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Defending the Roman Empire—A new strategy

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Abstract

Motivated by an article by Ian Stewart (Defend the Roman Empire!, Scientific American, Dec. 1999, pp. 136–138), we explore a new strategy of defending the Roman Empire that has the potential of saving the Emperor Constantine the Great substantial costs of maintaining legions, while still defending the Roman Empire. In graph theoretic terminology, let $G=(V,E)$ be a graph and let f be a function $f:V\to\{0,1,2\}$. A vertex u with $f(u)=0$ is said to be undefended with respect to f if it is not adjacent to a vertex with positive weight. The function f is a weak Roman dominating function (WRDF) if each vertex u with $f(u)=0$ is adjacent to a vertex v with $f(v)>0$ such that the function $f':V\to\{0,1,2\}$, defined by $f'(u)=1$, $f'(v)=f(v)-1$ and $f'(w)=f(w)$ if $w\in V-\{u,v\}$, has no undefended vertex. The weight of f is $w(f)=\sum_{v\in V}f(v)$. The weak Roman domination number, denoted $\gamma_r(G)$, is the minimum weight of a WRDF in G. We show that for every graph G, $\gamma(G)\leqslant\gamma_r(G)\leqslant 2\gamma(G)$. We characterize graphs G for which $\gamma_r(G)=\gamma(G)$ and we characterize forests G for which $\gamma_r(G)=2\gamma(G)$.

Keywords: Domination number; Forests; Weak Roman domination number

1. Introduction

Cockayne et al. [3] defined a *Roman dominating function* (RDF) on a graph $G=(V,E)$ to be a function $f:V\to\{0,1,2\}$ satisfying the condition that every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=2$. For a real-valued function $f:V\to R$ the *weight* of f is $w(f)=\sum_{v\in V}f(v)$, and for $S\subseteq V$

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we define $f(S)=\sum_{v\in S} f(v)$, so $w(f)=f(V)$. The *Roman domination number*, denoted $\gamma_R(G)$, is the minimum weight of an RDF in G; that is, $\gamma_R(G)=\min\{w(f)\mid f$ is a WRDF in $G\}$. An RDF of weight $\gamma_R(G)$ we call a $\gamma_R(G)$-function. Roman domination in graphs has been studied, for example, in [3,4,6,10].

This definition of a Roman dominating function was motivated by an article in Scientific American by Ian Stewart entitled "Defend the Roman Empire!" [11]. Each vertex in our graph represents a location in the Roman Empire. A location (vertex v) is considered *unsecured* if no legions are stationed there (i.e., $f(v)=0$) and *secured* otherwise (i.e., if $f(v)\in\{1,2\}$). An unsecured location (vertex v) can be secured by sending a legion to v from an adjacent location (an adjacent vertex u). But Emperor Constantine the Great, in the fourth century A.D., decreed that a legion cannot be sent from a secured location to an unsecured location if doing so leaves that location unsecured. Thus, two legions must be stationed at a location ($f(v)=2$) before one of the legions can be sent to an adjacent location. In this way, Emperor Constantine the Great can defend the Roman Empire. Since it is expensive to maintain a legion at a location, the Emperor would like to station as few legions as possible, while still defending the Roman Empire. A Roman dominating function of weight $\gamma_R(G)$ corresponds to such an optimal assignment of legions to locations.

In this paper we explore the potential of saving the Emperor substantial costs of maintaining legions, while still defending the Roman Empire (from a single attack). Let $G=(V,E)$ be a graph and let f be a function $f: V\to\{0,1,2\}$. Let V_0, V_1, and V_2 be the sets of vertices assigned the values 0, 1, and 2, respectively, under f. Note that there is a 1–1 correspondence between the functions $f: V\to\{0,1,2\}$ and the ordered partitions (V_0,V_1,V_2) of V. Thus we will write $f=(V_0,V_1,V_2)$.

