Linear Programming with Matrix Variables

B. D. Craven

Mathematics Department
University of Melbourne
Parkville, Victoria 3052, Australia

and

B. Mond

Department of Pure Mathematics
La Trobe University
Bundoora, Victoria 3083, Australia

Submitted by Hans Schneider

ABSTRACT

Linear programming is formulated with the vector variable replaced by a matrix variable, with the inner product defined using trace of a matrix. The theorems of Motzkin, Farkas (both homogeneous and inhomogeneous forms), and linear-programming duality thus extend to matrix variables. Duality theorems for linear programming over complex spaces, and over quaternion spaces, follow as special cases.

1. INTRODUCTION

Denote by $\mathbb{R}^{m\times r}$ the vector space of all real $m\times r$ matrices, equipped with the usual inner product: $\langle X,Y\rangle = \mathrm{Tr}(X^TY)$, where Tr denotes the *trace* of a square matrix. Various theorems of mathematical programming, including the Farkas and Motzkin theorems and linear-programming duality, extend to problems where the space $\mathbb{R}^n$ of variables is replaced by $\mathbb{R}^{m\times r}$. Results for programming over complex, or quaternion, spaces follow as special cases.

Any linear functional L on $\mathbf{Y}=\mathbb{R}^{m\times r}$ can be represented by some $H\in\mathbf{Y}$, by

$$(\forall Y\in\mathbf{Y})\quad L(Y)=\sum_{i=1}^{r}\sum_{j=1}^{m}H_{ij}Y_{ij}=\mathrm{Tr}(H^TY). \tag{1}$$

52 Vanderbilt Ave., New York, NY 10017 0024-3795/81/040073+8$02.50

The isomorphism $\theta:\mathbb{R}^{m\times r}\to\mathbb{R}^{mr}$ defined by $\theta(Y)=(Y_{11},\dots,Y_{1r},Y_{21},\dots,Y_{mr})$ preserves the trace; thus $\mathrm{Tr}(H^TY)=(\theta H)^T(\theta Y)$. Correspondingly [10], the inner product $\langle\theta(H),\theta(Y)\rangle=\mathrm{Tr}(H^TY)$ and $(AXB)=(A\otimes B^T)\theta(X)$, where $\otimes$ denotes Kronecker product. Let $\mathbf{S}$ be a closed convex cone in $Y=\mathbb{R}^{m\times r}$. An inequality $AX-K\in\mathbf{S}$, where $X\in\mathbb{R}^{n\times r}$, $A\in\mathbb{R}^{m\times n}$, $K\in\mathbb{R}^{m\times r}$, can be expressed as $A^{\#}\theta(X)-\theta(K)\in\theta(S)$, where $A^{\#}\in\mathbb{R}^{mr\times nr}$ has components $A^{\#}_{ij,ut}=\delta_{uj}A_{it}$, where δ_{uj} is the Kronecker delta, the indices i,j label rows of $A^{\#}$, and t,u label columns of $A^{\#}$. The (positive) *dual cone* of $\mathbf{S}$ is the set $\mathbf{S}^*$ of all linear functionals P on $\mathbf{Y}$ such that $P(Y)\geq 0$ for each $Y\in\mathbf{S}$. It follows from (1) that

$$\mathbf{S}^*=\left\{P\in\mathbf{Y}:(\forall Y\in\mathbf{S})\ \mathrm{Tr}(P^TY)\geq 0\right\}. \tag{2}$$

In what follows, I_r (or I) denotes the unit matrix in $\mathbb{R}^{r\times r}$; $\mathbb{R}_+=[0,\infty)$; symbols $\mathbf{X},\mathbf{S},\dots$ denote sets (vector spaces or cones), whereas $X,S,\dots$ denote matrices. Let $\mathbf{U}=\{\lambda I_r:\lambda\in\mathbb{R}_+\}$; then $\mathbf{U}^*=\{Z\in\mathbb{R}^{r\times r}:\mathrm{Tr}(Z)\geq 0\}$. Also let $\mathbf{U}_0=\{\lambda I_r:\lambda\in\mathbb{R}\}$ and $\mathbf{V}_0=\{\lambda I_n:\lambda\in\mathbb{R}\}$. The orthogonal complement of a subspace X is denoted by $X^{\perp}$. Let $\mathbf{V}=\{\lambda I_s:\lambda\in\mathbb{R}_+\}$.

