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The Decomposition Method for Cauchy Advection-Diffusion Problems

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Abstract—In this paper, the solution of Cauchy problems for the advection-diffusion equation is obtained using the decomposition method. In the case when the flow velocity is constant, an analytical solution can be derived, whilst for variable flow velocity, symbolic numerical computations need to be performed.

Keywords—Characteristic and noncharacteristic Cauchy problems, Inverse problems, Advection-diffusion equation, Decomposition method, Variable flow velocity.

1. INTRODUCTION

Inverse problems arise in advective (with prescribed flow velocity) or convective (with flow velocity unknown) diffusive flows when analysing the mechanism governing the release of hormones from secretory cells in response to a stimulus in a medium flowing past the cells and through a diffusion column, [1]. We presume that the layer of diffusion of length $l > 0$ (here, one-dimensional for simplicity, but the same question arises with more than one space dimension) is accompanied by forced convection (here, presumed constant, again for simplicity) and so the concentration $C(x,t)$ (of hormone) satisfies the advection-diffusion equation,

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2}, \qquad 0 < x < l, \quad t > 0, \tag{1}$$

where x is the distance measured down the column, t the time, u is the flow velocity down the column and $D > 0$ (constant) the diffusion coefficient. A similar situation arises in the forced convection cooling of flat electronic substrates, [2], or in the dispersion of pollutants in rivers.

At the initial time $t = 0$, the concentration can be prescribed as $h(x)$, i.e.,

$$C(x,0) = h(x), \qquad 0 \le x \le l. \tag{2}$$

Sometimes $h(x)$ can be taken to be uniform, i.e., a constant h_0, and further this constant can be taken to be zero, otherwise, we can work with the translated concentration $C - h_0$. With a finite

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column $(0, l)$, as presumed here, we take a prescribed flux condition at the bottom of the column $x = l$, i.e.,

$$D \frac{\partial C}{\partial x}(l,t) = g_l(t), \qquad t > 0, \tag{3}$$

where $g_l(t)$ is usually taken to be zero, i.e., zero-flux condition. Then, (3) becomes $\frac{\partial C}{\partial x}(l,t) = 0$, which is an insulation condition referred to as "Danckwert's boundary condition" and expresses the fact that there is purely advective flux out of the reaction section, [3]. The insulation condition, $\frac{\partial C}{\partial x}(l,t) = 0$ is valid in the double limit of slow time-dependences and of weak chemical reactions, [4].

By balancing the concentration flow across the inlet $x = 0$, we have [3]

$$\chi(t) = uC(0,t) - D \frac{\partial C}{\partial x}(0,t), \qquad t > 0, \tag{4}$$

where, in the elution chromatography experiments described in [1], $\chi(t)$ is the concentration-time profile of the hormone secreted at the top of the column $x = 0$. In these experiments the question arises as to the possibility of using the measured value of the concentration-time profile at the bottom of the layer, i.e.,

$$C(l,t) = f_l(t), \qquad t > 0, \tag{5}$$

to determine $C(0,t)$ and $\frac{\partial C}{\partial x}(0,t)$, or, with (4) in mind, to determine $\chi(t)$. The latter distinction is less important as described below, and we let

$$C(0,t) = f_0(t), \quad -D \frac{\partial C}{\partial x}(0,t) = g_0(t), \qquad t \geq 0. \tag{6}$$

To impose the three conditions (2), (3), and (5) is an overspecification and if all three conditions are applied, they must necessarily be consistent.

In this paper, we consider the solutions of the noncharacteristic Cauchy problem (1), (3), and (5), and the characteristic Cauchy problem (1) and (2), by the decomposition method as follows.

2. THE DECOMPOSITION METHOD

If we define the partial differential operators $L_t = \frac{\partial}{\partial} t$, $L_x = \frac{\partial}{\partial} x$, and $L_{xx} = \frac{\partial^2}{\partial x^2}$, then, equation (1) can be rewritten as

$$DL_{xx}(C) = L_t(C) + uL_x(C). \tag{7}$$

Let us define formally the inverse operators [5]

$$L_t = \int_0^t dt', \qquad L_{xx}^{-1} = \int_l^x dx' \int_l^{x'} dx''. \tag{8}$$

We seek the solution of equation (7) as given by the decomposition series,

$$C = \sum_{n=0}^{\infty} C_n, \tag{9}$$

where the sequence $\{C_n\}_{n \geq 0}$ satisfies

$$\begin{aligned} C_0(x,t) &= f_l(t) + \frac{x-l}{D} g_l(t), \\ C_{n+1}(x,t) &= \frac{1}{D} L_{xx}^{-1} \left[L_t(C_n(x,t)) + u(x,t) L_x(C_n(x,t)) \right], \qquad n \geq 0, \end{aligned} \tag{10}$$

for the noncharacteristic Cauchy problem (3), (5), and (7), and

$$\begin{aligned} C_0(x,t) &= h(x), \\ C_{n+1}(x,t) &= L_t^{-1}\left[DL_{xx}(C_n(x,t)) - u(x,t)\,L_x(C_n(x,t))\right], \qquad n \geq 0. \end{aligned} \tag{11}$$