We say that a vertex $u\in V_0$ is *undefended with respect to* f, or simply *undefended* if the function f is clear from the context, if it is not adjacent to a vertex in V_1 or V_2. We call the function f a *weak Roman dominating function* (WRDF) if each vertex $u\in V_0$ is adjacent to a vertex $v\in V_1\cup V_2$ such that the function $f': V\to\{0,1,2\}$, defined by $f'(u)=1$, $f'(v)=f(v)-1$ and $f'(w)=f(w)$ if $w\in V-\{u,v\}$, has no undefended vertex.

We define the weight $w(f)$ of f to be $|V_1|+2|V_2|$. The *weak Roman domination number*, denoted $\gamma_r(G)$, is the minimum weight of a WRDF in G; that is, $\gamma_r(G)=\min\{w(f)\mid f$ is a WRDF in $G\}$. A WRDF of weight $\gamma_r(G)$ we call a $\gamma_r(G)$-function. For a vertex v in V, we denote $f(N[v])$ by $f[v]$ for notational convenience.

This definition of a WRDF is motivated as follows. Using notation introduced earlier, we define a location to be *undefended* if the location and every location adjacent to it are unsecured (i.e., have no legion stationed there). Since an undefended location is vulnerable to an attack, we require that every unsecure location be adjacent to a secure location in such a way that the movement of a legion from the secure location to the unsecure location does not create an undefended location. Hence every unsecure location can be defended without creating an undefended location. In this way Emperor Constantine the Great can still defend the Roman Empire. Such a placement of legions corresponds to a WRDF and a minimum such placement of legions corresponds to a minimum WRDF. Since the potential exists to save the Emperor substantial costs of maintaining legions, while still defending the Roman Empire (from a single attack), this concept of weak Roman domination is an attractive alternative to Emperor Constantine's notion of Roman domination.

Notice that in a WRDF, every vertex in V_0 is dominated by a vertex in $V_1 \cup V_2$, while in an RDF every vertex in V_0 is dominated by at least one vertex in V_2 (this is more expensive). Furthermore, in a WRDF, every vertex in V_0 can be defended without creating an undefended vertex.

2. Notation

For notation and graph theory terminology we in general follow [8]. Specifically, let $G=(V,E)$ be a graph with vertex set V of order n and edge set E, and let v be a vertex in V. The open neighborhood of v is $N(v)=\{u \in V \mid uv \in E\}$ and the closed neighborhood of v is $N[v]=\{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* $N(S)=\bigcup_{v \in S} N(v)$ and its *closed neighborhood* $N[S]=N(S) \cup S$. A vertex u is called a *private neighbor of v with respect to S*, or simply an *S-pn of v*, if $N[u] \cap S=\{v\}$. The set $\mathrm{pn}(v,S)=N[v]-N[S-\{v\}]$ of all S-pns of v is called the *private neighbor set of v with respect to S*. We define the *external private neighbor set of v with respect to S* by $\mathrm{epn}(v,S)=\mathrm{pn}(v,S)-\{v\}$. Hence the set $\mathrm{epn}(v,S)$ consists of all S-pns of v that belong to $V-S$.

For ease of presentation, we mostly consider *rooted trees*. For a vertex v in a (rooted) tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v]=D(v) \cup \{v\}$. The maximal subtree at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A *leaf* of T is a vertex of degree 1, while a *support vertex* of T is a vertex adjacent to a leaf. A *strong support vertex* is adjacent to at least two leaves. In this paper, we denote the set of all strong support vertices of T by $S(T)$ and the set of leaves by $L(T)$.

Let $G=(V,E)$ be a graph and let $S \subseteq V$. A set S dominates a set U, denoted $S \succ U$, if every vertex in U is adjacent to a vertex of S. If $S \succ V-S$, then S is called a *dominating set* of G. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ we call a $\gamma(G)$-set. Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [8,9].

3. Properties

We begin with an inequality chain relating the domination number, the weak Roman domination number and the Roman domination number.

