b}{n}) - \psi_n^j(x+\frac{b}{n})\psi_{n-1}^j(x+\frac{a}{n})$ in the CD formula and the $A_1 A_2 x$ coefficient comes from the cross term. The same analyticity argument as the one given after (2.3) shows that convergence holds also for $a = b$ and is uniform in $\|A\| < C$, $x \in I$, and $|a|, |b| < C$.

Since, for $x \in I$, $|A_1 A_2 x| < d(A_1^2 + A_2^2)$ for some $d < 1$, we get that after dividing by $(A_1^2 + A_2^2 - A_1 A_2 x)$ the convergence is still uniform for $1/C < \|A\|$. □

Now, let $\mu^{(\ell)}$ be the measure associated with the Jacobi coefficients

$$a_n \equiv 1 \qquad b_n = \begin{cases} v_j & n = N_j, \quad j \le \ell \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

Let $K_n^{(\ell)}$ be the CD kernel, and $p_n^{(\ell)}$ the orthogonal polynomials associated with $\mu^{(\ell)}$, so that

$$K_n^{(\ell)}(x, y) = \sum_{j=1}^{n-1} p_j^{(\ell)}(x) p_j^{(\ell)}(y).$$

Let $A_n^{(\ell)}(x)$ be defined by

$$\begin{aligned} p_n^{\ell}(x) &= A_{n,1}^{(\ell)}(x)\psi_n^1(x) + A_{n,2}^{(\ell)}(x)\psi_n^2(x) \\ p_{n-1}^{(\ell)}(x) &= A_{n,1}^{(\ell)}(x)\psi_{n-1}^1(x) + A_{n,2}^{(\ell)}(x)\psi_{n-1}^2(x). \end{aligned} \tag{2.19}$$

Lemma 2.3. *For any $x \in (-2, 2)$, $a, b \in \mathbb{C}$,*

$$\lim_{n\to\infty} \frac{K_n^{(\ell)}\left(x + \frac{a}{n}, x + \frac{b}{n}\right)}{n(A_{n,1}^{(\ell)}(x)^2 + A_{n,2}^{(\ell)}(x)^2 - A_{n,1}^{(\ell)}(x)A_{n,2}^{(\ell)}(x)x)} = \frac{2\sin((\sqrt{4-x^2})^{-1}(b-a))}{\sqrt{4-x^2}(b-a)} \tag{2.20}$$

where, again, $\frac{\sin(0)}{0} = 1$. Moreover, for any m and $C > 0$, the convergence is uniform in $x \in I_m$, and in $|a|, |b| \le C$. That is, for any m, $C > 0$ and any $\varepsilon > 0$ there exists $N(\varepsilon, m)$ so that for any $x \in I_m$, any $|a|, |b| \le C$, and any $n \ge N(\varepsilon, m)$,

$$\left| \frac{K_n^{(\ell)}\left(x + \frac{a}{n}, x + \frac{b}{n}\right)}{n(A_{n,1}^{(\ell)}(x)^2 + A_{n,2}^{(\ell)}(x)^2 - A_{n,1}^{(\ell)}(x)A_{n,2}^{(\ell)}(x)x)} - \frac{2\sin((\sqrt{4-x^2})^{-1}(b-a))}{\sqrt{4-x^2}(b-a)} \right| < \varepsilon. \tag{2.21}$$

Proof. As in the proof of Lemma 2.2, it suffices to prove uniform convergence for $|a-b| > \delta$ for each $\delta > 0$. Since, for any $n \ge N_\ell + 1$, $A_n^{(\ell)}(x) = A_{N_\ell+1}^{(\ell)}(x)$ we use, for simplicity of notation, $\widetilde{A}(x) = A_{N_\ell+1}^{(\ell)}(x)$. Fix $x_0 \in I_m$ for some m and let

$$\varphi_n(x_0; x) = \widetilde{A}_1(x_0)\psi_n^1(x) + \widetilde{A}_2(x_0)\psi_n^2(x).$$

