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Partial metric monoids and semivaluation spaces

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Abstract

Stable partial metric spaces form a fundamental concept in Quantitative Domain Theory. Indeed, *all* domains have been shown to be quantifiable via a stable partial metric.

Monoid operations arise naturally in a quantitative context and hence play a crucial role in several applications. Here, we show that the structure of a stable partial metric monoid provides a suitable framework for a unified approach to some interesting examples of monoids that appear in Theoretical Computer Science. We also introduce the notion of a semivaluation monoid and show that there is a bijection between stable partial metric monoids and semivaluation monoids.

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1. Introduction

Throughout this paper the letters $\mathbb{R}$, $\mathbb{R}^+$ and ω will denote the set of real numbers, of nonnegative real numbers and of nonnegative integer numbers, respectively.

Matthews introduced in [14] the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks, and obtained, among other results, a nice relationship between partial metric spaces and the so-called weightable quasi-metric spaces. These structures have been applied to obtain an extensional treatment of lazy data flow deadlock in [15].

Let us recall that a *partial metric* on a (nonempty) set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(ii) $p(x,x) \leqslant p(x,y)$;
(iii) $p(x,y) = p(y,x)$;
(iv) $p(x,z) \leqslant p(x,y) + p(y,z) - p(y,y)$.

A *partial metric space* is a pair (X,p) such that X is a (nonempty) set and p is a partial metric on X.

Each partial metric p on X generates a T_0-topology $\mathcal{T}(p)$ on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon)\colon x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X\colon p(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Note that contrarily to the metric case, some open p-ball may be empty [14, p. 187].

The following is a simple but useful example of a partial metric space.

For each pair $x, y \in \mathbb{R}^+$ let $p(x,y) = x \vee y$. Then p is a partial metric on $\mathbb{R}^+$ and thus $(\mathbb{R}^+, p)$ is a partial metric space.

If (X,p) is a partial metric space, then $(X, \sqsubseteq_p)$ is clearly an ordered set, where $x \sqsubseteq_p y \Longleftrightarrow p(x,x) = p(x,y)$.

In the sequel $\sqsubseteq_p$ will be called the *associated* (*partial*) *order* of p.

In [23] it is shown that *all* domains are quantifiable via a partial metric induced by a suitable semivaluation.

We focus in the following on three well-known Computer Science examples of monoids for which the notion of a (stable) partial metric monoid will provide a unifying concept.

The interval domain (or the partial real line) forms a model for a programming language for higher-order exact real number computation [5]. It consists of the set $I(\mathbb{R})$ of all nonempty closed and bounded real intervals ordered by reverse inclusion, together with an artificial least element $\perp$. In [14] (see also [8,16]) a partial metric p is defined on $I(\mathbb{R})$ such that its associated order coincides with the reverse inclusion order and thus $(I(\mathbb{R}), \sqsubseteq_p)$ is a meet semilattice as it is observed in [22]. We shall denote by $I([0,1])$ the set of all nonempty closed and bounded intervals contained in $[0,1]$. It was proved in [6] that $I([0,1])$ can be equipped with a suitable structure of monoid for which $[0,1]$ is the neutral element. For simplicity and without essential loss of generality, we shall refer in the sequel to $I([0,1])$ as the interval domain.

If Σ^∞ denotes the set of all finite and infinite "words" over a nonempty alphabet Σ, then $(\Sigma^\infty, \sqsubseteq)$ is a meet semilattice where $\sqsubseteq$ is the prefix order on Σ^∞. Furthermore there is a partial metric p_Σ on Σ^∞ whose associated order coincides with $\sqsubseteq$ [25,12,14]. The domain of words $(\Sigma^\infty, p_\Sigma)$ appears in a natural way by modeling the streams of information in Kahn's model of parallel computation [10,14].

On the other hand, Schellekens introduced in [20] the complexity (quasi-metric) space to develop a topological foundation for the complexity analysis of algorithms. Via the study of its dual several quasi-metric properties of the complexity space, including Smyth completeness and total boundedness, were discussed in [18]. The dual complexity space is a weightable quasi-metric space that is a meet semilattice for its associated order [22].

In Section 3, we show that, indeed, both $(I[0,1], p)$, $(\Sigma^\infty, p_\Sigma)$ and the dual complexity space can be structured, in a natural way, as stable partial metric monoids (see Section 2 for definitions). In Section 4 we introduce the notion of a semivaluation monoid and show that there is a bijection between stable partial metric monoids and semivaluation monoids. This result provides an extension of the correspondence theorem of [22] to the context of monoids.

A natural property in the context of monoids $(X, \cdot)$ is that of m-invariance. A quasi-metric d on a monoid $(X, \cdot)$ is *m-invariant* ("monoid-invariant") when $d(x \cdot z, y \cdot z) \leqslant d(x, y)$ and $d(z \cdot x, z \cdot y) \leqslant d(x, y)$, for all $x, y, z \in X$.

We reserve the adjective "m-invariant" to indicate invariance with respect to the monoid operation. When we refer to invariance with respect to a (semi)lattice operation, we use the adjective "l-invariant" ("lattice-invariant"). It is of course possible to interpret a (semi)lattice operation as a monoid operation, in which case the two notions coincide. For our purposes, most examples involve a semilattice equipped with an additional monoid operation. We will see that in general l-invariance and m-invariance behave quite differently, hence it is useful to clearly separate the notions.

It has recently emerged that l-invariance plays a crucial role in Quantitative Domain Theory [22,24]. Hence, it is interesting to explore the notion of m-invariance for other monoid operations which arise naturally in Quantitative Domain Theory.

Aside from a notion of m-invariance for quasi-metrics, one can formulate a notion of m-invariance on partial metrics p as follows.

A partial metric p on a monoid $(X, \cdot)$ is called *m-invariant* if $p(x \cdot z, y \cdot z) \leqslant p(x, y)$ and $p(z \cdot x, z \cdot y) \leqslant p(x, y)$, for all $x, y, z \in X$.

When one studies the above examples, it quickly emerges that the two notions do not necessarily arise together.

