Applied Mathematics Letters 22 (2009) 264–267

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A kind of conditional vertex connectivity of star graphs

Min Wan, Zhao Zhang *

College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang, 830046, PR China
Shihezi University, Shihezi, Xinjiang, 832000, PR China

ARTICLE INFO

Article history:
Received 28 March 2008
Accepted 28 March 2008

Keywords:
Star graph
Conditional vertex connectivity

ABSTRACT

A subset $F \subset V(G)$ is called an R^2-vertex-cut of G if $G - F$ is disconnected and each vertex $u \in V(G) - F$ has at least two neighbors in $G - F$. The cardinality of a minimum R^2-vertex-cut of G, denoted by $\kappa^2(G)$, is the R^2-vertex-connectivity of G. In this work, we prove that $\kappa^2(S_n) = 6(n-3)$ for $n \geq 4$, where S_n is the n-dimensional star graph.

1. Introduction

Let $G = (V, E)$ be a finite graph without loops and parallel edges. We follow [3] for terminology not given here.

It is well known that the underlying topology of a computer interconnection network can be modeled by a graph G, and the connectivity $\kappa(G)$ of G is an important measure for fault tolerance of the network. In general, the larger $\kappa(G)$, the more reliable the network. However, $\kappa(G)$ is a worst case measure and thus underestimates the resilience of the network [9]. To overcome such shortcoming, Harary [5] introduced the concept of *conditional connectivity* by placing some requirements on the components of $G - F$. The R^k-vertex-connectivity follows this trend.

A subset $F \subset V(G)$ is called an *R^k-vertex-set* of G if each vertex $u \in V(G) - F$ has at least k neighbors in $G - F$. An *R^k-vertex-cut* of a connected graph G is a *R^k-vertex-set* F such that $G - F$ is disconnected. The *R^k-vertex-connectivity* of G, denoted by $\kappa^k(G)$, is the cardinality of a minimum R^k-vertex-cut of G. The idea behind this concept is that the probability that the failures concentrate around a vertex is small. For example, suppose G is a graph of order n which has t vertices of minimum degree k. If there are k faulty vertices in G, then the probability that these k vertices are exactly the neighbor set of some vertex is $t/\binom{n}{k}$, which is very small when n is large; while in the definition of the R^k-vertex-set, the requirement that there are at least k good neighbors around each vertex takes such resilience into account.

In [8], Latifi et al. proved that $\kappa^k(Q_n) = (n-k)2^k$, where Q_n is the n-dimensional hypercube. In [7], Hu and Yang proved that $\kappa^1(S_n) = 2n - 4$, where S_n is an n-dimensional star graph, the definition of which is given in the following.

Let X be a group and S be a subset of X. The *Cayley digraph* $Cay(X, S)$ is a digraph with vertex set X and arc set $\{(g, gs) \mid g \in X, s \in S\}$. The arc (g, gs) is labeled by s. Denote by Σ_n the group of all permutations on $\{1, \ldots, n\}$. An *n-dimensional star graph* S_n is the Cayley graph $Cay(\Sigma_n, S)$ with $S = \{(1i) \mid 1 < i \leqslant n\}$. It is well known that $Cay(X, S)$ is strongly connected if and only if S is a generating set of X. If $S = S^{-1}$, where $S^{-1} = \{s^{-1} \mid s \in S\}$, then $Cay(X, S)$ is an undirected graph. In particular, if all elements of S are involutions, as is the case for the star graph, $Cay(X, S)$ is undirected. Furthermore, S_n is $(n-1)$-regular (since $Cay(X, S)$ has degree $|S|$), bipartite (with the two parts of the bipartition containing even and odd permutations respectively), vertex transitive (since it is a Cayley graph) and edge transitive (see for example [6] Corollary 11).