Let, for $x \ne y$,

$$\widehat{K}_n(x_0; x, y) = \frac{\varphi_n(x_0; x)\varphi_{n-1}(x_0; y) - \varphi_n(x_0; y)\varphi_{n-1}(x_0; x)}{x - y}.$$

By Lemma 2.2

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{(\widetilde{A}_1(x_0)^2 + \widetilde{A}_2(x_0)^2 - \widetilde{A}_1(x_0)\widetilde{A}_2(x_0)x_0)} \frac{\widehat{K}_n\left(x_0; x_0 + \frac{a}{n}, x_0 + \frac{b}{n}\right)}{n} \\ &\quad = \frac{2\sin((\sqrt{4-x^2})^{-1}(b-a))}{\sqrt{4-x^2}(b-a)} \end{aligned} \tag{2.22}$$

uniformly.

It is clear that $f(z) = \widetilde{A}(z)$ is an analytic vector-valued function of z (its components are polynomials in z). Thus, $\widetilde{A}(x_0 + \frac{a}{n}) \to \widetilde{A}(x_0)$ as $n \to \infty$ uniformly for a in compact subsets of the plane. Using this, and (2.1), it is an easy (though somewhat tedious) computation to see that

$$\lim_{n\to\infty} \left| \frac{1}{(\widetilde{A}_1(x_0)^2 + \widetilde{A}_2(x_0)^2 - \widetilde{A}_1(x_0)\widetilde{A}_2(x_0)x_0)} \frac{K_n^{(\ell)}\left(x_0 + \frac{a}{n}, x_0 + \frac{b}{n}\right)}{n} - \frac{1}{(\widetilde{A}_1(x_0)^2 + \widetilde{A}_2(x_0)^2 - \widetilde{A}_1(x_0)\widetilde{A}_2(x_0)x_0)} \frac{\widehat{K}_n\left(x_0; x_0 + \frac{a}{n}, x_0 + \frac{b}{n}\right)}{n} \right| = 0 \tag{2.23}$$

uniformly in $x_0 \in I_m$ and $|a|, |b| < C$, $|a - b| > \delta$.

Combining (2.22) and (2.23) shows

$$\lim_{n\to\infty} \frac{K_n^{(\ell)}\left(x_0 + \frac{a}{n}, x_0 + \frac{b}{n}\right)}{n(\widetilde{A}_1(x_0)^2 + \widetilde{A}_2(x_0)^2 - \widetilde{A}_1(x_0)\widetilde{A}_2(x_0)x_0)} = \frac{2\sin((\sqrt{4 - x^2})^{-1}(b - a))}{\sqrt{4 - x^2}(b - a)} \tag{2.24}$$

uniformly. Noting that, for fixed a, the members of the sequence, as well as the limiting function, are all analytic in b, we obtain, as before, the limit for $a = b$. This ends the proof. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Given $\{v_n\}_{n=1}^\infty$ let $\{m_n\}_{n=1}^\infty$ satisfy $m_n \to \infty$ monotonically as $n \to \infty$ and $v_n m_n^2 M_{m_n}^4 \to 0$ where we recall that $M_m \equiv M_{I_m}$ and $I_m = [-2 + \frac{1}{m}, 2 - \frac{1}{m}]$ (recall the definition of M_I in (2.12) of Lemma 2.1). This is possible since $v_n \to 0$, so for any $r = 1, 2, \ldots,$ there exists N_r so that for any $n \geq N_r$, $|v_n| < \frac{1}{M_r^4 r^4}$. Thus, we can choose $m_1 = m_2 = \cdots = m_{N_2} = 1$, $m_{N_2+1} = m_{N_2+2} = \cdots = m_{N_3} = 2$ and generally, $m_{N_r+1} = m_{N_r+2} = \cdots = m_{N_{r+1}} = r$. In particular, $|v_n| M_{m_n}$ is bounded.