We summarize our findings in Table 1, where we use the following abbreviations: ID for the Interval Domain, (DW, $\oplus$) for Domain of Words equipped with the addition operation $\oplus$, (DW, conc) for the Domain of Words equipped with the concatenation operation and DC for the Dual Complexity Space. For each domain, we indicate whether the partial metric

Table 1
m-invariance properties

	ID	(DW, $\oplus$)	(DW, conc)	DC
m-invariance of partial metric	Yes	No	Yes	No
m-invariance of quasi-metric	No	Yes	No	Yes

and the associated weightable quasi-metric are m-invariant with respect to the monoid operation.

In the light of Table 1 it is interesting to point out that there exists an easy example of an m-invariant partial metric monoid whose associated weightable quasi-metric is m-invariant (see Remark 6 below).

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2. Basic notions and preliminary results

Our basic references for quasi-metric spaces are [7,12,13].

In our context, by a *quasi-metric* on a set X we mean a nonnegative real valued function d on $X \times X$ such that for all $x, y, z \in X$:

(i) $d(x, y) = d(y, x) = 0 \Longleftrightarrow x = y$;
(ii) $d(x, y) \leqslant d(x, z) + d(z, y)$.

A *quasi-metric space* is a pair (X, d) such that X is a (nonempty) set and d is a quasi-metric on X.

The *associated* (*partial*) *order* $\leqslant_d$ of a quasi-metric d on a set X is defined by $x \leqslant_d y \Longleftrightarrow d(x, y) = 0$.

Each quasi-metric d on X generates a T_0-topology $\mathcal{T}(d)$ on X which has as a base the family of open d-balls $\{B_d(x, r)\colon x \in X,\ r > 0\}$, where $B_d(x, r) = \{y \in X\colon d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

If d is a quasi-metric on X, then the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$, is a metric on X.

The following is a simple but useful example of a quasi-metric space.

For each pair $x, y \in \mathbb{R}$ let $u(x, y) = (y - x) \vee 0$. Then u is a quasi-metric on $\mathbb{R}$ called the upper quasi-metric on $\mathbb{R}$. Note that u^s is the usual metric on $\mathbb{R}$.

The weightable quasi-metric spaces were introduced by Matthews in [14]. A quasi-metric space (X, d) is called *weightable* if there exists a function $w : X \to \mathbb{R}^+$, such that for all $x, y \in X$, $d(x, y) + w(x) = d(y, x) + w(y)$. The function w is said to be a *weighting function* for (X, d) and the quasi-metric d is *weightable by the function* w.

The following result provides the precise relationship between partial metric spaces and weightable quasi-metric spaces.

Theorem 1 [14].

(a) *Let (X, p) be a partial metric space. Then, the function $d_p : X \times X \to \mathbb{R}^+$ defined by $d_p(x, y) = p(x, y) - p(x, x)$ for all $x, y \in X$ is a weightable quasi-metric on X with weighting function w given by $w(x) = p(x, x)$ for all $x \in X$. Furthermore $\mathcal{T}(p) = \mathcal{T}(d_p)$.*

(b) *Conversely, if* (X, d) *is a weightable quasi-metric space with weighting function* w, *then the function* $p_d : X \times X \to \mathbb{R}^+$ *defined by* $p_d(x, y) = d(x, y) + w(x)$ *is a partial metric on* X. *Furthermore* $\mathcal{T}(d) = \mathcal{T}(p_d)$.

Observe that the restriction to $\mathbb{R}^+$ of the quasi-metric u defined above is weightable with weighting function the identity function on $\mathbb{R}^+$.

Next we discuss the m-invariancy of partial metrics and (weightable) quasi-metrics on monoids. Although the results also hold for semigroups, it suffices to our purposes here, to state them in the realm of monoids. Let us recall that a *monoid* is a semigroup $(X, \cdot)$ with neutral element.

A (*quasi-*)*metric monoid* is a triple $(X, \cdot, d)$ such that $(X, \cdot)$ is a monoid and d is an m-invariant (quasi-)metric on X.

It is well known [11] that a (quasi-)metric d on a monoid $(X, \cdot)$ is m-invariant if and only if for all $x, y, a, b \in X$: $d(x \cdot a, y \cdot b) \leqslant d(x, y) + d(a, b)$.

We show that the situation is quite different for m-invariant partial metrics.

Proposition 1. *Let* $(X, \cdot)$ *be a monoid and let* p *be an m-invariant partial metric on* X. *Then* $p(x \cdot a, y \cdot b) \leqslant p(x, y) + p(a, b)$ *for all* $x, y, a, b \in X$.

Proof. Let $x, y, a, b \in X$. Then $p(x \cdot a, y \cdot b) \leqslant p(x \cdot a, y \cdot a) + p(y \cdot a, y \cdot b) - p(y \cdot a, y \cdot a) \leqslant p(x, y) + p(a, b)$. □

Remark 1. The converse of Proposition 1 does not hold in general. Indeed, consider the monoid $(\mathbb{R}^+, +)$, where $+$ is the usual addition, and let p be the partial metric on $\mathbb{R}^+$ given by $p(x, y) = x \vee y$. Then, for all $x, y, a, b \in \mathbb{R}^+$, $p(x + a, y + b) = (x + a) \vee (y + b) \leqslant (x \vee y) + (a \vee b) = p(x, y) + p(a, b)$. However, it is clear that for all $x, y, z \in \mathbb{R}^+$ with $z > 0$, one has $p(x + z, y + z) > p(x, y)$.

Let us recall that a real valued function f defined on a monoid $(X, \cdot)$ is *subadditive* provided that $f(x \cdot y) \leqslant f(x) + f(y)$ for all $x, y \in X$.

Proposition 2. *Let* $(X, \cdot, d)$ *be a weightable quasi-metric monoid with weighting function* w. *If* w *is subadditive, then* $p_d(x \cdot a, y \cdot b) \leqslant p_d(x, y) + p_d(a, b)$ *for all* $x, y, a, b \in X$.