The *Motzkin alternative theorem* for cone inequalities in finite-dimensional spaces (see [4, Lemma 2] or [5, Theorem 2.5.2]) translates by isomorphism θ to the following theorem for matrix variables.

THEOREM 1. *Let $A\in\mathbb{R}^{m\times n}$, $B\in\mathbb{R}^{q\times n}$, $C\in\mathbb{R}^{r\times s}$. Let $\mathbf{S}\subset\mathbb{R}^{m\times r}$ be a convex cone with interior; let $\mathbf{T}\subset\mathbb{R}^{q\times r}$ and $H\subset\mathbb{R}^{n\times s}$ be closed convex cones; let the cones $B^T(\mathbf{T}^*)$ and $\mathbf{H}^*C^T\equiv\{WC^T:W\in\mathbf{H}^*\}$ be closed. Then exactly one of the following systems has a solution (X or (P,Q,R), respectively):*

$$(\exists X)\quad AX\in\mathrm{int}\,\mathbf{S},\ BX\in\mathbf{T},\ XC\in\mathbf{H}; \tag{3}$$

$$(\exists Q\in\mathbf{T}^*,R\in\mathbf{H}^*,0\neq P\in\mathbf{S}^*)\quad P^TA+Q^TB+CR^T=0. \tag{4}$$

A direct corollary is the following matrix version of Farkas's theorem.

THEOREM 2. *Let $\mathbf{X}$ be a subspace of $\mathbb{R}^{n\times r}$; let $A\in\mathbb{R}^{m\times n}$, $M\in\mathbb{R}^{r\times s}$, $B_0\in\mathbb{R}^{n\times r}$; let $\mathbf{K}\subset\mathbb{R}^{m\times r}$ and $\mathbf{L}\subset\mathbb{R}^{n\times s}$ be convex cones, such that the cones $A^T(\mathbf{K}^*)+\mathbf{X}^{\perp}$ and $\mathbf{L}^*M^T+\mathbf{X}^{\perp}$ are closed in $\mathbb{R}^{n\times r}$. Then*

$$\left[(AX\in\mathbf{K},XM\in\mathbf{L},X\in\mathbf{X})\ \Rightarrow\ \mathrm{Tr}\left(B_0^TX\right)\geq 0\right]$$

$$\Leftrightarrow\quad\left[(\exists C\in\mathbf{K}^*,D\in\mathbf{L}^*,N\in\mathbf{X}^{\perp})\ (B_0+N)^T=C^TA+MD^T\right]. \tag{5}$$

The closed-cone hypothesis is fulfilled if $A^T(\mathbf{K}^*)$ and $\mathbf{L}^*M^T$ are closed in $\mathbf{X}$.

The cones in Theorems 1 and 2 need *not* be polyhedral. A more restrictive sufficient hypothesis [12] is that $AX_0 \in \operatorname{int}\mathbf{K}$ and $X_0 M \in \operatorname{int} L$ for some $X_0 \in \mathbf{X}$. Ben-Israel has given another matrix Farkas theorem [2]:

$$[AXB=C, \quad X\geqslant 0] \quad \Leftrightarrow \quad [A^T U B^T \geqslant 0, \quad \operatorname{Tr}(U^T C)\geqslant 0], \tag{6}$$

in which the cone considered is, however, a nonnegative orthant, and the subspace $\mathbf{X}$ is the whole space. The system (5) (with $N=0$) is obtainable from (6) by the replacements

$$A^T \to \begin{bmatrix} A \\ I_n \end{bmatrix}, \qquad B^T \to [I_r, M], \qquad C \to B. \tag{7}$$

Another version of Farkas's theorem, involving the trace inner product, was given in [6], Theorem 11.

The study of minimization problems with matrix variables was suggested to the authors by the matrix quadratic programs in [8] and [11].