The hypercube is an important network topology which has already been put into practice. The star graph is another popular topology which has many advantages over the hypercube. As can be seen from the following table (see [1,2,4]), if a

* Corresponding author at: College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang, 830046, PR China.
E-mail addresses: wanmin.0906@yahoo.com.cn (M. Wan), zhzhao@xju.edu.cn (Z. Zhang).

0893-9659/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2008.03.021

hypercube and a star graph have almost the same number of vertices, the star graph may have smaller degree (which reduces the production cost of the components), smaller diameter (which reduces the transmission delay), and higher connectivity (which increases the fault tolerance).

Graph	Dimension	Vertices	Degree	Diameter	Connectivity
n-cube	n	2^n	n	n	n
n-star	n	$n!$	$n-1$	$\lfloor \frac{3}{2}(n-1) \rfloor$	$n-1$

In this work, we study the fault tolerance measured by κ^k, and prove that $\kappa^2(S_n) = 6(n-3)$ for $n \geq 4$.

2. Preliminaries

For a vertex $v \in V$, $N(v)$ is the set of vertices adjacent to v in G. For a subset $U \subseteq V(G)$, $N(U) = (\bigcup_{u \in U} N(u)) - U$, and $G[U]$ is the subgraph of G induced by U. Sometimes, we use a graph itself to represent its vertex set, for instance, $N(G_1)$ means $N(V(G_1))$ where G_1 is a subgraph of G. Denote by $g(G)$ the *girth* of G, that is, the length of a shortest cycle of G. A cycle with length l is called an *l-cycle*.

The following notation should be distinguished: elements of Σ_n acting on the vertices of S_n are given in cycle format, for example $(i_1 i_2 i_3)$ or $(i_1 i_2)$, while vertices of S_n are given as a reordering $(i_1, i_2, \dots i_n)$ of $(1, 2, \dots, n)$, which is called the *label*. Moreover, the transposition (ij) means exchanging the 'positions' of the ith and jth elements in the label of a vertex (not exchanging element i and element j). That is, if a vertex u is labeled as $(p_1, \dots, p_i, \dots, p_j, \dots, p_n)$, then $u(ij) = (p_1, \dots, p_j, \dots, p_i, \dots, p_n)$.

Observe that two vertices $u, v \in V(S_n)$ are adjacent if and only if there exists some $(1i) \in S$ such that $v = u(1i)$. That is, u, v have the form $u = (p_1, \dots, p_i, \dots, p_n)$ and $v = (p_i, \dots, p_1, \dots, p_n)$. For an integer $i \in \{1, 2, \dots, n\}$, denote by S_{n-1}^i the $(n-1)$-dimensional sub-star graph of S_n induced by vertex set $\{(p_1, p_2, \dots, p_{n-1}, i) \mid (p_1, \dots, p_{n-1})$ ranges over all permutations of $\{1, \dots, n\} \setminus \{i\}\}$. Observe that S_n can be decomposed into n copies of S_{n-1}'s, namely $S_{n-1}^1, S_{n-1}^2, S_{n-1}^3, \dots, S_{n-1}^n$. For $u \in V(S_{n-1}^i)$, denote by $u' = u(1n)$ the unique neighbor of u in $S_n - S_{n-1}^i$, called the *outside neighbor* of u. For an R^2-vertex-set F of G, vertices in F are said to be *faulty* and vertices in $V(G) - F$ are called *good*.

The following lemma can be found in [7].

Lemma 1. $\kappa^1(S_n) = 2n - 4$.