Proof. Let $x, y, a, b \in X$. Then $p_d(x \cdot a, y \cdot b) = d(x \cdot a, y \cdot b) + w(x \cdot a) \leqslant d(x, y) + d(a, b) + w(x) + w(a) = p_d(x, y) + p_d(a, b)$. □

Remark 2. Related to Proposition 2, we note that $(\mathbb{R}^+, +, u)$ is an example of a quasi-metric monoid for which the identity function is a (sub)additive weighting function, and such that the partial metric u_p is not m-invariant (in fact, u_p is the partial metric of Remark 1 above).

In the light of the above propositions and remarks and for the sake of generality, we propose the following notion.

Definition 1. A *partial metric monoid* is a triple $(X, \cdot, p)$ such that $(X, \cdot)$ is a monoid and p is a partial metric on X such that for all $x, y, a, b \in X$, $p(x \cdot a, y \cdot b) \leqslant p(x, y) + p(a, b)$.

We clarify at this stage the relationship between weightable quasi-metric monoids with a subadditive weighting function and partial metric monoids.

From Proposition 2, it follows that every weightable quasi-metric monoid with a subadditive weighting function is a partial metric monoid. (The examples which we discuss below will show that it is not the case that this partial metric monoid is necessarily m-invariant.) The converse is not true, i.e. it is not the case that every partial metric monoid has a corresponding weightable quasi-metric monoid, since in general the quasi-metric is not m-invariant. We do obtain a subadditive weighting function however.

Let us recall that an ordered set $(X, \preceq)$ is a meet semilattice if every two elements $x, y \in X$ have an infimum $x \sqcap y$.

As a consequence of Theorem 1(a), we have the following.

Proposition 3. *Let (X, p) be a partial metric space. Then the following hold*:

(1) $\sqsubseteq_p = \leqslant_{d_p}$ *on X.*
(2) $(X, \sqsubseteq_p)$ *is a meet semilattice if and only if* $(X, \leqslant_{d_p})$ *is a meet semilattice.*

According to [21], a *quasi-metric meet semilattice* is a quasi-metric space which is a meet semilattice for its associated order. A quasi-metric meet semilattice (X, d) is called *l-invariant* if for all $x, y, z \in X$, $d(x \sqcap z, y \sqcap z) \leqslant d(x, y)$.

Lemma 1 [22]. *A quasi-metric meet semilattice (X, d) is l-invariant if and only if $d(x, x \sqcap y) = d(x, y)$ for all $x, y \in X$.*

Proposition 4. *Let (X, p) be a partial metric space. Then (X, d_p) is an l-invariant quasi-metric meet semilattice if and only if $(X, \sqsubseteq_p)$ is a meet semilattice such that $p(x, y) = p(x \sqcap y, x \sqcap y)$ for all $x, y \in X$.*

Proof. Suppose that (X, d_p) is an l-invariant quasi-metric meet semilattice. By Proposition 3, $(X, \sqsubseteq_p)$ is a meet semilattice. Let $x, y \in X$. From Theorem 1(a), Lemma 1 and the fact that $x \sqcap y \sqsubseteq_p x$, we obtain $p(x \sqcap y, x \sqcap y) = p(x, x \sqcap y) = d_p(x, x \sqcap y) + p(x, x) = d_p(x, y) + p(x, x) = p(x, y)$.

Conversely, by Proposition 3, $(X, \leqslant_{d_p})$ is a meet semilattice. Now let $x, y \in X$. Then $d_p(x, x \sqcap y) = p(x, x \sqcap y) - p(x, x) = p(x \sqcap y, x \sqcap y) - p(x, x) = p(x, y) - p(x, x) = d_p(x, y)$. $\square$

A partial metric space (X, p) satisfying the conditions of Proposition 4 will be called a *stable partial metric* (*meet semilattice*) in the sequel (cf. [26]).

Definition 2. A stable partial metric monoid is a partial metric monoid $(X, \cdot, p)$ such that (X, p) is a stable partial metric meet semilattice.

From Proposition 4 we deduce the following relationship between the monoid operation and the associated order for a stable partial metric monoid $(X, \cdot, p)$:

$$p\big((x \cdot a) \sqcap (y \cdot b), (x \cdot a) \sqcap (y \cdot b)\big) \leqslant p(x, y) + p(a, b) \quad \text{for all } x, y, a, b \in X.$$

3. Three examples: The interval domain, the domain of words and the dual complexity space

In this section we shall show that the interval domain, the domain of words and the dual complexity space can be modelled as stable partial metric monoids.

Example 1. As a first example we discuss the interval domain of [4,6]. Recall (see Section 1), that the interval domain $I([0,1])$ consists of the nonempty closed and bounded intervals of $[0,1]$ ordered by reverse inclusion. Let p be the partial metric on $I([0,1])$ (see [8,16]) given by $p([a,b],[c,d]) = (b \vee d) - (a \wedge c)$.

One can easily verify that the associated weightable quasi-metric space $(I([0,1]), d_p)$ is an l-invariant quasi-metric meet semilattice with a bounded weighting function (cf. [23]). Thus $(I([0,1]), p)$ is a stable partial metric meet semilattice by Proposition 4.

The interval domain forms a monoid with respect to the operation $\circ$, defined as follows (cf. [6]): One considers for every interval $x := [a,b] \subseteq [0,1]$, the unique increasing affine map: $\lambda \to r\lambda + s : [0,1] \to [0,1]$ with image x, namely, $\mathrm{cons}_x(\lambda) = (b-a)\lambda + a$. In practice x is assumed to have *rational* end-points so that one has countably many primitive operations [6].

From [6], we know that if M is a real PCF program of real number type, then one knows that the value $\mathrm{cons}_x(M)$ can be regarded as a partially evaluated program with partial result x. Partial results can be combined via a composition operation on the unit interval domain by $x \circ y = \mathrm{cons}_x(y)$. This makes the interval domain over $[0,1]$ into a semigroup with neutral element $[0,1]$. Associativity can be expressed as $\mathrm{cons}_x \circ \mathrm{cons}_y(\lambda) = \mathrm{cons}_{x \circ y}(\lambda)$. The information order on the interval domain is recovered by a refinement property: $x \supseteq y \Longleftrightarrow$ there is some z with $x \circ z = y$.