for the characteristic Cauchy problem (2) and (7). If all the conditions (2), (3), and (5) associated to (7) are specified, then, one can add (10) and (11) and divide by two, [6]. We only remark that if a direct problem, as that given by (1)–(3) and $(6)_1$ has to be solved, then, the decomposition method usually fails, [7]. It should also be stressed that the characteristic Cauchy problem is actually solved by assuming that the initial condition (2) is valid on the whole of the real axis, which is well-posed (for initial data $h(x)$, satisfying certain growth conditions). On the other hand, the noncharacteristic Cauchy problem (1), (3), and (5), often called the "sideways" problem is ill-posed and, even if the solution is unique, is unstable, i.e., small changes in the input data at $x = l$ will produce large changes in the desired solution at $x = 0$. Hence, in practice one has to mollify the data f_l and g_l, as described in [8], for $u = 0$, and in [9], for u =constant, prior to applying the decomposition method.

In the case u =constant, see Section 2.1, an analytical solution can be derived, whilst in the case $u \neq$constant, see Section 2.2, symbolic numerical computations need to be performed.

2.1. The Case u = constant

In this case, assuming conditions of Holmgren class two functions, [10], for the Cauchy data (f_l, g_l), the noncharacteristic Cauchy problem (1), (3), and (5) has formally the unique solution [11]

$$\begin{aligned} C(x,t) &= \sum_{n=0}^{\infty} \frac{(l-x)^{2n}}{D^n (2n)!}\, {}_1F_1\left(n; 2n+1; -\frac{u(l-x)}{D}\right) f_l^{(n)}(t) \\ &- \sum_{n=0}^{\infty} \frac{(l-x)^{2n+1}}{D^{n+1}(2n+1)!}\, {}_1F_1\left(n+1; 2n+2; -\frac{u(l-x)}{D}\right) g_l^{(n)}(t), \end{aligned} \tag{12}$$

where ${}_1F_1(a; b; X)$ is the confluent hypergeometric function, which, for $b \neq 0, -1, -2, \dots$, is given by [12, p. 276],

$${}_1F_1(a; b; X) = 1 + \sum_{k=0}^{\infty} \left(\prod_{j=0}^{k} \left(\frac{a+j}{b+j}\right)\right) \frac{X^{k+1}}{(k+1)!}. \tag{13}$$

In order to obtain the solution of the characteristic Cauchy problem (1) and (2), we follow the lead suggested by putting ϵ in the terms $\frac{\partial C}{\partial x}$ and $\frac{\partial^2 C}{\partial x^2}$ in (1) and seek

$$C(x,t) = \sum_{n=0}^{\infty} c_n(x,t)\,\epsilon^n. \tag{14}$$

Equating powers of ϵ, we obtain

$$\frac{\partial c_0}{\partial t} = 0, \qquad D\frac{\partial^2 c_{n-1}}{\partial x^2} - u\frac{\partial c_{n-1}}{\partial x} = \frac{\partial c_n}{\partial t}, \qquad n \geq 1. \tag{15}$$

Writing $c_n(x,t) = a_n(x) b_n(t)$, the differential-difference equation (15) separates and gives

$$b_0'(t) = 0, \qquad \frac{Da_{n-1}''(x) - ua_{n-1}'(x)}{a_n(x)} = \frac{b_n'(t)}{b_{n-1}(t)} = \lambda_n, \qquad n \geq 1 \tag{16}$$

where λ_n, for $n \geq 1$ are constants. Choosing $a_0(x) = h(x)$, which, based on (2), requires $b_0(0) = 1$, $b_n(0) = 0$, for $n \geq 1$, and solving equation (16), we obtain

$$\begin{aligned} b_0(t) &= 1, \qquad b_n(t) = \frac{t^n}{n!}\left(\prod_{j=1}^{n}\lambda_j\right), \qquad n \geq 1, \\ a_0(x) &= h(x), \qquad a_n(x) = \left(\prod_{j=1}^{n}\lambda_j\right)^{-1}\sum_{k=0}^{n} D^{n-k}(-u)^k C_n^k h^{(2n-k)}(x), \quad n \geq 1, \end{aligned} \tag{17}$$

where $C_n^k = n!/k!(n-k)!$. Putting $\epsilon = 1$ into (14), we obtain the explicit solution,

$$C(x,t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\left[\sum_{k=0}^{n} D^{n-k}(-u)^k C_n^k h^{(2n-k)}(x)\right]. \tag{18}$$

When $u = 0$, i.e., no convection, equation (1) becomes the classical heat conduction equation and then, (12) and (18) simplify to [13]

$$C(x,t)\,|_{u=0} = \sum_{n=0}^{\infty}\left[\frac{(l-x)^{2n}}{D^n(2n)!}f_l^{(n)}(t) - \frac{(l-x)^{2n+1}}{D^{n+1}(2n+1)!}g_l^{(n)}(t)\right], \tag{19}$$

$$C(x,t)\,|_{u=0} = \sum_{n=0}^{\infty}\frac{D^n t^n}{n!}h^{(2n)}(x). \tag{20}$$

Similar expressions are available for the one-dimensional diffusion of heat perpendicular to the surfaces of coaxial cylinders and concentric spheres [14].