Assume we have fixed $\{N_j\}_{j=1}^\ell$. Let $\tilde{I}_\ell = I_{m_{\ell+1} - 1/\ell}$ (a closed interval contained in the interior of $I_{m_{\ell+1}}$), and consider $a, b \in \mathbb{C}$ with $|a|, |b| \leq \ell$, $\Im(a), \Im(b) \leq 1$. By Lemma 2.3, there exists $\widehat{N}(\ell)$ such that for any $n \geq \widehat{N}(\ell)$, any $x \in \tilde{I}_\ell$ and any $|a|, |b| \leq \ell$,

$$\left| \frac{K_n^{(\ell)}\left(x + \frac{a}{n}, x + \frac{b}{n}\right)}{n(A_{n,1}^{(\ell)}(x)^2 + A_{n,2}^{(\ell)}(x)^2 - A_{n,1}^{(\ell)}(x)A_{n,2}^{(\ell)}(x)x)} - \frac{2\sin((\sqrt{4 - x^2})^{-1}(b - a))}{\sqrt{4 - x^2}(b - a)} \right| < \frac{1}{\ell}. \tag{3.1}$$

By taking $\widehat{N}(\ell)$ large enough, we may also assume that $\Re(x + \frac{a}{n}), \Re(x + \frac{b}{n}) \in I_{m_{\ell+1}}$ for any $n \geq \widehat{N}(\ell)$. Finally, we may assume that

$$\frac{\left|A_{n,1}^{(\ell)}\left(x + \frac{a}{n}\right)\right|^2 + \left|A_{n,2}^{(\ell)}\left(x + \frac{a}{n}\right)\right|^2}{|A_{n,1}^{(\ell)}(x)|^2 + |A_{n,2}^{(\ell)}(x)|^2} \leq 2 \tag{3.2}$$

and

$$\frac{\left|A_{n,1}^{(\ell)}\left(x + \frac{b}{n}\right)\right|^2 + \left|A_{n,2}^{(\ell)}\left(x + \frac{b}{n}\right)\right|^2}{|A_{n,1}^{(\ell)}(x)|^2 + |A_{n,2}^{(\ell)}(x)|^2} \leq 2 \tag{3.3}$$

for $x \in \tilde{I}_\ell$, $a, b \in \mathbb{C}$ with $|a|, |b| \leq \ell$ and $n \geq \widehat{N}(\ell)$. This is because $|A_{n,1}^{(\ell)}(x)|^2 + |A_{n,2}^{(\ell)}(x)|^2$ is a continuous, non-vanishing function which is independent of n for $n \geq N_\ell$ (recall (2.13)).

We will show that as long as we pick $N_{\ell+1} \geq \widehat{N}(\ell)$ (inductively), (1.11) holds uniformly for a, b in compact subsets of $\mathbb{R}$ and $x \in$ closed subintervals of $(-2, 2)$. Our strategy will be to first prove that

$$\left| \frac{K_n\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n(A_{n,1}(x)^2 + A_{n,2}(x)^2 - A_{n,1}(x)A_{n,2}(x)x)} - \frac{2\sin((\sqrt{4-x^2})^{-1}(b-a))}{\sqrt{4-x^2}(b-a)} \right| \to 0 \quad (3.4)$$

uniformly for complex a, b with $|a-b| > \delta$ and $\Im(a), \Im(b) \leq 1$ and deduce (1.11) using the analyticity argument used repeatedly above.