Next we show that p is m-invariant on $(I([0,1]), \circ)$, and thus $(I([0,1]), \circ, p)$ will be a partial metric monoid by Proposition 1. Indeed, let $x := [a,b]$, $y := [c,d]$ and $z := [s,t]$ be elements of the interval domain. Since $x \supseteq x \circ z$, and $y \supseteq y \circ z$, it immediately follows that $p(x \circ z, y \circ z) \leqslant (b \vee d) - (a \wedge c) = p(x,y)$. On the other hand $p(z \circ x, z \circ y) = ((t-s)(b \vee d) + s) - ((t-s)(a \wedge c) + s) = (t-s)((b \vee d) - (a \wedge c)) = (t-s)p(x,y) \leqslant p(x,y)$.

We conclude that p is m-invariant on $(I([0,1]), \circ)$ and hence $(I([0,1]), \circ, p)$ is a stable partial metric monoid.

Remark 3. $(I([0,1]), \circ, d_p)$ is not a quasi-metric monoid. In fact, let $x := [0,1]$, $y := [0,1/2]$ and $z := [s,t]$ with $s > 0$, $t \leqslant 1$. Then $x \circ z = z$ and $y \circ z := [s/2, t/2]$. Therefore $d_p(x \circ z, y \circ z) = p(x \circ z, y \circ z) - p(x \circ z, x \circ z) = (t - s/2) - (t - s) = s/2 > 0$. However $d_p(x,y) = p(x,y) - p(x,x) = 1 - 1 = 0$.

Observe also that the weighting function w for $(I([0,1]), d_p)$, given by $w(x) = p(x,x)$, is subadditive since $p(x \circ y, x \circ y) \leqslant p(x,x)$ for all $x, y \in I([0,1])$.

Recall that an *ordered monoid* is a triple $(X, \cdot, \sqsubseteq)$ such that $(X, \cdot)$ is a monoid and $\sqsubseteq$ is an order on X such that if $x \sqsubseteq y$, then $x \cdot z \sqsubseteq y \cdot z$ and $z \cdot x \sqsubseteq z \cdot y$ for all $z \in X$. Clearly, if $(X, \cdot, d)$ is a quasi-metric monoid, then $(X, \cdot, \leqslant_d)$ is an ordered monoid. In contrast to this fact, $(I([0,1], \circ, \sqsubseteq_p)$ is not an ordered monoid because, as we have shown above, one has for $x := [0,1]$, $y := [0,1/2]$ and $z := [s,t]$, with $s > 0$, $t \leqslant 1$, $p(x,x) = p(x,y)$ but $p(x \circ z, x \circ z) < p(x \circ z, y \circ z)$.

Since on a meet semilattice $(X, \sqsubseteq)$, the operation $\sqcap$ is associative, it clearly follows that for each l-invariant quasi-metric meet semilattice (X, d) with top$\top$, the triple $(X, \sqcap, d)$ is a quasi-metric monoid for which $\top$ is its neutral element. In our next examples we shall model both the domain of words and the dual complexity space as weightable quasi-metric monoids that are also l-invariant quasi-metric meet semilattices, in such a way that the monoid operation naturally given to the corresponding "support" set (an "alphabet" and $\mathbb{R}^+$, respectively) extend to the space. In this way, the two spaces will be stable partial metric monoids, of course. In particular, when the dual complexity space is equipped with the natural pointwise addition operation we obtain a weightable quasi-metric monoid with respect to this operation as well as with respect to its meet semilattice operation such that the weighting function is (sub)additive (see Example 3 below). We also explore extending the domain of words with an operation of addition. We show that a natural operation can be defined on this domain, which on undefined elements yields undefined and for which the domain of words forms a weightable quasi-metric monoid with subadditive weighting function.

Example 2. Let Σ be a nonempty alphabet. Let Σ^∞ be the set of all finite and infinite sequences ("words") over Σ, where we adopt the convention that the empty sequence ϕ is an element of Σ^∞.

Denote by $\sqsubseteq$ the prefix order on Σ^∞, i.e., $x \sqsubseteq y \Longleftrightarrow x$ is a prefix of y. Then $(\Sigma^\infty, \sqsubseteq)$ is an algebraic complete partial order which is a Scott domain if Σ is countable (see [25, Example 2.2]).

Now, for each $x, y \in \Sigma^\infty$ we define $x \sqcap y$ as the longest common prefix of x,and y, and for each $x \in \Sigma^\infty$ we denote by $\ell(x)$ the length of x. Then $\ell(x) \in [1, \omega]$ whenever $x \neq \phi$ and $\ell(\phi) = 0$.

Following Example 8 of [12] (see also [14, Example 3.3]), the function d defined on $\Sigma^\infty \times \Sigma^\infty$ by $d(x,y) = 2^{-\ell(x \sqcap y)} - 2^{-\ell(x)}$, is a quasi-metric on Σ^∞ and (Σ^∞, d) is a weightable quasi-metric space with weighting function w defined on Σ^∞ by $w(x) = 2^{-\ell(x)}$ for all $x \in \Sigma^\infty$, where we adopt the convention that $2^{-\omega} = 0$. (Other interesting quasi-metrics defined on the domain of words can be found in [17,1], etc.)

Furthermore, the associated order $\leqslant_d$ coincides with the prefix order $\sqsubseteq$ on Σ^∞. Thus $(\Sigma^\infty, \leqslant_d)$ is clearly a meet semilattice. Since for each $x, y \in \Sigma^\infty$ we have $d(x, x \sqcap y) = d(x,y)$, it follows from Lemma 1 that (Σ^∞, d) is an l-invariant quasi-metric meet semilattice.