Alternatively, when $u =$ constant, we can obtain equivalent expressions for the solution $C(x,t)$ by employing the transformation,

$$C(x,t) = V(x,t)\exp\left(\frac{ux}{2D} - \frac{u^2t}{4D}\right). \tag{21}$$

Then, equations (1)–(6) become

$$\frac{\partial V}{\partial t} = D\frac{\partial^2 V}{\partial x^2}, \qquad 0 < x < l, \quad t > 0, \tag{22}$$

$$V(x,0) = h(x)\exp\left(\frac{-ux}{(2D)}\right) := \theta(x), \qquad 0 \leq x \leq l, \tag{23}$$

$$D\frac{\partial V}{\partial x}(l,t) + \frac{u}{2}V(l,t) = g_l(t)\exp\left(\frac{u^2t}{4D} - \frac{Pe}{2}\right), \qquad t > 0, \tag{24}$$

$$\chi(t)\exp\left(\frac{u^2t}{(4D)}\right) = \frac{u}{2}V(0,t) - D\frac{\partial V}{\partial x}(0,t), \qquad t > 0, \tag{25}$$

$$V(l,t) = f_l(t)\exp\left(\frac{u^2t}{4D} - \frac{Pe}{2}\right) := \phi_l(t), \qquad t > 0, \tag{26}$$

$$\begin{aligned} V(0,t) &= f_0(t)\exp\left(\frac{u^2t}{(4D)}\right), \\ -D\frac{\partial V}{\partial x}(0,t) - \frac{u}{2}V(0,t) &= g_0(t)\exp\left(\frac{u^2t}{(4D)}\right), \qquad t \geq 0 \end{aligned} \tag{27}$$

where $Pe = ul/d$ is the longitudinal Peclet number. From (24) and (26), we obtain

$$D\frac{\partial V}{\partial x}(l,t) = \left(g_l(t) - \frac{u}{2}f_l(t)\right)\exp\left(\frac{u^2t}{4D} - \frac{Pe}{2}\right) := \psi_l(t), \qquad t > 0. \tag{28}$$

Then, applying formulae (19) and (20) to the Cauchy problems (22), (26), and (28) and (22) and (23), respectively, we obtain, via (21),

$$C(x,t) = \exp\left(\frac{ux}{2D} - \frac{u^2 t}{4D}\right) \sum_{n=0}^{\infty} \left[\frac{(l-x)^{2n}}{D^n (2n)!} \phi_l^{(n)}(t) - \frac{(l-x)^{2n+1}}{D^{n+1}(2n+1)!} \psi_l^{(n)}(t)\right], \tag{29}$$

$$C(x,t) = \exp\left(\frac{ux}{2D} - \frac{u^2 t}{4D}\right) \sum_{n=0}^{\infty} \frac{D^n t^n}{n!} \theta^{(2n)}(x). \tag{30}$$

Differentiating using Leibniz's rule and equations (23), (26), and (28), we obtain

$$\begin{aligned} \phi_l^{(n)}(t) &= \exp\left(\frac{u^2 t}{4D} - \frac{Pe}{2}\right) \sum_{k=0}^{n} \frac{u^{2k}}{(4D)^k} C_n^k f_l^{(n-k)}(t), \\ \psi_l^{(n)}(t) &= \exp\left(\frac{u^2 t}{4D} - \frac{Pe}{2}\right) \sum_{k=0}^{n} \frac{u^{2k}}{(4D)^k} C_n^k \left(g_l^{(n-k)}(t) - \frac{u}{2} f_l^{(n-k)}(t)\right), \end{aligned} \tag{31}$$

and

$$\theta^{(2n)}(x) = \exp\left(-\frac{ux}{2D}\right) \sum_{n=0}^{2n} \left(-\frac{u}{2D}\right)^k C_{2n}^k h^{(2n-k)}(x). \tag{32}$$

Introducing (31) and (32) into (29) and (30), respectively, we obtain

$$\begin{aligned} C(x,t) = {} & \exp\left(\frac{u(x-l)}{2D}\right) \sum_{n=0}^{\infty} \left[\left(\frac{(l-x)^{2n}}{D^n (2n)!} + \frac{u(l-x)^{2n+1}}{2D^{n+1}(2n+1)!}\right) \sum_{k=0}^{n} \frac{u^{2k}}{(4D)^k} C_n^k f_l^{(n-k)}(t)\right] \\ & - \exp\left(\frac{u(x-l)}{2D}\right) \sum_{n=0}^{\infty} \left[\frac{(l-x)^{2n+1}}{D^{n+1}(2n+1)!} \sum_{k=0}^{n} \frac{u^{2k}}{(4D)^k} C_n^k g_l^{(n-k)}(t)\right] \end{aligned} \tag{33}$$

and

$$C(x,t) = \exp\left(-\frac{u^2 t}{4D}\right) \sum_{n=0}^{\infty} \left[\frac{D^n t^n}{n!} \sum_{k=0}^{2n} \left(-\frac{u}{2D}\right)^k C_{2n}^k h^{(2n-k)}(x)\right]. \tag{34}$$

Again, when $u = 0$, equations (33) and (34) reduce to (19) and (20), respectively.