2. NONHOMOGENEOUS FARKAS THEOREM FOR MATRIX SPACES

THEOREM 3. *Let* $A\in\mathbb{R}^{m\times n}$, $D\in\mathbb{R}^{m\times r}$, $B\in\mathbb{R}^{n\times r}$, $c\in\mathbb{R}$. *Let* $\mathbf{K}\subset\mathbb{R}^{m\times r}$ *be a closed polyhedral cone. Assume that* $AX_0 - D\in\mathbf{K}$ *for some* $X_0\in\mathbb{R}^{n\times r}$. *Then*

$$\begin{aligned} &[AX-D\in\mathbf{K} \quad \Rightarrow \quad \operatorname{Tr}(B^T X)\geqslant c \quad (X\in\mathbb{R}^{n\times r})] \\ &\qquad \Leftrightarrow \quad [(\exists C\in\mathbf{K}^*)\ B^T = C^T A, \quad \operatorname{Tr}(D^T C)\geqslant c]. \end{aligned} \tag{8}$$

Proof. The system

$$\begin{bmatrix} A & -D \\ 0 & I_r \end{bmatrix}\begin{bmatrix} Y \\ \Lambda \end{bmatrix} \in \begin{bmatrix} \mathbf{K} \\ \mathbf{U} \end{bmatrix} \quad \Rightarrow \quad \operatorname{Tr}\left\{[B^T, -cr^{-1}I_r]\begin{bmatrix} Y \\ \Lambda \end{bmatrix}\right\}\geqslant 0 \tag{9}$$

is equivalent, by Theorem 2, to the system

$$(\exists C\in\mathbf{K}^*, N\in\mathbf{U}^*)[B^T, -cr^{-1}I_r] = [C^T, N^T]\begin{bmatrix} A & -D \\ 0 & I_r \end{bmatrix}. \tag{10}$$

Substituting for U^* shows that (10) is equivalent to $B^T = C^T A$, $D^T C - cr^{-1} I_r \in \mathbf{U}^*$, for some $C \in \mathbf{K}^*$. Hence (9) is equivalent to the right side of (8).

Suppose that the right side of (8) does not hold. Then (9) does not hold; so there exist Y' and Λ' satisfying $AY' - D\Lambda' \in \mathbf{K}$, $\Lambda' \in \mathbf{U}$, $\mathrm{Tr}(B^T Y' - cr^{-1}\Lambda') < 0$, and $\Lambda' = \beta I_r$ for some $\beta \geq 0$. If $\beta > 0$, then $X = \beta^{-1} Y'$ satisfies $AX - D \in \mathbf{K}$, $\mathrm{Tr}(B^T X) < c$. If $\beta = 0$, then $AY' \in \mathbf{K}$, $\mathrm{Tr}(B^T Y') < 0$. For $\alpha > 0$,

$$A(X_0 + \alpha Y') - D = (AX_0 - D) + \alpha AY' \in \mathbf{K} + \alpha \mathbf{K} \subset \mathbf{K}; \tag{11}$$

but

$$\mathrm{Tr}\{B^T(X_0 + \alpha Y')\} = \mathrm{Tr}(B^T X_0) + \alpha \mathrm{Tr}(B^T Y') < c$$

for sufficiently large $\alpha > 0$, since $\mathrm{Tr}(B^T Y') < 0$. In either case, the left side of (8) is contradicted. Hence the left side of (8) implies the right side.

Conversely, let the right side of (8) hold. If $AX - D \in K$, then

$$\mathrm{Tr}(B^T X) = \mathrm{Tr}(C^T AX) \geq \mathrm{Tr}(C^T D) = \mathrm{Tr}(D^T C) \geq c;$$

hence the left side of (8) holds. ■

Remark. If K is a closed convex cone, not polyhedral, then the result of Theorem 3 remains valid, provided that the convex cone

$$\begin{bmatrix} A^T & 0 \\ -D^T & I_r \end{bmatrix} \begin{bmatrix} \mathbf{K}^* \\ \mathbf{U}^* \end{bmatrix} \tag{12}$$

is assumed closed. Bellman and Fan [1] have given a nonhomogeneous Farkas theorem for positive semidefinite Hermitian matrices only.

4. LINEAR PROGRAMMING FOR MATRIX VARIABLES

Let $\mathbf{X}$ be a subspace of $\mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{n\times r}$, $A \in \mathbb{R}^{m\times r}$, $J \in \mathbb{R}^{n\times s}$, $B \in \mathbb{R}^{m\times r}$, $M \in \mathbb{R}^{r\times s}$. Let $\mathbf{K} \subset \mathbb{R}^{m\times r}$ and $\mathbf{L} \subset \mathbb{R}^{n\times s}$ be closed convex cones, not necessarily polyhedral. Consider the pair of programming problems:

$$\underset{X \in \mathbf{X}}{\text{Minimize}}\ \mathrm{Tr}(C^T X) \quad \text{subject to} \quad AX - B \in \mathbf{K}, \quad XM - J \in \mathbf{L}; \tag{13}$$

$$\underset{Z, W}{\text{Maximize}}\ \mathrm{Tr}(B^T Z) + \mathrm{Tr}(JW^T)$$

$$\text{subject to} \quad Z \in \mathbf{K}^*, \quad W \in \mathbf{L}^*, \quad C^T = Z^T A + MW^T. \tag{14}$$