Now suppose that there exists an operation $+$ on Σ for which $(\Sigma, +)$ is an (Abelian) monoid with neutral element e. We shall prove that, then, Σ^∞ can be endowed with the structure of an (Abelian) monoid $(\Sigma^\infty, \oplus)$ such that d is m-invariant for $\oplus$, and thus $(\Sigma^\infty, \oplus, d)$ is a weightable quasi-metric monoid.

Denote by e the infinite word such that $\mathsf{e}(k) = e$ for all $k \in \omega \setminus \{0\}$. For each $x \in \Sigma^\infty$ we define $x \oplus \phi = \phi \oplus x = \phi$. For each $x, y \in \Sigma^\infty \setminus \phi$, we define $x \oplus y$ as the element of Σ^∞ of length $\ell(x \oplus y) = \min\{\ell(x), \ell(y)\}$ such that for each $k \leqslant \ell(x \oplus y)$, $(x \oplus y)(k) = x(k) + y(k)$.

It is straightforward to show that for each $x, y, z \in \Sigma^\infty$, $x \oplus \mathsf{e} = \mathsf{e} \oplus x = x$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$, and $x \oplus y = y \oplus x$ whenever $(\Sigma, +)$ is an Abelian monoid. Therefore, $(\Sigma^\infty, \oplus)$ is an (Abelian) monoid.

Observe that if each "letter" a of Σ,is identified with the word $x_a \in \Sigma^\infty$ such that $\ell(x_a) = 1$ and $x_a(1) = a$, then the restriction of d to Σ,is exactly the discrete metric on Σ and the restriction of $\oplus$ to Σ is the operation $+$ (cf. [1])

Next we prove that the quasi-metric d is m-invariant for $\oplus$. Let $x, y, z \in \Sigma^\infty$. If $\ell(z) \leqslant \ell(x \sqcap y)$, then $x \oplus z = y \oplus z$, whence $d(x \oplus z, y \oplus z) = 0 \leqslant d(x, y)$. If $\ell(z) > \ell(x \sqcap y)$, then $\ell((x \oplus z) \sqcap (y \oplus z)) \geqslant \ell(x \sqcap y)$. Together with $\ell(x \oplus z) \leqslant \ell(x)$, this implies

$$d(x \oplus z, y \oplus z) = 2^{-\ell((x\oplus z)\sqcap(y\oplus z))} - 2^{-\ell(x\oplus z)} \leqslant 2^{-\ell(x\sqcap y)} - 2^{-\ell(x)} = d(x, y).$$

The proof that $d(z \oplus x, z \oplus y) \leqslant d(x, y)$ is analogous. We conclude that d is m-invariant and, consequently, $(\Sigma^\infty, \oplus, d)$ is a weightable quasi-metric monoid.

Moreover, the weighting function w is subadditive. Indeed, since for each $x, y \in \Sigma^\infty$, $\ell(x \oplus y) = \min\{\ell(x), \ell(y)\}$, it follows that $w(x \oplus y) = \max\{w(x), w(y)\}$.

Now let p_Σ the partial metric on Σ^∞ induced by d (see Theorem 1). Then, by Propositions 2 and 4, $(\Sigma^\infty, \oplus, p_\Sigma)$ is a stable partial metric monoid. (Note that $p_\Sigma(x, y) = 2^{-\ell(x\sqcap y)}$ for all $x, y \in \Sigma^\infty$.)

One might motivate the summation $\oplus$ defined above as follows (see [1]). If we interpret a finite list $z = z_1 z_2 \cdots z_n$ as an infinite list of which only finitely many elements are defined, i.e., $z = z_1 z_2 \cdots z_n \bot\bot\bot \cdots$, where $\bot$ is the symbol for undefined value, then the sum makes sense: adding a defined value to an undefined one should give undefined; therefore if we add a finite list $z = z_1 z_2 \cdots z_n \bot\bot\bot \cdots$ with an infinite list $y = y_1 y_2 \cdots y_n y_{n+1} \cdots$, we obtain the finite list $z \oplus y = (z_1 + y_1)(z_2 + y_2) \cdots (z_n + y_n)\bot\bot\bot \cdots$.

Remark 4. The partial metric p_Σ is not m-invariant on Σ^∞. Indeed, let $x \in \Sigma^\infty$ be such that $\ell(x) = 1$, let $y = x$ and let $z = \phi$. Then $p_\Sigma(x \oplus z, y \oplus z) = 1$ and $p_\Sigma(x, y) = 1/2$.

Remark 5. It is interesting to discuss in this context the domain of words (Σ^∞, d) of Example 2, equipped with the concatenation operation. Thus Σ^∞ is a monoid with neutral element the empty word ϕ. Now suppose that Σ has at least two letters. Since for each $x, y, z \in \Sigma^\infty$, $\ell(xz \sqcap yz) \geqslant \ell(x \sqcap y)$ and $\ell(zx \sqcap zy) \geqslant \ell(x \sqcap y)$, it follows that p_Σ is m-invariant with respect to concatenation, so we obtain a stable partial metric monoid. However d is not m-invariant since for $a, b \in \Sigma\setminus\{\phi\}$, with $a \neq b$, it suffices to take $x, y, z \in \Sigma^\infty$ such that $x := a$, $y := aa$ and $z := b$, to obtain $d(xz, yz) = 2^{-1} - 2^{-2}$ and $d(x, y) = 0$.

Example 3. As a third example of a stable partial metric monoid, we mention the dual complexity (quasi-metric) space (cf. [18]). The *dual complexity space* is the quasi-metric space $(\mathcal{C}^*, d_{\mathcal{C}^*})$, where $\mathcal{C}^* = \{f : \omega \to \mathbb{R}^+ \colon \sum_{n=0}^\infty 2^{-n} f(n) < \infty\}$, and $d_{\mathcal{C}}$ is the quasi-metric on $\mathcal{C}$ defined by $d_{\mathcal{C}^*}(f, g) = \sum_{n=0}^\infty 2^{-n}[(g(n) - f(n)) \vee 0]$, for all $f, g \in \mathcal{C}^*$. Furthermore $(\mathcal{C}^*, d_{\mathcal{C}^*})$ is weightable by the weighting function w defined on $\mathcal{C}^*$ by

$w(f) = \sum_{n=0}^{\infty} 2^{-n} f(n)$ for all $f \in \mathcal{C}^*$. Several quasi-metric properties of the dual complexity space are discussed in [18,19].