Equation (29) can also be derived by using the decomposition method as follows. By defining the initial starting term,

$$V_0(x,t) = \phi_l(t) + \frac{(x-l)}{D} \psi_l(t), \tag{35}$$

we can seek the solution $V(x,t)$ of the inverse noncharacteristic Cauchy problem (22), (26), and (28) based on Adomian's decomposition approach, as

$$V = \sum_{n=0}^{\infty} V_n, \tag{36}$$

where

$$V_{n+1} = \frac{1}{D} L_{xx}^{-1}\left[L_t(V_n)\right], \qquad n \geq 0. \tag{37}$$

The recurrence relation (37) with the starting term (35), successively yields

$$\begin{aligned} V_1 &= \frac{1}{D} L_{xx}^{-1}\left[L_t(V_0)\right] = \frac{(x-l)^2}{2!D} \phi_l'(t) + \frac{(x-l)^3}{3!D^2} \psi_l'(t), \\ V_2 &= \frac{1}{D} L_{xx}^{-1}\left[L_t(V_1)\right] = \frac{(x-l)^4}{4!D^2} \phi_l''(t) + \frac{(x-l)^5}{5!D^3} \psi_l''(t), \end{aligned} \tag{38}$$

and, in general,

$$V_n = \frac{1}{D} L_{xx}^{-1} \left[L_t \left(V_{n-1} \right) \right] = \frac{(x-l)^{2n}}{(2n)! D^n} \phi_l^{(n)} (t) + \frac{(x-l)^{2n+1}}{(2n+1)! D^{n+1}} \psi_l^{(n)} (t), \qquad n \geq 0. \tag{39}$$

Based on (36) and (39), via (21), we obtain the general representation of the solution (29).

Similarly, in order to obtain (30) using the decomposition method, we define [11] the initial starting term,

$$V_0 (x,t) = \theta (x), \tag{40}$$

and we can seek the solution $V(x,t)$ of the characteristic Cauchy problem (22) and (23) in the form of series (36), where

$$V_{n+1} = D L_t^{-1} \left[L_{xx} \left(V_n \right) \right], \qquad n \geq 0. \tag{41}$$

The recurrence relation (41) with the starting term (40) successively yields

$$\begin{aligned} V_1 &= D L_t^{-1} \left[L_{xx} \left(V_0 \right) \right] = D t \theta'' (x), \\ V_2 &= D L_t^{-1} \left[L_{xx} \left(V_1 \right) \right] = D^2 \frac{t^2}{2!} \theta'''' (x), \end{aligned} \tag{42}$$

and, in general,

$$V_n = D L_t^{-1} \left[L_{xx} \left(V_{n-1} \right) \right] = D^n \frac{t^n}{n!} \theta^{(2n)} (x), \qquad n \geq 0. \tag{43}$$

Based on (36) and (43), via (21), we obtain the general representation of the solution (30).

At this stage, we remark that the decomposition method can be applied directly to equation (1) using equations (9)–(11), rather than to equation (22) obtained via the transformation (21). For this, for the noncharacteristic Cauchy problem (1), (3), and (5), we define the initial starting term,

$$C_0 (x,t) = \left[1 - \frac{u}{D} (x-l) \right] f_l (t) + \frac{x-l}{D} g_l (t), \tag{44}$$

and then, the recurrence relation (10) for $u =$ constant recasts as

$$C_{n+1} = \frac{1}{D} \left(L_{xx}^{-1} L_t + u L_x^{-1} \right) C_n = \frac{1}{D} L_x^{-1} \left(L_x^{-1} L_t + u I \right) C_n, \qquad n \geq 0, \tag{45}$$

where $L_x^{-1} = \int_l^x dx'$ and I is the identity operator. The recurrence relation (45) with the starting term (44) yields, successively,

$$\begin{aligned} C_1 &= \left[\frac{u(x-l)}{D} - \frac{u^2 (x-l)^2}{D^2 2!} \right] f_l (t) + \frac{u (x-l)^2}{D^2 2!} g_l (t) \\ &\quad + \left[\frac{(x-l)^2}{D 2!} - \frac{u (x-l)^3}{D^2 3!} \right] f_l' (t) + \frac{(x-l)^3}{D^2 3!} g_l' (t), \\ C_2 &= \left[\frac{u^2 (x-l)^2}{D^2 2!} - \frac{u^3 (x-l)^3}{D^3 3!} \right] f_l (t) + \frac{u^2 (x-l)^3}{D^3 3!} g_l (t) \\ &\quad + 2 \left[\frac{u (x-l)^3}{D^2 3!} - \frac{u^2 (x-l)^4}{D^3 4!} \right] f_l' (t) + 2 \frac{u (x-l)^4}{D^3 4!} g_l' (t) \\ &\quad + \left[\frac{(x-l)^4}{D^2 4!} - \frac{u (x-l)^5}{D^3 5!} \right] f_l'' (t) + \frac{(x-l)^5}{D^3 5!} g_l'' (t), \end{aligned} \tag{46}$$

etc., and, in general, using MAPLE, one can obtain that series (9) is equivalent to (12).

Similarly, for the characteristic Cauchy problem (1) and (2), defining the recurrence relation,

$$C_0(x,t) = h(x), \quad C_{n+1} = L_t^{-1}(DL_{xx} - uL_x)C_n, \qquad n \geq 0, \tag{47}$$

we obtain

$$C_1 = t(Dh''(x) - uh'(x)), \qquad C_2 = \frac{t^2}{2}\left(D^2h''''(x) - 2Duh'''(x) + u^2h''(x)\right), \tag{48}$$

etc., and based on (9), we retrieve the series expansion (18).