Observation 1. *Every* RDF *in a graph G is also a* WRDF *of G.*

Proof. Let $f=(V_0,V_1,V_2)$ be an RDF of G. Let $u \in V_0$. Then u is adjacent to a vertex $v \in V_2$. Let $f': V \to \{0,1,2\}$ be the function defined by $f'(u)=1$, $f'(v)=1$, and $f'(w)=f(w)$ if $w \in V-\{u,v\}$ (i.e., f' corresponds to the movement of a legion from

v to u). Then, $f' = (V_0 - \{u\}, V_1 \cup \{u, v\}, V_2 - \{v\})$. Now each $w \in V_0 - \{u\}$ is adjacent to v or to a vertex in $V_2 - \{v\}$ and is therefore a defended vertex. Thus, f' has no undefended vertex and is therefore a WRDF of G. □

Theorem 2. *For any graph* G,

$$\gamma(G) \leqslant \gamma_r(G) \leqslant \gamma_R(G) \leqslant 2\gamma(G).$$

Proof. It is shown in [3] that $\gamma_R(G) \leqslant 2\gamma(G)$. For completeness, we provide this short proof. Let S be a $\gamma(G)$-set. Let $f = (V_0, V_1, V_2)$ be the function defined by $V_0 = V(G) - S$, $V_1 = \emptyset$ and $V_2 = S$. Since $V_2 \succ V_0$, f is an RDF, and therefore $\gamma_r(G) \leqslant w(f) = 2|V_2| = 2|S| = 2\gamma(G)$. To show that $\gamma_r(G) \leqslant \gamma_R(G)$, let g be a $\gamma_R(G)$-function. Then, by Observation 1, g is a WRDF of G, and so $\gamma_r(G) \leqslant w(g) = \gamma_R(G)$. Finally, to show that $\gamma(G) \leqslant \gamma_r(G)$, let $h = (V_0, V_1, V_2)$ be a $\gamma_r(G)$-function. Then, since $V_1 \cup V_2 \succ V_0$, $V_1 \cup V_2$ is a dominating set of G, and so $\gamma(G) \leqslant |V_1 \cup V_2| = |V_1| + |V_2| \leqslant |V_1| + 2|V_2| = w(h) = \gamma_r(G)$. □

Note that if $G = P_5$, then $\gamma(G) = 2$, $\gamma_r(G) = 3$ and $\gamma_R(G) = 4$. Hence there exist connected graphs G with $\gamma(G) < \gamma_r(G) < \gamma_R(G)$.

4. Paths and cycles

The Roman domination number of a path P_n and a cycle C_n on n vertices is established in [3].

Proposition 3. *For* $n \geqslant 3$, $\gamma_R(P_n) = \gamma_R(C_n) = \lceil 2n/3 \rceil$.

In this section we determine the weak Roman domination number of paths and cycles. We begin with paths.

4.1. Paths

To determine $\gamma_r(P_n)$, we first present two lemmas.

Lemma 4. *If G is a graph that contains a path P of order* 7, *every internal vertex of which has degree* 2 *in* G, *then* $f(V(P)) \geqslant 3$ *for any* WRDF f *of* G.