If $f, g \in \mathcal{C}$, then $f \vee g \in \mathcal{C}^*$. Thus $(\mathcal{C}^*, d_{\mathcal{C}^*})$ is an l-invariant quasi-metric meet semilattice, where $\sqsubseteq$ is here the dual order of the usual pointwise order, i.e., $f \sqsubseteq g \Longleftrightarrow g \leqslant f$ [22].

Since the complexity of a given program is frequently obtained by a summation of complexity functions, we endow $\mathcal{C}$ with a suitable structure of monoid by the usual addition operation $+$. Thus if $f, g \in \mathcal{C}^*$, $f + g \in \mathcal{C}^*$, and clearly we have that $d_{\mathcal{C}^*}(f + g, h + l) \leqslant d_{\mathcal{C}^*}(f, h) + d_{\mathcal{C}^*}(g, l)$, for all $f, g, h, l \in \mathcal{C}^*$. Therefore $(\mathcal{C}^*, +, d_{\mathcal{C}^*})$ is a weightable quasi-metric monoid. Since the weighting function w is (sub)additive, it follows from Proposition 2 that $(\mathcal{C}^*, +, p_{d_{\mathcal{C}^*}})$ is a stable partial metric monoid.

Observe that, however, $p_{d_{\mathcal{C}^*}}$ is not m-invariant because for $f, g, h \in \mathcal{C}^*$ with $h(k) > 0$ for some $k \in \omega$, we clearly have $p_{d_{\mathcal{C}^*}}(f + h, g + h) > p_{d_{\mathcal{C}^*}}(f, g)$.

In the following we shall denote by $\mathcal{SPM}$ the class of all stable partial metric monoids.

4. Semivaluation monoids and the class $\mathcal{SPM}$

In this section we characterize the spaces of the class $\mathcal{SPM}$ in terms of semivaluation monoids which will be defined below. The key of such a characterization is our next result.

Proposition 5. *Let (X, p) be a stable partial metric monoid and let $f : X \to \mathbb{R}^+$ be given by $f(x) = p(x, x)$ for all $x \in X$. Then for each $x, y, a, b \in X$ the following condition holds*:

$$f\big((x \cdot a) \sqcap (y \cdot b)\big) \leqslant f(x \sqcap y) + f(a \sqcap b). \tag{$*$}$$

Proof. Let $x, y, a, b \in X$. Taking into account Proposition 4 we obtain

$$\begin{aligned} f\big((x \cdot a) \sqcap (y \cdot b)\big) &= p\big((x \cdot a) \sqcap (y \cdot b), (x \cdot a) \sqcap (y \cdot b)\big) = p(x \cdot a, y \cdot b) \\ &\leqslant p(x, y) + p(a, b) = p(x \sqcap y, x \sqcap y) + p(a \sqcap b, a \sqcap b) \\ &= f(x \sqcap y) + f(a \sqcap b). \qquad \square \end{aligned}$$

Condition $(*)$ above is a quite natural requirement in the framework of meet semilattices, when they are equipped with some usual monoid operations.

Indeed, first note that if $(X, \sqsubseteq)$ is a meet semilattice and $\cdot$ is a monoid operation on X such that the function $f : X \to \mathbb{R}$ satisfies condition $(*)$, then f is subadditive, because for all $x, y \in X$, we obtain

$$f(x \cdot y) = f\big((x \cdot y) \sqcap (x \cdot y)\big) \leqslant f(x \sqcap x) + f(y \sqcap y) = f(x) + f(y).$$

Conversely, we have the two following results.

Proposition 6. *Let $(X, \cdot, \sqsubseteq)$ be an ordered monoid such that $(X, \sqsubseteq)$ is a meet semilattice. Then, every decreasing subadditive function on X satisfies condition $(*)$.*

Proof. Let $f : X \to \mathbb{R}$ be decreasing and subadditive and let $x, y, a, b \in X$. Then $(x \sqcap y) \cdot (a \sqcap b) \sqsubseteq x \cdot (a \sqcap b) \sqsubseteq x \cdot a$, and, similarly, $(x \sqcap y) \cdot (a \sqcap b) \sqsubseteq y \cdot b$. Therefore $(x \sqcap y) \cdot (a \sqcap b) \sqsubseteq (x \cdot a) \sqcap (y \cdot b)$. So

$$f\big((x \cdot a) \sqcap (y \cdot b)\big) \leqslant f\big((x \sqcap y) \cdot (a \sqcap b)\big) \leqslant f(x \sqcap y) + f(a \sqcap b). \qquad \square$$

Proposition 7. *Let $(X, \sqsubseteq)$ be a meet semilattice and let $\cdot$ be a monoid operation on X such that $x \sqsubseteq x \cdot y$ for all $x, y \in X$ (i.e., the monoid operation is monotone). Then, every decreasing nonnegative real valued function on X satisfies condition $(*)$.*

Proof. Let $f : X \to \mathbb{R}$ be decreasing and let $x, y, a, b \in X$. Then $f((x \cdot a) \sqcap (y \cdot b)) \leqslant f(x \sqcap y)$. We conclude that f satisfies condition $(*)$. $\square$

The domain of words with addition operation and the dual complexity space are examples of spaces where Proposition 6 applies (see Examples 2 and 3). The interval domain and the domain of words with concatenation operation are examples of spaces where Proposition 7 applies (see Example 1 and Remark 5).

Remark 6. In connection with the examples of Section 3 and Propositions 6 and 7 above, we give an example of a stable partial metric monoid $(X, \cdot, p)$ such that $(X, \cdot, \sqsubseteq)$ is an ordered monoid, the monoid operation is monotone and both p and d_p are m-invariant.