EXAMPLE 1. CASE u = CONSTANT. Let us consider a one-dimensional advection-diffusion experiment with $D = u = 1$ within a sample of length $l = 1$. Consider first the noncharacteristic Cauchy problem (1), (3), and (5) in which the concentration $f_1(t) = e - 1 + t$ and the flux $g_1(t) = e - 1$ are both prescribed at the end $x = 1$, and second, the characteristic Cauchy problem (1),(2) in which only the initial concentration at $t = 0$ is prescribed as $h(x) = e^x - x$. For both problems, the exact solution is given by $C(x,t) = e^x - x + t$.

For the first problem, based on equations (26) and (28), we have

$$\phi_1(t) = (e - 1 + t)\exp\left(\frac{t}{4} - \frac{1}{2}\right), \qquad \psi_1(t) = \frac{(e-1-t)}{2}\exp\left(\frac{t}{4} - \frac{1}{2}\right), \tag{49}$$

and, in general, using Leibniz's rule of differentiation,

$$\begin{aligned} \phi_1^{(n)}(t) &= \frac{(e-1+4n+t)}{4^n}\exp\left(\frac{t}{4} - \frac{1}{2}\right), \\ \psi_1^{(n)}(t) &= \frac{(e-1-4n-t)}{2\cdot 4^n}\exp\left(\frac{t}{4} - \frac{1}{2}\right), \quad n \geq 0. \end{aligned} \tag{50}$$

Introducing these expressions into (29), we obtain

$$\begin{aligned} C(x,t) &= \exp\left(\frac{x-1}{2}\right)\sum_{n=0}^{\infty} 4^{-n} \\ &\quad \times \left[\frac{(1-x)^{2n}(e-1+4n+t)}{(2n)!} - \frac{(1-x)^{2n+1}(e-1-4n-t)}{(2n+1)!\cdot 2}\right] \\ &= t + \exp\left(\frac{x-1}{2}\right)\sum_{n=0}^{\infty}\frac{4^{-n}(1-x)^{2n}}{(2n)!}\left[e - 1 + 4n - \frac{(1-x)(e-1-4n)}{2(2n+1)}\right] \\ &= e^x - x + t, \end{aligned} \tag{51}$$

as required.

Alternatively, we can use the decomposition series (9) with the terms C_n given by the recurrence relations (44) and (45). Since $f_1(t) = e - 1 + t$, $g_1(t) = e - 1$, $f_1'(t) = 1$, $f_1^{(n)}(t) = g_1^{(n-1)}(t) = 0$, for $n \geq 2$, from (44) and (46) by cancelling some terms in (9), we obtain

$$\begin{aligned} C = \sum_{n=0}^{\infty} C_n &= f_1(t) + g_1(t)\sum_{k=1}^{\infty}\frac{(x-1)^k}{k!} + f_1'(t)\sum_{k=2}^{\infty}\frac{(x-1)^k}{k!} \\ &= e - 1 + t + (e-1)\left(e^{x-1} - 1\right) + \left(e^{x-1} - 1 - (x-1)\right) = e^x - x + t, \end{aligned} \tag{52}$$

as required.

For the second problem, based on equation (23), we have

$$\theta(x) = e^{x/2} - xe^{-x/2} \tag{53}$$

and, in general,

$$\theta^{(n)}(x) = 2^{-n}\left(e^{x/2} + (2n - x)\,e^{-x/2}\right), \qquad n \geq 0. \tag{54}$$

Introducing (54) into (30), we obtain

$$\begin{aligned} C(x,t) &= e^{(x/2-t/4)}\sum_{n=0}^{\infty}\frac{4^{-n}t^n}{n!}\left(e^{x/2} + (4n - x)\,e^{-x/2}\right) \\ &= e^x - x + e^{-t/4}\sum_{n=1}^{\infty}\frac{4^{n-1}t^n}{(n-1)!} = e^x - x + t, \end{aligned} \tag{55}$$

as required.

Alternatively, we can use the decomposition series (9) with the terms C_n given by the recurrence relation (47). Since $h(x) = e^x - x$, $h'(x) = e^x - 1$, and $h^{(n)}(x) = e^x$, for $n \geq 2$, from (47) and (48), we obtain $C_0 = e^x - x$, $C_1 = t(e^x - e^x + 1) = t$, $C_2 = (t^2/2)(e^x - 2e^x + e^x) = 0$, and in general, $C_n = 0$, for all $n \geq 2$. Then, (9) gives $C = C_0 + C_1 = e^x - x + t$. Thus, the decomposition method yields the exact solution within only two terms of the series expansion (9) showing a much faster rate of convergence than (55).

2.2. The Case $u \neq$ constant

When u is spacewise and/or time-dependent one cannot apply the transformation (21), however, the decomposition method based on equations (9)–(11) can still be made applicable.

EXAMPLE 2. CASE $u = u(x)$. We take $D = l = 1$, $u(x) = 2(x-1)$, and the exact solution $C(x,t) = e^{(x-1)^2+2t}$.