Lemma 2. *The girth of S_n is 6. Any 6-cycle in S_n has the form $u_1, u_2, u_3, u_4, u_5, u_6, u_1$ where $u_2 = u_1(1i)$, $u_3 = u_2(1j)$, $u_4 = u_3(1i)$, $u_5 = u_4(1j)$, $u_6 = u_5(1i)$, $u_1 = u_6(1j)$ for some i, j with $i \neq j$.*

Proof. It is well known that the girth of S_n is 6. In the following, we show the second part of the lemma. Note that each vertex v at distance 3 from u may have only two forms: either $v = u(1i)(1j)(1k) = u(1kji)$ for three different integers i, j, k, or $v = u(1i)(1j)(1i) = u(ij)$ for two different integers i, j. Since the way to decompose $(1kji)$ into the form $(1i_1)(1i_2)(1i_3)$ is unique, namely, $(1kji)$ can only be decomposed into $(1i)(1j)(1k)$, we see that there is exactly one path of length 3 from u to $u(1kji)$. Hence, if u and v are in a 6-cycle, then v can only have the form $u(ij)$. Since there are exactly two ways to decompose (ij) into the form $(1i_1)(1i_2)(1i_3)$, namely, $(ij) = (1i)(1j)(1i) = (1j)(1i)(1j)$ (note that $(ij) = (ji)$), we have two (u, v)-paths of length 3, which form a 6-cycle as described in the lemma. □

Lemma 3. *For any path $P = u_0 u_1 u_2 u_3$ which is in some S_{n-1}^ℓ,*

(1) u_0', u_1' and u_2' are in three different S_{n-1}^k's;

(2) u_0', u_1', u_2' and u_3' are in four different S_{n-1}^k's unless $u_3 = u_0(1i)(1j)(1i)$ for some $i \neq j$, in which case u_0' and u_3' are in the same S_{n-1}^k.

Proof. Suppose $u_1 = u_0(1i)$ $(i \neq n)$, $u_2 = u_1(1j)$ $(j \neq i, n)$, $u_3 = u_2(1r)$ $(r \neq j, n)$. We assume, without loss of generality, that the path P is in S_{n-1}^1, and $i < j$. Write $u_0 = (p_1, \dots, p_i, \dots, p_j, \dots, p_{n-1}, 1)$. Then $u_1 = (p_i, \dots, p_1, \dots, p_j, \dots, p_{n-1}, 1)$, $u_2 = (p_j, \dots, p_1, \dots, p_i, \dots, p_{n-1}, 1)$. The first part follows since $u_0' \in S_{n-1}^{p_1}$, $u_1' \in S_{n-1}^{p_i}$, $u_2' \in S_{n-1}^{p_j}$. If $u_3 = u_0(1i)(1j)(1i)$, then $u_3 = (p_1, \dots, p_j, \dots, p_i, \dots, p_{n-1}, 1)$ and thus $u_3' \in S_{n-1}^{p_1}$. Otherwise, $u_3' \in S_{n-1}^{p_r}$ for $r \neq i, j$. □

Lemma 4. *$N(S_{n-1}^j)$ $(j = 1, \dots, n)$ is an independent set of cardinality $(n-1)!$, and $|N(S_{n-1}^j) \cap V(S_{n-1}^k)| = (n-2)!$ for $k \neq j$.*

Proof. For each vertex $u = (i_1, i_2, \dots, j)$ in S_{n-1}^j, u has a unique neighbor u' outside of S_{n-1}^j with the form $u' = (j, i_2, \dots, i_1)$. Note that no two vertices in S_{n-1}^j have the same outside neighbor, and no outside neighbors of two different vertices of S_{n-1}^j can be adjacent (since they have the same first element j); the first part of the lemma follows from the observation that $|V(S_{n-1}^j)| = (n-1)!$. For each vertex $u \in V(S_{n-1}^j)$, its outside neighbor $u' \in V(S_{n-1}^k)$ if and only if u has the form $(k, p_2, \dots, p_{n-1}, j)$. So, $N(S_{n-1}^j) \cap V(S_{n-1}^k) = \{(j, p_2, \dots, p_{n-1}, k) \mid (p_2, \dots, p_{n-1})$ ranges over all permutations of $\{1, \dots, n\} \setminus \{j, k\}\}$. The second part follows. □