Note that, for any $N_\ell \leq n < N_{\ell+1}$, $K_n = K_n^{(\ell)}$ and $A_n = A_n^{(\ell)}$. Thus, for any $N_{\ell+1} \leq n < N_{\ell+2}$

$$\begin{aligned} &\frac{K_n\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n(A_{n,1}(x)^2 + A_{n,2}(x)^2 - A_{n,1}(x)A_{n,2}(x)x)} - \frac{K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n(A_{n,1}^{(\ell)}(x)^2 + A_{n,2}^{(\ell)}(x)^2 - A_{n,1}^{(\ell)}(x)A_{n,2}^{(\ell)}(x)x)} \\ &= \frac{K_n^{(\ell+1)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n(A_{n,1}^{(\ell+1)}(x)^2 + A_{n,2}^{(\ell+1)}(x)^2 - A_{n,1}^{(\ell+1)}(x)A_{n,2}^{(\ell+1)}(x)x)} \\ &\quad - \frac{K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n(A_{n,1}^{(\ell)}(x)^2 + A_{n,2}^{(\ell)}(x)^2 - A_{n,1}^{(\ell)}(x)A_{n,2}^{(\ell)}(x)x)}, \end{aligned}$$

and since (3.1) holds for any $n \geq N_{\ell+1}$, $x \in \tilde{I}_\ell$ and $|a|, |b| < \ell$, it will be enough to show that

$$\begin{aligned} \max_{N_{\ell+1} \leq n < N_{\ell+2},\ x \in \tilde{I}_\ell} &\left| \frac{K_n^{(\ell+1)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n(A_{n,1}^{(\ell+1)}(x)^2 + A_{n,2}^{(\ell+1)}(x)^2 - A_{n,1}^{(\ell+1)}(x)A_{n,2}^{(\ell+1)}(x)x)} \right. \\ &\left. - \frac{K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n(A_{n,1}^{(\ell)}(x)^2 + A_{n,2}^{(\ell)}(x)^2 - A_{n,1}^{(\ell)}(x)A_{n,2}^{(\ell)}(x)x)} \right| \to 0 \end{aligned} \quad (3.5)$$

as $\ell \to \infty$, uniformly in $|a|, |b| < C$, $|a-b| > \delta$.

For notational simplicity, let $\kappa_n^{(\ell)}(x) = A_{n,1}^{(\ell)}(x)^2 + A_{n,2}^{(\ell)}(x)^2 - A_{n,1}^{(\ell)}(x)A_{n,2}^{(\ell)}(x)x$, and write

$$\begin{aligned} &\left| \frac{K_n^{(\ell+1)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n\kappa_n^{(\ell+1)}(x)} - \frac{K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n\kappa_n^{(\ell)}(x)} \right| \\ &\leq \left| \frac{K_n^{(\ell+1)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n\kappa_n^{(\ell+1)}(x)} - \frac{K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n\kappa_n^{(\ell+1)}(x)} \right| \\ &\quad + \left| \frac{K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n\kappa_n^{(\ell+1)}(x)} - \frac{K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n\kappa_n^{(\ell)}(x)} \right| \\ &= \left| \frac{K_n^{(\ell+1)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right) - K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n\kappa_n^{(\ell+1)}(x)} \right| \\ &\quad + \left| \frac{K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n\kappa_n^{(\ell)}(x)} \right| \left| \frac{\kappa_n^{(\ell)}(x) - \kappa_n^{(\ell+1)}(x)}{\kappa_n^{(\ell+1)}(x)} \right|. \end{aligned} \quad (3.6)$$

We treat the summands on the right hand side one by one, starting with $\left|\frac{K_n^{(\ell+1)}(x+\frac{a}{n},x+\frac{b}{n})-K_n^{(\ell)}(x+\frac{a}{n},x+\frac{b}{n})}{n\kappa_n^{(\ell+1)}(x)}\right|$. Note that for any $n < N_{\ell+1}$, $p_n^{(\ell)}(x) = p_n^{(\ell+1)}(x)$. For $n = N_{\ell+1}$, by (1.4),

$$\begin{aligned} p_n^{(\ell+1)}(x) &= x p_{n-1}^{(\ell+1)}(x) - p_{n-2}^{(\ell+1)}(x) - v_{\ell+1} p_{n-1}^{(\ell+1)}(x) \\ &= x p_{n-1}^{(\ell)}(x) - p_{n-2}^{(\ell)}(x) - v_{\ell+1} p_{n-1}^{(\ell)}(x) = p_n^{(\ell)}(x) - v_{\ell+1} p_{n-1}^{(\ell)}(x). \end{aligned}$$