For the problem (1), (3), and (5), $f_1(t) = e^{2t}$, $g_1(t) = 0$, and the recurrence relation (10) gives $C_0(x,t) = e^{2t}$, $C_1(x,t) = (x-1)^2e^{2t}$, $C_2(x,t) = (x-1)^4/2e^{2t}$, and in general, $C_n(x,t) = (x-1)^{2n}/n!e^{2t}$. The decomposition series (9) yields immediately the exact solution.

For the problem (1) and (2), $h(x) = e^{(x-1)^2}$ and the recurrence relation (11) gives $C_0(x,t) = e^{(x-1)^2}$, $C_1(x,t) = 2te^{(x-1)^2}$, $C_2(x,t) = (2t)^2/2e^{(x-1)^2}$, and in general, $C_n(x,t) = (2t)^n/n!e^{(x-1)^2}$. The decomposition series (9) yields immediately the exact solution.

EXAMPLE 3. CASE $u = u(t)$. We take $D = l = 1$, $u(t) = 1 - 2t$, and the exact solution $C(x,t) = e^{x-1+t^2}$.

For the problem (1), (3), and (5), $f_1(t) = g_1(t) = e^{t^2}$, and the recurrence relation (10) gives

$$\begin{aligned} C_0 &= (1 + (x-1))\,e^{t^2}, \\ C_1 &= \left[\frac{(x-1)^2}{2!} + 2t\frac{(x-1)^3}{3!}\right]e^{t^2}, \\ C_2 &= \left[(1-2t)\frac{(x-1)^3}{3!} + \left(4t - 4t^2\right)\frac{(x-1)^4}{4!} + \left(2 + 4t^2\right)\frac{(x-1)^5}{5!}\right]e^{t^2}, \\ C_3 &= \left[\left(1 - 4t + 4t^2\right)\frac{(x-1)^4}{4!} + \left(-2 + 6t - 16t^2 + 8t^3\right)\frac{(x-1)^5}{5!}\right]e^{t^2} \\ &\quad + \left[\left(6 - 12t + 12t^2 - 16t^3\right)\frac{(x-1)^6}{6!} + \left(12t + 8t^2\right)\frac{(x-1)^7}{7!}\right]e^{t^2}, \end{aligned} \tag{56}$$

and so on. Summing the series (9), we observe that some alternating terms add and subtract each other resembling the noisy convergence phenomena in the decomposition method [16] and finally yielding

$$C(x,t) = e^{t^2}\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \dots\right] = e^{t^2+x-1}, \tag{57}$$

as required.

For the problem (1),(2), $h(x) = e^{x-1}$ and the recurrence relation (11) gives

$$\begin{aligned} C_0(x,t) &= e^{x-1}, \\ C_1(x,t) &= t^2 e^{x-1}, \\ C_2(x,t) &= \frac{t^4}{2} e^{x-1}, \end{aligned}$$

and in general,

$$C_n(x,t) = \frac{t^{2n}}{n!} e^{x-1}.$$

The decomposition series (9) yields immediately the exact solution.

EXAMPLE 4. CASE $u = u(x,t)$. We take $D = l = 1$, $u(x,t) = 2(x-1) + (1-t)/(x-1)$ and the exact solution $C(x,t) = e^{(x-1)^2+t^2}$.

For the problem (1), (3), and (5),

$$f_1(t) = e^{t^2}, \qquad g_1(t) = 0,$$

and the recurrence relation (10) gives

$$\begin{aligned} C_0 &= e^{t^2}, \\ C_1 &= (x-1)^2 t e^{t^2}, \\ C_2 &= (x-1)^2 \left[\frac{(x-1)^2}{12} \left(1 + 4t + 2t^2\right) + t(1-t) \right] e^{t^2}, \\ C_3 &= (x-1)^2 \left[\frac{(x-1)^4}{180} \left(6 + 19t + 12t^2 + 2t^3\right) \right. \\ &\quad \left. + \frac{(x-1)^2}{36} \left(4 + 9t - 8t^2 - 8t^3\right) + t(1-t)^2 \right] e^{t^2}, \end{aligned} \tag{58}$$

etc., and, in general, for any $n \geq 0$, we observe that

$$C_n(x,t) = e^{t^2} \left(\sum_{k=0}^{n} a_k^n(t) (x-1)^{2k} \right), \tag{59}$$

where $a_k^n(t)$ are functions of t to be determined. From equations (10) and (59), after some calculus, we obtain

$$\begin{aligned} & a_0^0(t) = 1, \qquad a_0^n(t) = 0, \quad a_1^n(t) = t(1-t)^{n-1}, \qquad n \geq 1, \\ a_{k+1}^{n+1}(t) &= \frac{(2t+4k) a_k^n + (2k+2)(1-t) a_{k+1}^n(t) + (a_k^n)'(t)}{(2n+1)(2n+2)}, \qquad n \geq 2, \quad k = \overline{1,(n-1)}, \\ a_{n+1}^{n+1}(t) &= \frac{(2t+4n) a_n^n(t) + (a_n^n)'(t)}{(2n+1)(2n+2)}, \qquad n \geq 1. \end{aligned} \tag{60}$$