3. Main result

Theorem 3.1. *For any integer $n \geq 4$, $\kappa^2(S_n) = 6(n-3)$.*

Proof. First, we show that $\kappa^2(S_n) \leq 6(n-3)$. Let $F = N(G_1)$, where G_1 is a sub-star graph of dimension 3, say the subgraph of S_n induced by $\{(i_1, i_2, i_3, 4, 5, 6, \ldots, n) \mid (i_1, i_2, i_3)$ ranges over all permutations of $\{1, 2, 3\}\}$. Then $S_n - F$ is disconnected, and $|F| = 6(n-3)$ (since $g(S_n) = 6$, no two vertices in G_1 have a same neighbor in F). Furthermore, every vertex in $S_n - F$ has at least two good neighbors. This is true for vertex in G_1 because G_1 is 2-regular. For every vertex $v = (p_1, p_2, p_3, \ldots p_n)$ in $S_n - G_1 - F$, suppose $v(1i) \in F$. If $i \geq 4$, then $\{p_2, p_3\} \subseteq \{1, 2, 3\}$, and thus $v \in F$, a contradiction. So, $i = 2$ or 3. If $v(12)$ and $v(13)$ are both in F, then $\{p_1, p_2, p_3\} = \{1, 2, 3\}$, contradicting that $v \notin G_1$. So, v is adjacent to at most one good neighbor in F. Since the regularity of S_n is $n - 1 \geqslant 3$, v has at least two good neighbors. It follows that F is a R^2-vertex-cut and thus $\kappa^2(S_n) \leqslant |F| = 6(n-3)$.

Next we show that $\kappa^2(S_n) \geq 6(n-3)$. For this purpose, we show that for any R^2-vertex-set F with $|F| < 6(n-3)$, $S_n - F$ is still connected. Write $F_i = F \cap S^i_{n-1}$. We consider two cases.

Case 1. $|F_i| \leq 2n - 7$ for all i.

Note that F_i is an R^1-vertex-set of S^i_{n-1}. Since $\kappa^1(S_{n-1}) = 2(n-1) - 4 = 2n - 6$ by Lemma 1, we see that $S^i_{n-1} - F_i$ is connected for every i. Suppose there are two vertices $u, v \in V(S_n - F)$ which are disconnected by F. Then u, v belong to different copies of S^i_{n-1}, say $u \in S^j_{n-1}$ and $v \in S^k_{n-1}$ for $j \neq k$.

Note that $N(S^j_{n-1} - F_j) \cap V(S^k_{n-1}) \subseteq F_k$, since otherwise $S^j_{n-1} - F_j$ could be connected with $S^k_{n-1} - F_k$ through a vertex in $N(S^j_{n-1} - F_j) \cap V(S^k_{n-1} - F_k)$. By Lemma 4, we have $(n-2)! - (2n-7) \leq (n-2)! - |F_j| \leq |F_k| \leq 2n-7$, which is impossible for $n \geq 6$. So, $n = 4$ or 5. In both cases, the above inequalities become equalities. In particular, $|F_k| = |F_j| = 2n - 7$, $N(S^j_{n-1} - F_j) \cap V(S^k_{n-1}) = F_k$ and $N(S^k_{n-1} - F_k) \cap V(S^j_{n-1}) = F_j$ (notice the symmetry of j and k). So vertices in F_j have the form $(k, \ldots, j)$ and vertices in F_k have the form $(j, \ldots, k)$. Let i be an integer different from j, k. Then the $(n-2)!$ vertices in $N(S^i_{n-1}) \cap V(S^j_{n-1})$ are all good. Since $|N(S^i_{n-1}) \cap V(S^j_{n-1})| = (n-2)! > 2n-7 \geq |F_i|$, we see that $S^j_{n-1} - F_j$ and $S^i_{n-1} - F_i$ are connected. Similarly, $S^k_{n-1} - F_k$ and $S^i_{n-1} - F_i$ are connected. But then u is connected to v through $S^i_{n-1} - F_i$, a contradiction.

Case 2. $|F_i| \geq 2n - 6$ for some i.