The problems (13) and (14) have linear objective functions, and linear constraints with convex cones. The unconventional restriction of X to a subspace has later application. Linear programs with matrix variables are applicable when a conventional linear program has nr variables, which may relevantly be arranged as a $n\times r$ matrix. For example, in a transportation problem, the constraints are $e_n^TX=B$, $Xe_r=J$, $X\geqslant 0$, where e_r, e_n are columns of ones, and (13) extends this to more general weighting matrices. The following theorem proves duality for (13) and (14).

THEOREM 4. *Let* (13) *reach a minimum at* $X=\bar{X}$; *let* $Q=A\bar{X}-B$ *and* $R=\bar{X}M-J$; *assume that the cones*

$$\begin{bmatrix} A & Q \\ 0 & I_r \end{bmatrix}^T \begin{bmatrix} \mathbf{K}^* \\ \mathbf{U}^* \end{bmatrix} \quad \text{and} \quad \left\{ \begin{bmatrix} G \\ H \end{bmatrix} \begin{bmatrix} M & 0 \\ R & I_s \end{bmatrix}^T : \begin{bmatrix} G \\ H \end{bmatrix} \in \begin{bmatrix} \mathbf{L}^* \\ \mathbf{V}^* \end{bmatrix} \right\} \tag{15}$$

are closed (*in particular,* K *and* L *may be polyhedral*). *Then* (14) *reaches a maximum at some* $Z=\bar{Z}$, $W=\bar{W}$, *and*

$$\operatorname{Tr}(C^T\bar{X})=\operatorname{Tr}(B^T\bar{Z})+\operatorname{Tr}(J\bar{W}^T);$$

$$C^T-\bar{Z}^TA+M\bar{W}^T\in\mathbf{X}^\perp, \qquad \operatorname{Tr}\left(\bar{Z}^T(A\bar{X}-B)\right)=0,$$

$$\operatorname{Tr}\left((\bar{X}M-J)\bar{W}^T\right)=0. \tag{16}$$

Also $\operatorname{Tr}(C^TX)\geqslant\operatorname{Tr}(B^TZ)+\operatorname{Tr}(JW^T)$ *whenever* X *is feasible for* (13), *and* Z,W *for* (14).

Proof. Let X and Z,W satisfy the constraints of (13) and (14), respectively. Then

$$\operatorname{Tr}(C^TX)-\left[\operatorname{Tr}(B^TZ)+\operatorname{Tr}(JW^T)\right]=\operatorname{Tr}\left[Z^T(AX-B)\right]$$
$$+\operatorname{Tr}\left[(XM-J)W^T\right]\geqslant 0. \tag{17}$$

Now let $Y\in X$, $\Lambda=\lambda I_r\in\mathbf{U}_0$, $\Xi=\xi I_n\in\mathbf{V}_0$ satisfy

$$\begin{bmatrix} A & Q \\ 0 & I_r \end{bmatrix}\begin{bmatrix} Y & \Xi \\ \Lambda & 0 \end{bmatrix}\in\begin{bmatrix} \mathbf{K} & \mathbb{R}^{m\times n} \\ \mathbf{U}_0 & \mathbb{R}^{r\times n} \end{bmatrix},$$

$$\begin{bmatrix} Y & \Xi \\ \Lambda & 0 \end{bmatrix}\begin{bmatrix} M & 0 \\ R & I_n \end{bmatrix}\in\begin{bmatrix} \mathbf{L} & \mathbf{V}_0 \\ \mathbb{R}^{r\times s} & \mathbb{R}^{r\times n} \end{bmatrix}. \tag{18}$$