Thus, for any $N_{\ell+1} \le n < N_{\ell+2}$,

$$\begin{aligned} \begin{pmatrix} p_n^{(\ell+1)}(x) \\ p_{n-1}^{(\ell+1)}(x) \end{pmatrix} &= T_{n-N_{\ell+1}}(x) \begin{pmatrix} p_{N_{\ell+1}}^{(\ell+1)}(x) \\ p_{N_{\ell+1}-1}^{(\ell+1)}(x) \end{pmatrix} \\ &= T_{n-N_{\ell+1}}(x) \begin{pmatrix} p_{N_{\ell+1}}^{(\ell)}(x) - v_{\ell+1} p_{N_{\ell+1}-1}^{(\ell)}(x) \\ p_{N_{\ell+1}-1}^{(\ell)}(x) \end{pmatrix} \\ &= T_{n-N_{\ell+1}}(x) \begin{pmatrix} p_{N_{\ell+1}}^{(\ell)}(x) \\ p_{N_{\ell+1}-1}^{(\ell)}(x) \end{pmatrix} - v_{\ell+1} p_{N_{\ell+1}-1}^{(\ell)}(x) T_{n-N_{\ell+1}}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned} \tag{3.7}$$

which implies

$$\begin{aligned} \begin{pmatrix} p_n^{(\ell+1)}(x) \\ p_{n-1}^{(\ell+1)}(x) \end{pmatrix} &= \begin{pmatrix} p_n^{(\ell)}(x) \\ p_{n-1}^{(\ell)}(x) \end{pmatrix} - v_{\ell+1} p_{N_{\ell+1}-1}^{(\ell)}(x) T_{n-N_{\ell+1}}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} p_n^{(\ell)}(x) \\ p_{n-1}^{(\ell)}(x) \end{pmatrix} - v_{\ell+1} p_{N_{\ell+1}-1}^{(\ell)}(x) \begin{pmatrix} \psi_{n-N_{\ell+1}}^1(x) \\ \psi_{n-1-N_{\ell+1}}^1(x) \end{pmatrix}. \end{aligned} \tag{3.8}$$

Using Lemma 2.1 and $2|\alpha|\,|\beta| \le |\alpha|^2 + |\beta|^2$, it follows that, for $N_{\ell+1} \le n < N_{\ell+2}$, $x \in \tilde{I}_\ell$ and a, b with $\Im(a), \Im(b) \le 1$, and $|a|, |b| \le \ell$,

$$\begin{aligned} &\left| \left(p_n^{(\ell+1)}\left(x+\frac{a}{n}\right) p_{n-1}^{(\ell+1)}\left(x+\frac{b}{n}\right) - p_n^{(\ell+1)}\left(x+\frac{b}{n}\right) p_{n-1}^{(\ell+1)}\left(x+\frac{a}{n}\right) \right) \right. \\ &\quad \left. - \left(p_n^{(\ell)}\left(x+\frac{a}{n}\right) p_{n-1}^{(\ell)}\left(x+\frac{b}{n}\right) - p_n^{(\ell)}\left(x+\frac{b}{n}\right) p_{n-1}^{(\ell)}\left(x+\frac{a}{n}\right) \right) \right| \\ &\le 4|v_{\ell+1}|^2 M_{I_{m_{\ell+1}}}^2 \left(\left| p_{N_{\ell+1}-1}^{(\ell)}\left(x+\frac{a}{n}\right) \right|^2 + \left| p_{N_{\ell+1}-1}^{(\ell)}\left(x+\frac{b}{n}\right) \right|^2 \right) \\ &\quad + 4|v_{\ell+1}| M_{I_{m_{\ell+1}}} \left(\left| p_{N_{\ell+1}-1}^{(\ell)}\left(x+\frac{a}{n}\right) \right|^2 + \left| p_{N_{\ell+1}-1}^{(\ell)}\left(x+\frac{b}{n}\right) \right|^2 \right. \\ &\quad + \left| p_n^{(\ell)}\left(x+\frac{a}{n}\right) \right|^2 + \left| p_{n-1}^{(\ell)}\left(x+\frac{a}{n}\right) \right|^2 \\ &\left. + \left| p_n^{(\ell)}\left(x+\frac{b}{n}\right) \right|^2 + \left| p_{n-1}^{(\ell)}\left(x+\frac{b}{n}\right) \right|^2 \right). \end{aligned} \tag{3.9}$$