Define $I = \{i \mid |F_i| \geq 2n - 6\}$. Since $|F| < 6(n-3)$, we have $|I| \leq 2$. Note that for any $j \notin I$, $S^j_{n-1} - F_j$ is connected.

First we claim that the subgraph of $S_n - F$ induced by $\bigcup_{j \notin I}(V(S^j_{n-1}) - F_j)$, denoted by $\tilde{G}$, is connected. Suppose this is not true. Then there exist two indices $j, k \notin I$ and two vertices $u \in S^j_{n-1} - F_j$ and $v \in S^k_{n-1} - F_k$, such that there is no path from u to v in $S_n - F$. Like in the deduction of Case 1, this is possible only for $n = 4$ or $n = 5$, and $|F_k| = |F_j| = 2n - 7$. If there is an index $\ell \notin I$ such that $\ell \neq j, k$, then also by an argument similar to that in Case 1, u and v are connected through the connected subgraph $S^\ell_{n-1} - F_\ell$. So, we may assume that $n = 4$ and $|I| = 2$. But then $|F| \geq 2(2n-6) + 2(2n-7) = 6(n-3) > |F|$, a contradiction.

Next, we show that any connected component C of $S_n[\bigcup_{i \in I}(V(S^i_{n-1}) - F_i)]$ is connected to $\tilde{G}$. If there is a good vertex $v \in N(C)$, then $v \in V(\tilde{G})$ and we are done. So, suppose $N(C) \subseteq F$.

For simplicity of notation, suppose $I = \{1\}$ if $|I| = 1$, and $I = \{1, 2\}$ if $|I| = 2$.

First, consider the case where C is completely contained in, say, $S^1_{n-1} - F_1$. By the assumption $N(C) \subseteq F$, every outside neighbor v' of a vertex $v \in V(C)$ is faulty, and thus all good neighbors of v are in C. It follows that $\delta(C) \geq 2$, and hence C has a cycle D. By Lemma 2, the length of D is at least 6. Let $u_1, \ldots, u_6$ be six vertices on D, sequentially. Since $g(S_n) = 6$ and S_n is bipartite (so there is no odd cycle in S_n), the only pairs of vertices that may have a common neighbor are $\{u_1, u_5\}$ and $\{u_2, u_6\}$. It follows that among $\{u'_1, \ldots, u'_6\}$, only u'_1 may coincide with u'_5, or u'_2 may coincide with u'_6, in which case a 6-cycle goes through them. But by the structure of 6-cycles in Lemma 2, this is impossible. Hence $u'_1, \ldots, u'_6$ are all distinct, and thus $F' = F - \{u'_1, \ldots, u'_6\}$ satisfies $|F'| \leq 6(n-3) - 1 - 6 = 6n - 25$. Also by Lemma 2, if u_1, u_5 have a common neighbor, then u_2, u_6 do not have, and vice versa. So, $X = (N(u_1) \cap N(u_5)) \cup (N(u_2) \cap N(u_6))$ satisfies $|X| \leq 1$. Furthermore, if u_1 is adjacent to u_6, then $|X| = 0$. Write $Y = N(\{u_1, \ldots, u_6\}) \cap V(S^1_{n-1})$. Then $|Y| \geq 6n - 24 > |F'|$. So there is at least one good vertex v in Y whose outside neighbor v' is also good (note that the correspondence between outside neighbors and the vertices in Y is one to one), a contradiction.