If $Y=0$, then $\mathrm{Tr}(C^TY)\geqslant 0$. If $Y\neq 0$, then for some $K\in\mathbf{K}$ and $L\in\mathbf{L}$,

$$A(\bar{X}+\alpha Y)-B=(1-\alpha\lambda)Q+\alpha K\in\mathbf{K},$$

$$(\bar{X}+\alpha Y)M-J=(1-\alpha\xi)R+\alpha L\in\mathbf{L}$$

whenever $\alpha\in\mathbb{R}_+$ is sufficiently small. Since $\bar{X}$ minimizes (13), $\mathrm{Tr}[C^T(\bar{X}+Y)]\geqslant\mathrm{Tr}(C^T\bar{X})$. So, in either case, (18) implies that

$$\mathrm{Tr}(C^TY)=\mathrm{Tr}\left\{\begin{bmatrix} C^T & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} Y & \Xi\\ \Lambda & 0\end{bmatrix}\right\}\geqslant 0.$$

From Theorem 2, given that the cones (15) are closed, there follows

$$\begin{bmatrix} C^T & 0\\ 0 & 0\end{bmatrix}+P=\begin{bmatrix} \bar{Z}^T & N\\ 0 & 0\end{bmatrix}\begin{bmatrix} A & Q\\ 0 & I_r\end{bmatrix}+\begin{bmatrix} M & 0\\ R & I_n\end{bmatrix}\begin{bmatrix} \bar{W}^T & 0\\ S^T & 0\end{bmatrix} \tag{20}$$

for some matrices

$$\begin{bmatrix} \bar{Z}^T & N\\ 0 & 0\end{bmatrix}\in\begin{bmatrix} \mathbf{K} & \mathbb{R}^{m\times n}\\ \mathbf{U}_0 & \mathbb{R}^{r\times n}\end{bmatrix}^*=\begin{bmatrix} \mathbf{K}^* & \mathbf{U}_0^*\\ \{0\} & \{0\}\end{bmatrix}$$

and

$$\begin{bmatrix} \bar{W}^T & 0\\ S^T & 0\end{bmatrix}\in\begin{bmatrix} \mathbf{L} & \mathbf{V}_0^*\\ \mathbb{R}^{r\times s} & \mathbb{R}^{r\times n}\end{bmatrix}=\begin{bmatrix} \mathbf{L}^* & \{0\}\\ \mathbf{V}_0^* & \{0\}\end{bmatrix};$$

$$P\in\begin{bmatrix} \mathbf{X} & \mathbf{V}_0\\ \mathbf{U}_0 & \{0\}\end{bmatrix}^\perp.$$

Since $N\in\mathbf{U}^*\cap(-\mathbf{U})^*$, $\mathrm{Tr}(N)=0$; similarly $\mathrm{Tr}(S)=0$. Hence (20) gives $C^T+E=\bar{Z}^TA+M\bar{W}^T$, $E\in\mathbf{X}^\perp$, together with $\mathrm{Tr}(\bar{Z}^TQ)=0=\mathrm{Tr}(R\bar{W}^T)$. Therefore

$$\begin{aligned}\mathrm{Tr}(C^T\bar{X})-\left[\mathrm{Tr}(B^T\bar{Z})+\mathrm{Tr}(J\bar{W}^T)\right]&=\mathrm{Tr}(\bar{Z}^TA\bar{X})+\mathrm{Tr}(M\bar{W}^T\bar{X})\\ &\quad-\mathrm{Tr}(B^T\bar{Z})-\mathrm{Tr}(J\bar{W}^T)\\ &=\mathrm{Tr}\left[\bar{Z}^T(A\bar{X}-B)\right]+\mathrm{Tr}\left[(\bar{X}M-J)\bar{W}^T\right]=0,\end{aligned} \tag{21}$$

using $\mathrm{Tr}(M\overline{W}^T\overline{X})=\mathrm{Tr}(\overline{X}M\overline{W}^T)$. This, with (17), proves that $(\overline{Z},\overline{W})$ maximizes (14). ∎

5. COMPLEX AND QUATERNION LINEAR PROGRAMMING

A vector in complex space $\mathbb{C}^n$ can be represented isomorphically by a column of n real submatrices

$$\begin{bmatrix} x_j & y_j \\ -y_j & x_j \end{bmatrix} \qquad (j=1,2,\ldots,n;\quad x_j, y_j \in \mathbb{R}), \tag{22}$$

and thus by a $2n\times 2$ real matrix. Theorem 5 may be applied, taking $\mathbf{X}$ as the subspace of such matrices. The complex inner product $\langle z, w\rangle = \mathrm{Re}(\bar{z}^T w)$ differs only by a constant factor from $\mathrm{Tr}(Z^T W)$, where $Z, W \in \mathbb{R}^{2n\times 2}$ represent $z, w \in \mathbb{C}^n$. Programming in quaternion spaces may be handled similarly, representing each quaternion $q = a\mathbf{I}+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$ by a real 4×4 matrix

$$\begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & b \\ -d & -c & -b & a \end{bmatrix}. \tag{23}$$