Now, by (2.19) and Lemma 2.1, for any $x \in I$, $I \subseteq (-2, 2)$ closed and any $-1 \leq t \leq 1$,

$$\left|p_n^{(\ell)}\left(x+\frac{it}{n}\right)\right|^2 + \left|p_{n+1}^{(\ell)}\left(x+\frac{it}{n}\right)\right|^2 \leq M_I^2\left(\left|A_{n,1}^{(\ell)}\left(x+\frac{it}{n}\right)\right|^2 + \left|A_{n,2}^{(\ell)}\left(x+\frac{it}{n}\right)\right|^2\right). \tag{3.10}$$

Moreover, by (2.15), for any $n \geq N_\ell$,

$$(|A_{n,1}^{(\ell)}(x)|^2 + |A_{n,2}^{(\ell)}(x)|^2) \leq (1+|v_{\ell+1}|M_I)^2(|A_{n,1}^{(\ell+1)}(x)|^2 + |A_{n,2}^{(\ell+1)}(x)|^2). \tag{3.11}$$

Combining (3.2), (3.3), (3.10) and (3.11) with (3.9) we get (recall $M_{I_{m_{\ell+1}}} \geq 1$)

$$\begin{aligned}
&\left|\left(p_n^{(\ell+1)}\left(x+\frac{a}{n}\right)p_{n-1}^{(\ell+1)}\left(x+\frac{b}{n}\right) - p_n^{(\ell+1)}\left(x+\frac{b}{n}\right)p_{n-1}^{(\ell+1)}\left(x+\frac{a}{n}\right)\right)\right.\\
&\quad \left. - \left(p_n^{(\ell)}\left(x+\frac{a}{n}\right)p_{n-1}^{(\ell)}\left(x+\frac{b}{n}\right) - p_n^{(\ell)}\left(x+\frac{b}{n}\right)p_{n-1}^{(\ell)}\left(x+\frac{a}{n}\right)\right)\right|\\
&\quad \leq (8|v_{\ell+1}|^2 M_{I_{m_{\ell+1}}}^4 + 24|v_{\ell+1}|M_{I_{m_{\ell+1}}}^4)(1+|v_{\ell+1}|M_{I_{m_{\ell+1}}})^2\\
&\qquad \times (|A_{n,1}^{(\ell+1)}(x)|^2 + |A_{n,2}^{(\ell+1)}(x)|^2)\\
&\quad \leq D|v_{\ell+1}|M_{I_{m_{\ell+1}}}^4(|A_{n,1}^{(\ell+1)}(x)|^2 + |A_{n,2}^{(\ell+1)}(x)|^2)
\end{aligned} \tag{3.12}$$

where D is some constant which is independent of n, ℓ and x.