Proof. Let P: $v_1, v_2, \ldots, v_7$ denote the path P of order 7. Then, $\deg v_i = 2$ for $i = 2, 3, \ldots, 6$. Let $f = (V_0, V_1, V_2)$ be a WRDF of G. Then, $f[v] \geqslant 1$ for every vertex v of G. In particular, $f[v_2] \geqslant 1$ and $f[v_6] \geqslant 1$. If $f(v_4) \geqslant 1$; then $f(V(P)) = f[v_2] + f(v_4) + f[v_6] \geqslant 3$. On the other hand, suppose that $f(v_4) = 0$. Since $v_4 \in V_0$, v_4 must be adjacent to a vertex $v \in V_1 \cup V_2$ such that the movement of a legion from v to v_4 will not create an undefended vertex, i.e., the function $f' : V \to \{0, 1, 2\}$ defined by $f'(v_4) = 1$, $f'(v) = f(v) - 1$, and $f'(w) = f(w)$ if $w \in V(G) - \{v, v_4\}$ has no undefended vertex. We may assume that $v = v_3$. Thus, $f(v_3) \geqslant 1$. If $f(v_3) = 1$, then since the movement of a legion from v_3 to v_4 does not create an undefended vertex, it follows that $f(v_1)+f(v_2) \geqslant 1$, and so $f[v_2] \geqslant 2$. If $f(v_3)=2$, then clearly $f[v_2] \geqslant 2$. Hence we must have $f[v_2] \geqslant 2$ irrespective of whether $f(v_3)=1$ or $f(v_3)=2$. Thus, $f(V(P)) = f[v_2] + f(v_4) + f[v_6] \geqslant 2+0+1 = 3$. □

Lemma 5. *If G is any graph and u is any vertex of G, then the graph H obtained from G by attaching a path of length 7 to u satisfies $\gamma_r(H) = \gamma_r(G)+3$.*

Proof. Let $u, v_1, v_2, \ldots, v_7$ denote the path of length 7 added to G, and let P denote the path $v_1, v_2, \ldots, v_7$. Any $\gamma_r(G)$-function can be extended to a WRDF of H by assigning the value 1 to v_2, v_4 and v_6 and the value 0 to v_1, v_3, v_5 and v_7. Hence, $\gamma_r(H) \leqslant \gamma_r(G)+3$.

On the other hand, suppose $f = (V_0, V_1, V_2)$ is a $\gamma_r(H)$-function. By Lemma 4, $f(V(P)) \geqslant 3$. Suppose $f(V(P)) \geqslant 4$. Then the function $f' : V \to \{0,1,2\}$ defined by $f'(v_2)=f'(v_4)=f'(v_6)=1$, $f'(v_1)=f'(v_3)=f'(v_5)=f'(v_7)=0$, $f'(u)=2$ if $f(u)=2$ and $f'(u) = f(u)+1$ if $f(u) \leqslant 1$, and $f'(w) = f(w)$ if $w \in V(G) - \{u\}$ is a WRDF of weight $w(f)$ that satisfies $f'(V(P))=3$. Hence, we may assume that $f(V(P))=3$.

Suppose $f(v_1) \geqslant 1$. Then, $3 = f(V(P)) = f[v_7] + f[v_4] + f(v_2) + f(v_1) \geqslant 3$ with equality if and only if $f[v_7] = 1$, $f[v_4] = 1$, and $f(v_2) = 0$. If $f(v_7) = 1$ (and so $f(v_6) = 0$), then we can simply interchange the values of v_6 and v_7 to produce a new $\gamma_r(G)$-function. Hence, we may assume that $f(v_7)=0$ (and so $f(v_6)=1$). Since $f[v_3] \geqslant 1$, $f(v_5) = 0$. If the legion at v_6 is moved to v_5, then the vertex v_7 would become undefended. It follows that $f(v_4)=1$, and so $f(v_3)=0$. However if the legion at v_4 is moved to v_5, then the vertex v_3 would become undefended. Hence there is no vertex $v \in V_1 \cup V_2$ adjacent to the vertex $v_5 \in V_0$ such that the movement of a legion from v to v_5 will not create an undefended vertex. This contradicts the fact that f is a WRDF. Hence we must have $f(v_1)=0$.

Since $f(v_1)=0$, the restriction of f to G is a WRDF of G. Hence, $\gamma_r(G) \leqslant f(V(G)) = w(f)-3 = \gamma_r(H)-3$, or, equivalently, $\gamma_r(H) \geqslant \gamma_r(G)+3$. Consequently, $\gamma_r(H)=\gamma_r(G)+3$. □

Using Lemma 4, we can determine the weak Roman domination number of a path.