From equation (60), we infer that $a_k^n(t)$ is a polynomial of degree n in t, thus, we seek

$$a_k^n(t) = \sum_{l=0}^{n} \alpha(n,k,l) t^l, \qquad k = \overline{0,n}, \tag{61}$$

where $\alpha(n,k,l)$ are unknown coefficients to be determined. From equations (60) and (61), we obtain the first few terms as given by

$$\begin{gathered}
\alpha(0,0,0)=1, \quad \alpha(n,0,l)=0, \quad n\geq 0, \quad l=\overline{0,n}, \\
\alpha(n,1,0)=0, \quad \alpha(n,1,l)=(-1)^{l-1}C_{n-1}^{l-1}, \quad n\geq 1, \quad l=\overline{1,n}, \\
\alpha(2,0,0)=\alpha(2,0,1)=\alpha(2,0,2)=\alpha(2,1,0)=0, \quad \alpha(2,1,1)=1, \\
\alpha(2,1,2)=-1, \quad \alpha(2,2,0)=\frac{1}{12}, \quad \alpha(2,2,1)=\frac{1}{3}, \quad \alpha(2,2,2)=\frac{1}{6},
\end{gathered} \tag{62}$$

and, in general, for $n\geq 2$ and $k,l=\overline{1,(n-1)}$,

$$\begin{aligned}
\alpha(n+1,k+1,0) &= \frac{4k\alpha(n,k,0)+(2k+2)\alpha(n,k+1,0)+\alpha(n,k,1)}{(2k+1)(2k+2)}, \\
\alpha(n+1,k+1,l) &= \frac{2\alpha(n,k,l-1)+4k\alpha(n,k,l)+(l+1)\alpha(n,k,l+1)}{(2k+1)(2k+2)} \\
&\quad + \frac{\alpha(n,k+1,l)-\alpha(n,k+1,l-1)}{2k+1}, \\
\alpha(n+1,k+1,n) &= \frac{2\alpha(n,k,n-1)+4k\alpha(k,n,n)+\alpha(n,k+1,n)}{(2k+1)(2k+2)} \\
&\quad - \frac{\alpha(n,k+1,n-1)}{2k+1}, \\
\alpha(n+1,k+1,n+1) &= \frac{2\alpha(n,k,n)-(2k+2)\alpha(n,k+1,n)}{(2k+1)(2k+2)}, \\
\alpha(n+1,n+1,0) &= \frac{4n\alpha(n,n,0)+\alpha(n,n,1)}{(2n+1)(2n+2)}, \\
\alpha(n+1,n+1,l) &= \frac{4n\alpha(n,n,l)+2\alpha(n,n,l-1)+(l+1)\alpha(n,n,l+1)}{(2n+1)(2n+2)}, \\
\alpha(n+1,n+1,n) &= \frac{4n\alpha(n,n,n)+2\alpha(n,n,n-1)}{(2n+1)(2n+2)}, \\
\alpha(n+1,n+1,n+1) &= \frac{2\alpha(n,n,n)}{(2n+1)(2n+2)}.
\end{aligned} \tag{63}$$

Numerically evaluating the recurrence relation (63) for $\alpha(n,k,l)$, we obtain the general term of the decomposition (9) as

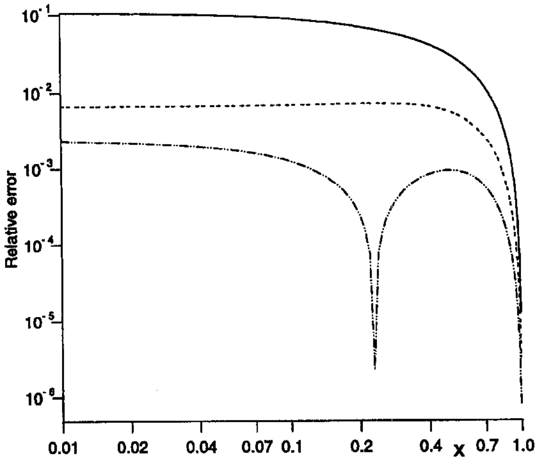
$$C_n(x,t)=e^{t^2}\left(\sum_{k=0}^{n}(x-1)^{2k}\left\{\sum_{l=0}^{n}\alpha(n,k,l)\,t^l\right\}\right). \tag{64}$$

Defining now the sequence of partial sums of the series (9) divided by e^{t^2}, namely,

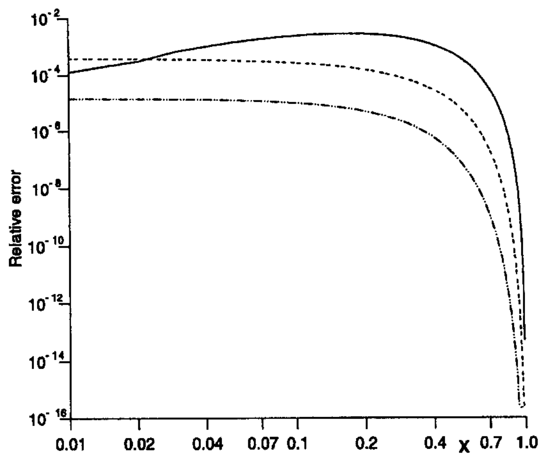
$$S_N(x,t):=e^{-t^2}\sum_{n=0}^{N}C_n(x,t), \tag{65}$$