On the other hand, it is easy to see that

$$|\kappa_n^{(\ell+1)}(x)| \geq \left(1-\frac{|x|}{2}\right)(|A_{n,1}^{(\ell+1)}(x)|^2 + |A_{n,2}^{(\ell+1)}|^2). \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\left|\frac{K_n^{(\ell+1)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right) - K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n\kappa_n^{(\ell+1)}(x)}\right| \leq \frac{2D|v_{\ell+1}|M_{I_{m_{\ell+1}}}^4}{(2-|x|)|b-a|} \tag{3.14}$$

which implies, by the choice of m_n, that

$$\max_{N_{\ell+1}\leq n<N_{\ell+2},\ x\in\tilde{I}_\ell}\left|\frac{K_n^{(\ell+1)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right) - K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n\kappa_n^{(\ell+1)}(x)}\right| \to 0 \tag{3.15}$$

as $\ell \to \infty$, uniformly for a, b with $|a|, |b| \leq C$ and $\Im(a), \Im(b) \leq 1$ and satisfying $|b-a| > \delta$.

As for the second term on the right hand side of (3.6), we write

$$\begin{aligned}
&((A_{n,1}^{(\ell+1)}(x))^2 + (A_{n,2}^{(\ell+1)}(x))^2 - A_{n,1}^{(\ell+1)}(x)A_{n,2}^{(\ell+1)}x)\\
&\quad - ((A_{n,1}^{(\ell)}(x))^2 + (A_{n,2}^{(\ell)}(x))^2 - A_{n,1}^{(\ell)}(x)A_{n,2}^{(\ell)}x)\\
&= (A_{n,1}^{(\ell+1)}(x) - A_{n,1}^{(\ell)}(x))(A_{n,1}^{(\ell+1)}(x) + A_{n,1}^{(\ell)}(x))\\
&\quad + (A_{n,2}^{(\ell+1)}(x) - A_{n,2}^{(\ell)}(x))(A_{n,2}^{(\ell+1)}(x) + A_{n,2}^{(\ell)}(x))\\
&\quad - xA_{n,1}^{(\ell+1)}(x)(A_{n,2}^{(\ell+1)}(x) - A_{n,2}^{(\ell)}(x)) - xA_{n,2}^{(\ell)}(x)(A_{n,1}^{(\ell+1)}(x) - A_{n,1}^{(\ell)}(x)),
\end{aligned} \tag{3.16}$$

which implies, by (2.13) and (2.15),

$$
\begin{aligned}
|\kappa_n^{(\ell+1)}(x) - \kappa_n^{(\ell)}(x)| &= |((A_{n,1}^{(\ell+1)}(x))^2 + (A_{n,2}^{(\ell+1)}(x))^2 - A_{n,1}^{(\ell+1)}(x)A_{n,2}^{(\ell+1)}x) \\
&\quad - ((A_{n,1}^{(\ell)}(x))^2 + (A_{n,2}^{(\ell)}(x))^2 - A_{n,1}^{(\ell)}(x)A_{n,2}^{(\ell)}x)| \\
&\le |v_{\ell+1}|M_{I_{m_{\ell+1}}}^2 (|A_{n,1}^{(\ell+1)}(x)|^2 + |A_{n,1}^{(\ell+1)}(x)A_{n,1}^{(\ell)}(x)| \\
&\quad + |A_{n,2}^{(\ell+1)}(x)|^2 + |A_{n,2}^{(\ell+1)}A_{n,2}^{(\ell)}(x)| + |x|\, |A_{n,1}^{(\ell+1)}(x)A_{n,2}^{(\ell+1)}(x)| \\
&\quad + |x|\, |A_{n,1}^{(\ell)}(x)A_{n,2}^{(\ell)}(x)|),
\end{aligned}
$$

so that

$$
\begin{aligned}
|\kappa_n^{(\ell+1)}(x) - \kappa_n^{(\ell)}(x)| &\le |v_{\ell+1}|M_{I_{m_{\ell+1}}}^2 (2+|x|)(|A_{n,1}^{(\ell+1)}(x)|^2 + |A_{n,1}^{(\ell+2)}(x)|^2 \\
&\quad + |A_{n,1}^{(\ell)}(x)|^2 + |A_{n,1}^{(\ell)}(x)|^2) \\
&\le |v_{\ell+1}|M_{I_{m_{\ell+1}}}^2 (4 + (1+|v_{\ell+1}|M_{I_{m_{\ell+1}}})^2) \\
&\quad \times (|A_{n,1}^{(\ell+1)}(x)|^2 + |A_{n,1}^{(\ell+2)}(x)|^2),
\end{aligned} \tag{3.17}
$$

by (3.11) (recall $|x| \le 2$).