Proposition 6. *For $n \geqslant 1$,*

$$\gamma_r(P_n) = \left\lceil \frac{3n}{7} \right\rceil.$$

Proof. We proceed by induction on n. It is straightforward to verify the result for small n, $1 \leqslant n \leqslant 7$. Assume that the result holds for all paths of order less than n, where $n \geqslant 8$. Let T: $v_1, v_2, \ldots, v_n$ be a path of order n. Let T' denote the path $v_8, \ldots, v_n$ of order $n-7 \geqslant 1$. Applying the inductive hypothesis to T', $\gamma_r(T') = \lceil 3(n-7)/7 \rceil$. By Lemma 5, $\gamma_r(T) = \gamma_r(T')+3$, and so $\gamma_r(T) = \lceil 3(n-7)/7 \rceil + 3 = \lceil 3n/7 \rceil$. The result now follows by mathematical induction. □

As a consequence of Propositions 3 and 6 we note that the cost savings of weak Roman domination over Roman domination for a path on $n \geqslant 1$ vertices is $\gamma_R(P_n) - \gamma_r(P_n) = \lceil 2n/3 \rceil - \lceil 3n/7 \rceil$, which is either $\lceil 5n/21 \rceil$ or $\lfloor 5n/21 \rfloor$.

4.2. Cycles

Next we consider the weak Roman domination number of a cycle. For this purpose, we shall need the following two observations.

Observation 7. *If H is a spanning subgraph of a graph G, then $\gamma_r(G) \leqslant \gamma_r(H)$.*

Proof. The proof follows immediately from the observation that any WRDF of H is also a WRDF of G. □

Observation 8. *If a graph G has a $\gamma_r(G)$-function that assigns the value 0 to two adjacent vertices u and v, then $\gamma_r(G) = \gamma_r(G - uv)$.*

Proof. Let f be a $\gamma_r(G)$-function for which $f(u) = f(v) = 0$. Then, f is a WRDF of $G - uv$, and so $\gamma_r(G - uv) \leqslant w(f) = \gamma_r(G)$. However, by Observation 7, $\gamma_r(G) \leqslant \gamma_r(G - uv)$. Consequently, $\gamma_r(G) = \gamma_r(G - uv)$. □

We are now in a position to determine the weak Roman domination number of a cycle. Clearly, $\gamma_r(C_3) = 1$.

Proposition 9. *For $n \geqslant 4$,*

$$\gamma_r(C_n) = \gamma_r(P_n) = \left\lceil \frac{3n}{7} \right\rceil.$$

Proof. It is straightforward to verify the result for small n, $4 \leqslant n \leqslant 12$. Suppose that $n \geqslant 13$. By Proposition 6 and Observation 7, $\gamma_r(C_n) \leqslant \gamma_r(P_n) = \lceil 3n/7 \rceil$. Let f be a $\gamma_r(C_n)$-function. Since $w(f) \leqslant \lceil 3n/7 \rceil$ and $n \geqslant 13$, the function f must assign the value 0 to two adjacent vertices u and v of C_n. Hence, by Observation 8, $\gamma_r(C_n) = \gamma_r(C_n - uv) = \gamma_r(P_n)$. Consequently, $\gamma_r(C_n) = \gamma_r(P_n)$. □

5. Graphs G with $\gamma_r(G) = \gamma(G)$

Our aim in this section is to characterize graphs G for which $\gamma_r(G) = \gamma(G)$.

Theorem 10. *For any graph G, $\gamma(G) = \gamma_r(G)$ if and only if there exists a $\gamma(G)$-set S such that*

(1) $\mathrm{pn}(v, S)$ *induces a clique for every $v \in S$,*
(2) *for every vertex $u \in V(G) - S$ that is not a private neighbor of any vertex of S, there exists a vertex $v \in S$ such that $\mathrm{pn}(v, S) \cup \{u\}$ induces a clique.*

Proof. Suppose $G = (V, E)$ and $\gamma(G) = \gamma_r(G)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_r(G)$-function. Then, $\gamma_r(G) = \gamma(G) \leqslant |V_1| + |V_2| \leqslant |V_1| + 2|V_2| = w(f) = \gamma_r(G)$. Hence we must have equality throughout the above inequality chain. In particular, it follows that $V_2 = \emptyset$. Thus, $S = V_1$ is a $\gamma(G)$-set.