If quaternion vectors q, r are thus represented by real matrices Q, R, then the inner product $\frac{1}{4}\mathrm{Tr}(Q^T R)$ represents the scalar part of $\bar{q}r$, where $\bar{q}$ denotes the conjugate of q (thus $a\mathbf{I}-b\mathbf{i}-c\mathbf{j}-d\mathbf{k}$). (The scalar part of q is a.)

Denote by $\langle\cdot,\cdot\rangle$ the inner product, for either complex or quaternion n-space. Denote by $\mathbf{W}$ either $\mathbb{C}$, or quaternion space; let $M:\mathbf{W}^n \to \mathbf{W}^n$ be a linear map; let $b, c \in \mathbf{W}^n$; let $\mathbf{S} \subset \mathbf{W}^m$ be a closed convex cone; denote by $\mathbf{H}$ the orthogonal complement of $\mathbb{R}$ in $\mathbf{W}$. Consider the pair of linear programs (considered as either in complex or in quaternion space)

$$\underset{x\in\mathbf{W}^n}{\mathrm{Minimize}}\,\langle c, x\rangle \qquad \text{subject to} \quad Mx-b\in\mathbf{S}; \tag{24}$$

$$\underset{z\in\mathbf{W}^m}{\mathrm{Maximize}}\,\langle b, z\rangle \qquad \text{subject to} \quad z\in\mathbf{S}^*, \quad M^Tz=c. \tag{25}$$

THEOREM 5. *If x and z satisfy the constraints of* (24) *and* (25), *respectively, then $\langle c, x\rangle \geqslant \langle b, z\rangle$. If* (24) *reaches a minimum at $x=\bar{x}$, and the* *convex cone*

$$\begin{bmatrix} M & M\bar{x}-b \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \mathbf{S}^* \\ \mathbf{H} \end{bmatrix} \tag{26}$$

is closed, then (25) *reaches a maximum at some* $\bar{z}$, *where* $\langle c, \bar{x} \rangle = \langle b, \bar{z} \rangle$.

Proof. Apply Theorem 4. ∎

For linear programming in complex spaces, with vector variables, see [9] and [7].

The authors thank two referees for several references and amendments.

REFERENCES

1 R. Bellman and Ky Fan, On systems of linear inequalities in Hermitian matrix variables, in *Proceedings of the Symposium on Pure Mathematics 7* (V. Klee, Ed.), Amer. Math. Soc., 1963, pp. 1–11.
2 A. Ben-Israel, On decompositions of matrix spaces with applications to matrix equations, *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I* 44 (2):122–127 (1968).
3 A. Berman and A. Ben-Israel, More on linear inequalities with applications to matrix theory, *J. Math. Anal. Appl* 33:482–495 (1971).
4 B. D. Craven, Lagrangean conditions and quasiduality, *Bull. Austral. Math. Soc.* 16:325–339 (1977).
5 B. D. Craven, *Mathematical Programming and Control Theory*, Chapman & Hall, London, 1978.
6 B. D. Craven and J. J. Koliha, Generalizations of Farkas's theorem, *SIAM J. Math. Anal.* 8:983–997 (1977).
7 B. D. Craven and B. Mond, On duality in complex linear programming, *J. Austral. Math. Soc.* 16:172–175 (1973).
8 D. G. Kabe, A note on a quadratic programming problem, *J. Indust. Math. Soc.* 23:61–66 (1973).
9 N. Levinson, Linear programming in complex space, *J. Math. Anal. Appl.* 14:44–62 (1966).
10 M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Prindle, Weber, & Schmidt, Boston, 1964.
11 P. Scobey and D. G. Kabe, Vector quadratic programming problems and inequality constrained least squares estimation, *J. Indust. Math. Soc.* 28:37–49 (1978).
12. V. A. Sposito and H. T. David, A note on Farkas' theorem over cone domains, *SIAM J. Appl. Math.* 22:356–358 (1972).

Received 27 July 1979; revised 26 July 1980