Next, suppose $V(C) \cap (V(S^i_{n-1}) - F_i) \neq \emptyset$ holds for $i = 1, 2$. In this case, we can find a path $u_1 u_2 \ldots u_6$ in C with $u_1, u_2, u_3 \in V(S^1_{n-1}) - F_1$, $u_4, u_5, u_6 \in V(S^2_{n-1}) - F_2$, and $\{u'_1, u'_2, u'_5, u'_6\} \subseteq F$. In fact, let $u_3 u_4$ be an edge of S_n with $u_3 \in V(S^1_{n-1}) \cap V(C)$ and $u_4 \in V(S^2_{n-1}) \cap V(C)$. Let u_2 be another good neighbor of u_3. Then $u_2 \in V(S^1_{n-1}) \cap V(C)$. By Lemma 3(1), $u'_2 \notin V(S^2_{n-1})$; hence $u'_2 \in N(C)$ is faulty. Let u_1 be another good neighbor of u_2 different from u_3. Also by Lemma 3(1), $u_1 \in V(S^1_{n-1}) \cap V(C)$ and $u'_1 \in F$. Similarly, u_5 and u_6 can be found as required.

Let $X_1 = N(\{u_1, u_2, u_3\}) \cap V(S^1_{n-1})$, $X_2 = N(\{u_4, u_5, u_6\}) \cap V(S^2_{n-1})$, and $X = X_1 \cup X_2$. By Lemma 3(2), at most one outside neighbor of X_1 may be in S^2_{n-1}, and at most one outside neighbor of X_2 may be in S^1_{n-1}. Define $F' = F - \{u'_1, u'_2, u'_5, u'_6\}$. Then $|F'| \leq 6(n-3) - 1 - 4 = 6n - 23 < 6n - 22 = 2(3(n-2) - 4) - 2 = |X| - 2$. So, there is at least one good vertex $v \in X$ whose outside neighbor $v' \in V(S_n) - \bigcup_{i=1,2} V(S^i_{n-1})$ is also good. Since $v' \in N(C)$, we have arrived at the contradiction that $N(C) \subseteq F$. $\square$

4. Conclusion

In this work, we have proved that the κ^2-vertex-connectivity of the n-dimensional star graph is $\kappa^2(S_n) = 6(n-3)$. Note that this value is exactly $|N(S_3)|$. Combining this with the observation that $\kappa^0(S_n) = n-1 = N(S_1)$ and $\kappa^1(S_n) = 2(n-2) = |N(S_2)|$, we may guess that $\kappa^k(S_n) = (k+1)!(n-1-k) = |N(S_{k+1})|$.

Acknowledgements

We would thank the anonymous referees for their valuable suggestions for improving the clarity and style of this work. The second author's research is supported by NSFC (60603003) and XJEDU.

References

[1] S.B. Akers, D. Harel, B. Krishnamurthy, The star graph: An attractive alternative to the n-cube, Proc. Int. Conf. Parallel Process. (1987) 393–400.
[2] S.B. Akers, B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, IEEE Trans. Comput. 38 (4) (1989) 555–566.
[3] J.A. Bondy, U.S.R. Murty, Graph Theory with Application, Macmillan, London, 1976.
[4] K. Day, A. Tripathi, A comparative study of topological properties of hypercubes and star graphs, IEEE Trans. Comput. 5 (1) (1994) 31–38.
[5] F. Harary, Conditional connectivity, Networks 13 (1983) 347–357.
[6] M.C. Heydemann, B. Ducourthial, Cayley graphs and interconnection networks, in: G. Hahn, G. Sabidussi (Eds.), Graph Symmetry (Montreal, PQ, 1996), NATO Advanced Science Institutes Series C, in: Mathematica and Physical Sciences, vol. 497, Kluwer Academic Publishers, Dordrecht, 1997, pp. 167–224.
[7] S.C. Hu, C.B. Yang, Fault tolerance on star graphs, in: Proceedings of the First Aizu International Symposium on Parallel Algorithms/Architecture Synthesis, 1995, pp. 176–182.
[8] S. Latifi, M. Hegde, M. Naraghi-Pour, Conditional connectivity measures for large multiprocessor systems, IEEE Trans. Comput. 43 (2) (1994) 218–222.
[9] W. Najjar, J.L. Gaudiot, Network resilience: A measure of network fault tolerance, IEEE Trans. Comput. C-39 (2) (1990) 174–181.