Suppose $u, w \in \text{epn}(v, S)$. Then, $u, w \in V - S = V_0$. Since v is the only vertex of V_1 adjacent to u, the movement of a legion from v to u cannot create an undefended vertex, i.e., the function $f' : V \to \{0, 1, 2\}$ defined by $f'(u) = 1$, $f'(v) = 0$, and $f'(w) = f(w)$ if $w \in V - \{u, v\}$ has no undefended vertex. But since $N(w) \cap (V_1 - \{v\}) = \emptyset$, we must have $uw \in E$. It follows that $\text{epn}(v, S)$ induces a clique. This establishes (1).

Suppose $u \in V - S$ is not a private neighbor of any vertex of S. Since $u \in V_0$, u must be adjacent to a vertex $v \in V_1$ such that the movement of a legion from v to u will not create an undefended vertex. But then $u \succ \text{pn}(v, S)$, and so $\text{pn}(v, S) \cup \{u\}$ induces a clique. This establishes (2) and proves the necessity.

To prove the sufficiency, suppose there exists a $\gamma(G)$-set S satisfying conditions (1) and (2) in the statement of the theorem. Let $g = (V_0, V_1, V_2)$ be a function defined by $V_0 = V - S$, $V_1 = S$ and $V_2 = \emptyset$. Then, g is a WRDF, and so $\gamma_r(G) \leqslant |S| = \gamma(G)$. Consequently, $\gamma(G) = \gamma_r(G)$. □

6. Forests F with $\gamma_r(F) = 2\gamma(F)$

By Theorem 2, $\gamma_r(G) \leqslant 2\gamma(G)$ for all graphs G. Our aim in this section is to characterize forests F for which $\gamma_r(F) = 2\gamma(F)$. We begin with the following lemma.

Lemma 11. *If G is a graph satisfying $\gamma_r(G) = 2\gamma(G)$, then for every $\gamma(G)$-set S and every $v \in S$, the set* $\text{epn}(v, S)$ *contains two nonadjacent vertices.*

Proof. Suppose $G = (V, E)$. Let S be a $\gamma(G)$-set and let $v \in S$. Suppose $\text{epn}(v, S)$ induces a clique. Let $f = (V_0, V_1, V_2)$ be a function defined by $V_0 = V - S$, $V_1 = \{v\}$ and $V_2 = S - \{v\}$. Since $\text{epn}(v, S)$ induces a clique, the movement of a legion from v to any vertex in $\text{epn}(v, S)$ cannot create an undefended vertex. If $w \in V_2$ and $u \in V_0$ with $u \in N(w)$, then the movement of a legion from w to u cannot create an undefended vertex. In particular, if $u \in N(v) - \text{pn}(v, S)$, then since u is adjacent to at least one vertex $w \in S - \{v\} = V_2$, the movement of a legion from w to u cannot create an undefended vertex. It follows that f is a WRDF, and so $\gamma_r(G) \leqslant |V_1| + 2|V_2| = 2|S| - 1 = 2\gamma(G) - 1$, contrary to assumption. Hence, $\text{epn}(v, S)$ cannot induce a clique, i.e., $\text{epn}(v, S)$ contains two nonadjacent vertices. □

The necessary condition in Lemma 11 for a graph G satisfying $\gamma_r(G) = 2\gamma(G)$ is not sufficient. For example, for $k \geqslant 2$ an integer, let G be the path $v_1, v_2, \ldots, v_{3k}$ on $3k$ vertices. Then, $S = \bigcup_{i=1}^{k} \{v_{3i-1}\}$ is the unique $\gamma(G)$-set, and for each $i = 1, 2, \ldots, k$, v_{3i-2} and v_{3i} are two nonadjacent vertices in $\text{epn}(v_{3i-1}, S)$. However, by Proposition 6, $\gamma_r(G) = \lceil 9k/7 \rceil < 2k = 2\gamma(G)$.