Let $X = [0, 1]$ and let p be the restriction to X of the usual partial metric on $\mathbb{R}^+$, i.e. $p(x, y) = x \vee y$ for all $x, y \in X$ (see Section 1). Then d_p is the upper quasi-metric on X, i.e. $d_p(x, y) = (y - x) \vee 0$ for all $x, y \in X$.

For each $x, y \in X$ let $x \cdot y = x \wedge y$. Obviously $(X, \cdot)$ is a monoid, with neutral element 1. Furthermore (X, d_p) an l-invariant quasi-metric meet semilattice, and $(X, \cdot, \leqslant_{d_p})$ (equivalently $(X, \cdot, \sqsubseteq_p)$) is clearly an ordered monoid. (Note that $x \sqcap y = x \vee y$ for all $x, y \in X$.) On the other hand, since for each $x, y \in X$ one has $p(x, x) = x = x \vee (x \wedge y) = p(x, x \cdot y)$, the monoid operation is monotone. Finally, for each $x, y, z \in X$, we have $p(x \cdot z, y \cdot z) = (x \wedge z) \vee (y \wedge z) \leqslant x \vee y = p(x, y)$, and $d_p(x \cdot z, y \cdot z) = [((y \wedge z) - (x \wedge z)) \vee 0] \leqslant [(y - x) \vee 0] = d_p(x, y)$. Since $\cdot$ is commutative, we conclude that both p and d_p are m-invariant.

Now let $(X, \sqsubseteq)$ be a meet semilattice and let $\cdot$ be a monoid operation on X. The following condition is referred as *left-continuity* (see [4]):

$$(z \cdot x) \sqcap (z \cdot y) = z \cdot (x \sqcap y).$$

We shall briefly discuss left-continuity and other related conditions due to their relevance in connection to condition $(*)$. Thus, condition

$$(x \cdot z) \sqcap (y \cdot z) = (x \sqcap y) \cdot z$$

is referred to as *right-continuity.*

A more restrictive version of these conditions is the following

$$(x \cdot u) \sqcap (y \cdot v) = (x \sqcap y) \cdot (u \sqcap v).$$

We refer to this condition as *continuity*.

The following assertions are easily seen:

(i) Continuity implies left-continuity and right-continuity;
(ii) left-continuity implies $z \cdot x \sqsubseteq z \cdot y$ whenever $x \sqsubseteq y$;
(iii) right-continuity implies $x \cdot z \sqsubseteq y \cdot z$ whenever $x \sqsubseteq y$;
(iv) left-continuity plus right-continuity imply that $(X, \cdot, \sqsubseteq)$ is an ordered monoid;
(v) the dual complexity space and the domain of words satisfy continuity.

On the other hand, it is proved in [4] that the interval domain satisfies left-continuity. However, it does not satisfy right-continuity and hence not continuity:

In fact, let x, y, z be the elements of $I([0,1])$ considered in Remark 3 above. We showed that $p(x,x) = p(x,y)$ but $p(x \circ z, x \circ z) < p(x \circ z, y \circ z)$. Hence, by assertion (iii), the interval domain does not satisfy right-continuity.

Now, we show that under continuity, we have that subadditivity of f is equivalent to condition $(*)$. (Compare with Proposition 6.)

Indeed, assuming subadditivity of f and continuity, we obtain $f((x \cdot a) \sqcap (y \cdot b)) = f((x \sqcap y) \cdot (a \sqcap b)) \leqslant f(x \sqcap y) + f(a \sqcap b)$.

The converse is obvious because condition $(*)$ implies subadditivity as we have observed above.

Next we recall the definition of a valuation on a lattice $(L, \preceq)$. A function $f : L \to \mathbb{R}^+$ is said to be a *valuation* if (1) f is increasing, and (2) for each $x, y \in L$, $f(x \sqcap y) + f(x \sqcup y) = f(x) + f(y)$.

In case the function f is decreasing and satisfies (2), we refer to f as a *co-valuation*. If f only satisfies (2) we say that f is *modular*.

Actually, there does not seem to be a consistent terminology in the literature. Valuations, also called evaluations, as used in computer science (e.g., [3] or [9]) typically satisfy (1) and (2) above. In the classical mathematical literature a valuation only needs to satisfy (2) (e.g., [2]).

Since in our context we work with meet semilattices rather than lattices, we only need to consider meet co-valuations.

Recall that a *meet co-valuation* [22] on a meet semilattice $(X, \preceq)$ is a function $f : X \to \mathbb{R}^+$ such that for each $x, y, z \in X$,

$$f(x \sqcap z) \leqslant f(x \sqcap y) + f(y \sqcap z) - f(y).$$

A meet co-valuation f on a meet semilattice $(X, \preceq)$ is called strictly decreasing if for each $x, y \in X$, $x \prec y \Longrightarrow f(y) < f(x)$.

A *semivaluation space* is (compare [22]) a pair (X, f) such that X is a meet semilattice and f is a decreasing meet co-valuation on X.

Recall that semivaluations arise in many different contexts in Quantitative Domain Theory [22,23]. The Baire quasi-metric spaces [14], the (dual) complexity space [22] and the interval domain [4] are well-known examples of spaces that are semivaluation spaces.

Definition 3. A *semivaluation monoid* is a triple $(X, \cdot, f)$ such that $(X, \cdot)$ is a monoid and (X, f) is a semivaluation space with f a strictly decreasing meet co-valuation on X satisfying the condition $(*)$ of Proposition 5.

We denote by $\mathcal{SVM}$ the class of all semivaluation monoids. Our final result shows that stable partial metric monoids can be characterized as semivaluation monoids.

Theorem 2. *There exists a bijection $\Psi : \mathcal{SVM} \to \mathcal{SPM}$ defined to be the function which associates to each $(X, \cdot, f) \in \mathcal{SVM}$ the space $(X, \cdot, p_f) \in \mathcal{SPM}$, such that for each $x, y \in X$, $p_f(x, y) = f(x \sqcap y)$. The inverse of Ψ is the function which to each $(X, \cdot, p) \in \mathcal{SPM}$ associates the space $(X, \cdot, f_p) \in \mathcal{SVM}$, where $f_p(x) = p(x, x)$ for all $x \in X$.*

Proof. Le $(X, \cdot, f) \in \mathcal{SVM}$. Then $(X, \cdot)$ is a monoid and $(X, f) := (X, \preceq, f)$ is a semivaluation space where f is a strictly decreasing meet co-valuation on X satisfying $(*)$. Define the function p_f on $X \times X$ by $p_f(x, y) = f(x \sqcap y)$, for all $x, y \in X$.