Using (3.13) again, we deduce that

$$
\left| \frac{\kappa_n^{(\ell)}(x) - \kappa_n^{(\ell+1)}(x)}{\kappa_n^{(\ell+1)}(x)} \right| \le \frac{\widetilde{D}|v_{\ell+1}|M_{I_{m_{\ell+1}}}^2}{(2-|x|)}, \tag{3.18}
$$

where $\widetilde{D}$ is some constant that is independent of x and n. Since

$$
\left| \frac{K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n\kappa_n^{(\ell)}(x)} \right| \le \frac{\widetilde{C}}{(2-|x|)}
$$

for any $n \ge N_{\ell+1}$, with $\widetilde{C}$ some universal constant (recall (3.1)), we see that

$$
\begin{aligned}
&\max_{N_{\ell+1} \le n < N_{\ell+2},\ x \in \tilde{I}_\ell} \left| \frac{K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n\kappa_n^{(\ell)}(x)} \right| \left| \frac{\kappa_n^{(\ell)}(x) - \kappa_n^{(\ell+1)}(x)}{\kappa_n^{(\ell+1)}(x)} \right| \\
&\quad \le \frac{\widetilde{C}\widetilde{D}|v_{\ell+1}|M_{I_{m_{\ell+1}}}^2}{(2-|x|)^2}.
\end{aligned} \tag{3.19}
$$

This implies, by the choice of m_n, that

$$
\max_{N_{\ell+1} \le n < N_{\ell+2},\ x \in \tilde{I}_\ell} \left| \frac{K_n^{(\ell)}\left(x+\frac{a}{n}, x+\frac{b}{n}\right)}{n\kappa_n^{(\ell)}(x)} \right| \left| \frac{\kappa_n^{(\ell)}(x) - \kappa_n^{(\ell+1)}(x)}{\kappa_n^{(\ell+1)}(x)} \right| \to 0 \tag{3.20}
$$

as $\ell \to \infty$.

Combining (3.6), (3.15) and (3.20) we obtain (3.5), for $a \ne b$, uniformly for $x \in$ compact subsets of $(-2, 2)$, $|a|, |b| \le C$ with $\Im(a), \Im(b) \le 1$, and $|a-b| > \delta$. This, in turn, implies (3.4) under the same conditions.

Now, having obtained the limit for $a \ne b$ we note, as before, that, for fixed a, the limiting function, as well as the members of the sequence, is analytic in a strip and has an analytic extension to $b = a$. By integrating along a closed path around a (recall the convergence is uniform for $|b-a|>\delta$) we see there is convergence at $b=a$ in the interior of the strip as well. Taking $a=b=0$ in (3.4) we obtain that

$$\lim_{n\to\infty}\frac{K_n(x,x)}{n(A_{n,1}(x)^2+A_{n,2}(x)^2-A_{n,1}(x)A_{n,2}(x)x)}=\frac{2}{4-x^2}$$

which implies immediately

$$\lim_{n\to\infty}\frac{K_n\left(x+\frac{a}{n},x+\frac{b}{n}\right)}{K_n(x,x)}=\frac{\sin((\sqrt{4-x^2})^{-1}(b-a))}{\sqrt{4-x^2}^{-1}(b-a)}$$

for any a,b in the interior of the strip of width 1 around $\mathbb{R}$. In particular, this holds for $a,b\in\mathbb{R}$. This concludes the proof of the theorem. □

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