Gunther et al. [7] presented the following characterization of trees with unique minimum dominating sets.

Theorem 12 (Gunther et al. [7]). *Let T be a tree of order at least* 3. *Then, T has a unique $\gamma(T)$-set if and only if T has a $\gamma(T)$-set S such that* $|\text{epn}(v, S)| \geqslant 2$ *for every vertex $v \in S$.*

As an immediate consequence of Lemma 11 and Theorem 12, we have the following result.

Corollary 13. *If T is a tree satisfying $\gamma_r(T)=2\gamma(T)$, then T has a unique $\gamma(T)$-set.*

The necessary condition in Corollary 13 for a tree T satisfying $\gamma_r(T)=2\gamma(T)$ is not sufficient as may be seen by considering a path P_{3k} with $k \geqslant 2$.

Recall that the set of all strong support vertices of T is denoted by $S(T)$. Every strong support vertex of a tree T belongs to every $\gamma(T)$-set. Hence, if a tree T has a unique $\gamma(T)$-set S, then $S(T) \subseteq S$. We state this as an observation.

Observation 14. *If T is a tree with a unique $\gamma(T)$-set S, then $S(T) \subseteq S$.*

Lemma 15. *If T is a tree with a unique $\gamma(T)$-set S, and if every vertex of S is a strong support vertex, then $\gamma_r(T) = 2\gamma(T)$.*

Proof. By Observation, $S(T) \subseteq S$. By assumption, every vertex of S is a strong support vertex, and so $S \subseteq S(T)$. Consequently, $S=S(T)$. Let f be a $\gamma_r(T)$-function. We show that $w(f) \geqslant 2\gamma(T)$. For each $v \in S$, let N_v consist of v and every leaf adjacent to v. If $f(v)=0$, then $f(u)=1$ for each $u \in N_v - \{v\}$, and so $f(N_v) \geqslant 2$. If $f(v)=1$, then $f(u)=1$ for all except possibly one vertex in $N_v-\{v\}$, and so $f(N_v) \geqslant 2$. If $f(v)=2$, then $f(N_v) \geqslant 2$. Hence in all cases, $f(N_v) \geqslant 2$. Since the sets $\bigcup_{v\in S} N_v$ are disjoint sets in T, it follows that $w(f) \geqslant \sum_{v\in S} f(N_v) \geqslant 2|S| = 2\gamma(T)$. On the other hand, the function $g=(V_0,V_1,V_2)$ defined by $V_0 = V-S$, $V_1=\emptyset$ and $V_2=S$ is a WRDF, and so $\gamma_r(T) \leqslant w(g) = 2|V_2| = 2|S| = 2\gamma(G)$. Consequently, $\gamma_r(T)=2\gamma(T)$. □

Lemma 16. *If T is a tree with a unique $\gamma(T)$-set S, and if no vertex of S is a strong support vertex, then $\gamma_r(T) < 2\gamma(T)$.*

Proof. We proceed by induction on $\gamma(T)$. Suppose $\gamma(T)=1$ and $S=\{v\}$. Then, T is a star on at least three vertices with v as the central vertex. But then $v \in S(T)$, and so there is no tree satisfying the hypothesis in the statement of the lemma. Hence we may assume $\gamma(T) \geqslant 2$.