We first show that p_f is a partial metric on X:

Indeed, suppose that $p_f(x, y) = p_f(x, x) = p_f(y, y)$. Then $f(x \sqcap y) = f(x) = f(y)$. Since f is strictly decreasing it follows that $x \sqcap y = x = y$.

On the other hand, it is clear that for each $x, y \in X$, $p_f(x, y) = p_f(y, x)$.

Now let $x, y, z \in X$. Since f is a meet co-valuation we have

$$\begin{aligned} p_f(x, z) = f(x \sqcap z) &\leqslant f(x \sqcap y) + f(y \sqcap z) - f(y) \\ &= p_f(x, y) + p_f(y, z) - p_f(y, y). \end{aligned}$$

We have shown that p_f is a partial metric on X.

Next we show that the partial order $\sqsubseteq_{p_f}$ coincides with $\preceq$. Indeed, for $x, y \in X$, one has:

$$\begin{aligned} x \sqsubseteq_{p_f} y &\iff p_f(x, y) = p_f(x, x) \iff f(x \sqcap y) = f(x) \\ &\iff x \sqcap y = x \iff x \preceq y. \end{aligned}$$

Hence $(X, \sqsubseteq_{p_f})$ is a meet semilattice. Moreover, for each $x, y \in X$:

$$p_f(x \sqcap y, x \sqcap y) = f(x \sqcap y) = p_f(x, y).$$

So (X, p_f) is a stable partial metric meet semilattice. It remains to show that $(X, \cdot, p_f)$ is a partial metric monoid. Indeed, let $x, y, a, b \in X$. By condition $(*)$,

$$p_f(x \cdot a, y \cdot b) = f\big((x \cdot a) \sqcap (y \cdot b)\big) \leqslant f(x \sqcap y) + f(a \sqcap b) = p_f(x, y) + p_f(a, b).$$

Therefore $(X, \cdot, p_f)$ is a partial metric monoid. We conclude that $(X, \cdot, p_f) \in \mathcal{SPM}$.

Conversely, let $(X, \cdot, p) \in \mathcal{SPM}$. We shall prove that $(X, \cdot, f_p) \in \mathcal{SVM}$, where f_p is the function defined on the meet semilattice $(X, \sqsubseteq_p)$ by $f_p(x) = p(x, x)$ for all $x \in X$.

Indeed, by assumption $(X, \cdot)$ is a monoid. Furthermore f_p is a meet co-valuation for $(X, \sqsubseteq_p)$ because for $x, y, z \in X$, we obtain

$$\begin{aligned} f_p(x \sqcap z) = p(x \sqcap z, x \sqcap z) &= p(x, z) \\ &\leqslant p(x, y) + p(y, z) - p(y, y) \\ &= p(x \sqcap y, x \sqcap y) + p(y \sqcap z, y \sqcap z) - p(y, y) \\ &= f_p(x \sqcap y) + f_p(y \sqcap z) - f_p(y). \end{aligned}$$

In addition f_p is strictly decreasing because if $x \sqsubset_p y$,we have

$$f_p(x) = p(x, x) = p(x, y) > p(y, y) = f_p(y).$$

Moreover f_p satisfies condition $(*)$ by Proposition 5, because for $x, y, a, b \in X$ we have

$$\begin{aligned} f_p\big((x \cdot a) \sqcap (y \cdot b)\big) &= p\big((x \cdot a) \sqcap (y \cdot b), (x \cdot a) \sqcap (y \cdot b)\big) = p(x \cdot a, y \cdot b) \\ &\leqslant p(x, y) + p(a, b) = p(x \sqcap y, x \sqcap y) + p(a \sqcap b, a \sqcap b) \\ &= f_p(x \sqcap y) + f_p(a \sqcap b). \end{aligned}$$

We conclude that $(X, \cdot, f_p) \in \mathcal{SVM}$.

It remains to show that Ψ is bijective. Let $(X, \cdot, f), (Y, \star, g) \in \mathcal{SVM}$ such that $\Psi((X, \cdot, f)) = \Psi((Y, \star, g))$. Then $(X, \cdot, p_f)$ and $(Y, \star, p_g)$ coincide. In particular $X = Y$. Since $p_f = p_g$, it follows that for each $x \in X$, $p_f(x, x) = p_g(x, x)$, i.e. $f(x) = g(x)$, so $f = g$. Thus Ψ is injective. Now let $(X, \cdot, p) \in \mathcal{SPM}$. Then $(X, \cdot f_p) \in \mathcal{VPM}$ as we have proved above and $\Psi((X, \cdot, f_p)) = (X, \cdot, p)$ because $p_{f_p}(x, y) = f_p(x \sqcap y) = p(x \sqcap y, x \sqcap y) = p(x, y)$ for all $x, y \in X$. So Ψ is surjective. This completes the proof. □

Remark 7. It seems interesting to note that the unique partial metric which allows one to quantify a domain equipped with certain monoid operations, behaves in our context. In fact, from the notion of a quantification of a domain (cf. [23]), one can define in a natural way, a *quantitative monoid* as a quantification of a domain equipped with a monoid operation; an *ordered quantitative monoid* as a quantitative monoid for which the associated monoid is ordered and a *monotone quantitative monoid* as a quantitative monoid for which the monoid operation is monotone.

Thus, by using the techniques of the proof of Proposition 27 of [23] in combination with Propositions 6 and 7, respectively, we can prove the following facts:

(i) *each ordered quantitative monoid with a decreasing subadditive selfdistance is a partial metric monoid*;
(ii) *each monotone quantitative monoid is a partial metric monoid.*

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