we enquire whether $S_N(x,t)$ converges to the exact solution,

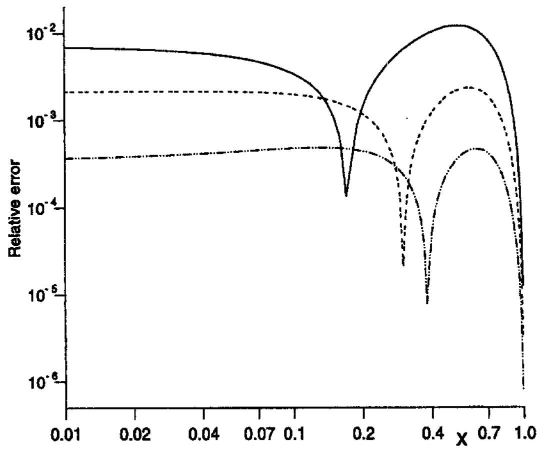
$$e^{-t^2}C(x,t)=e^{(x-1)^2}.$$



(a)



(b)



(c)

Figure 1. The relative errors, on a log-log plot, between the terms $S_3(x,t)$ (——), $S_5(x,t)$ (– – –), and $S_7(x,t)$ (–..–) and the exact function $e^{(x-1)^2}$, as a function of $x \in [0,1]$, for various values of t, namely, (a) $t = 0.5$, (b) $t = 1$, and (c) $t = 1.5$, see also equation (65).

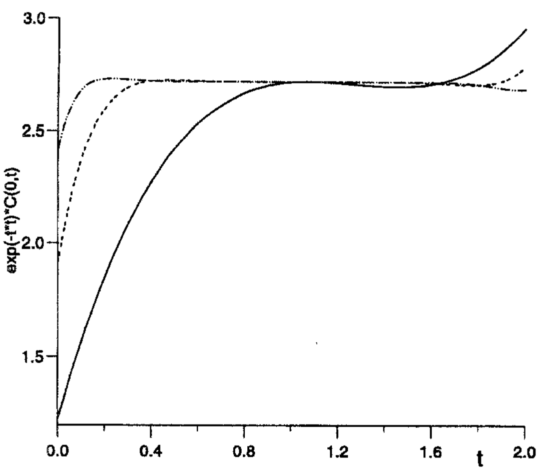


Figure 2. The terms $S_3(0,t)$ (—), $S_7(0,t)$ (– – –) and $S_{11}(0,t)$ (–..–), as a function of $t \in [0,2]$, in comparison with the analytical value for $e^{-t^2}C(0,t) = e$ (–...–).

Figure 1 presents the relative errors between the terms $S_3(x,t)$, $S_5(x,t)$ and $S_7(x,t)$ and the exact function $e^{(x-1)^2}$, as a function of $x \in [0,1]$, for various values of $t \in \{0.5, 1, 1.5\}$, respectively. From these figures, it can be seen that an excellent relative error of less than 0.1% can be achieved by taking only $N+1 = 4$ to 6 terms (65). Some "spiked" behaviour of the relative error shown in Figures 1a and 1c, for $t = 0.5$ and $t = 1.5$, respectively, is not explicable at present, and it is probably example-dependent. Moreover, the sequence $\{S_N\}_{N \geq 0}$ is not convergent for all values of t, and in fact, the time coefficients $a_1^n(t) = t(1-t)^{n-1}$, for the power $(x-1)^2$ in equations (58) and (60) suggest that the series is convergent only for $t \in [0,2)$. This is numerically illustrated in Figure 2 which presents the terms $S_3(0,t)$, $S_7(0,t)$, and $S_{11}(0,t)$, as a function of $t \in [0,2]$, in comparison with the analytical value for $e^{-t^2}C(0,t) = e$. We present the results only at the boundary $x = 0$ since at this location the problem is the most ill-posed. From this figure, it can be seen that as N increases the sequence $S_N(0,t)$ converges to the exact limit e, for $t \in [0,2)$, but, for $t \geq 2$ it starts to diverge. To summarise, it was found that the decomposition series (9) converges rapidly to the exact solution, for all $(x,t) \in [0,1] \times [0,2)$.

For problems (1) and (2), $h(x) = e^{(x-1)^2}$ and the recurrence relation (11) gives

$$C_0(x,t) = e^{(x-1)^2},$$
$$C_1(x,t) = t^2 e^{(x-1)^2},$$
$$C_2(x,t) = \frac{t^4}{2} e^{(x-1)^2},$$

and in general, $C_n(x,t) = (t^{2n}/n!)e^{(x-1)^2}$. The decomposition series (9) yields immediately the exact solution.

3. CONCLUSIONS

In this paper, the solution of Cauchy problems for the advection-diffusion equation in which the flow velocity may be zero, a nonzero constant, spacewise and/or time dependent is obtained using the decomposition method. This method can also be applied to nonlinear fluid velocities, i.e., $u = u(C)$, as described in [17]. Further, extensions to higher-dimensional advection-diffusion equations (which may also include reaction terms) will be investigated in a future work using the Green's decomposition method, [18]. At this stage of the research, the paper was orientated to the application of the decomposition method to several synthetic examples. More interesting physical problems will be investigated in future work.

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