Suppose $\gamma(T)=2$ and $S=\{u,v\}$. By Theorem 12, $|\text{epn}(u,S)| \geqslant 2$. Since $u \notin S(T)$, at most one vertex in $\text{epn}(u,S)$ is a leaf. Let $u' \in \text{epn}(u,S)-L(T)$ and let $x \in N(u')-\{u\}$. Similarly, let $v' \in \text{epn}(v,S)-L(T)$ and let $y \in N(v')-\{v\}$. Since S dominates $V(T)$, $x \neq y$. If $u'v' \notin E(T)$, then, since S dominates $V(T)$, we must have $xv \in E(T)$. But then y is not dominated by S, a contradiction. Hence, $u'v' \in E(T)$. It follows that $T=P_6$, and so, by Proposition 6, $\gamma_r(T)=3<4=2\gamma(T)$. Hence the result of the lemma is true when $\gamma(T) \leqslant 2$.

Suppose that the result of the lemma is true for all trees T' with $\gamma(T')<k$, where $k \geqslant 3$, that satisfy the hypothesis in the statement of the lemma. Let $T=(V,E)$ be a tree with $\gamma(T)=k$ and with a unique $\gamma(T)$-set S such that $v \notin S(T)$ for every vertex $v \in S$. Let T be rooted at an end-vertex r of a longest path. Let w be a vertex at distance $\text{diam}(T)-2$ from r on a longest path starting at r, and let v be the child of w on this path. Let x denote the parent of w, and let y denote the parent of x.

By Observation 14, $S(T) \subset S$. Hence, since S is the unique $\gamma(T)$-set and no vertex of S is a strong support vertex, $S(T)=\emptyset$. In particular, $\deg v=2$. Let u denote the child of v. By Theorem 12, no leaf belongs to S, and so $v \in S$. Furthermore, $|\text{epn}(v,S)| \geqslant 2$ and therefore $\text{epn}(v,S) = N(v) = \{u,w\}$. It follows that $\deg w = 2$ and that $w,x \notin S$. Thus, x cannot be a support vertex.

Suppose x has a child w' that is a support vertex. Then it follows from Theorem 12 that $w' \in S$. If w' has a child v' that is a support vertex, then, since $S(T) = \emptyset$, $\deg v' = 2$. But then, $v' \in S$ and $|\text{epn}(v',S)| = 1$, contradicting Theorem 12. Hence, $\deg w' = 2$ and so, by Theorem 12, $\text{epn}(w',S) = N(w')$. Let $f = (V_0, V_1, V_2)$ be the function defined by $V_0 = V - S$, $V_1 = \{v, w', x\}$ and $V_2 = S - \{v, w'\}$. Then f is a WRDF, and so $\gamma_r(T) \leqslant w(f) = |V_1| + 2|V_2| = 3 + 2(|S| - 2) = 2|S| - 1 = 2\gamma(G) - 1$, contrary to assumption. Hence, no child of x is a support vertex.

Suppose $\deg x \geqslant 3$. Let $w' \in C(x) - \{w\}$. Then, w' is neither a leaf nor a support vertex. Let v' be a child of w' and let u' be a child of v'. As shown earlier, $\deg v'=2$, $v' \in S$ and $\deg w' = 2$. Let $g = (V_0, V_1, V_2)$ be the function defined by $V_0 = V - S$, $V_1 = \{v, v', x\}$ and $V_2 = S - \{v, v'\}$. Then g is a WRDF, and so $\gamma_r(T) \leqslant w(g) = |V_1| + 2|V_2| = 3 + 2(|S| - 2) = 2|S| - 1 = 2\gamma(G) - 1$, contrary to assumption. Hence, $\deg x = 2$. Since $w,x \notin S$, we must therefore have $y \in S$.

Let $T' = T - \{u, v, w, x\}$. Since $y \in S$, $S - \{v\}$ is a dominating set of T', and so $\gamma(T') \leqslant |S| - 1 = \gamma(T) - 1$. Let h' be a $\gamma_r(T')$-function and let $h : V \to \{0, 1, 2\}$ be the function defined by $h(z)=h'(z)$ if $z \in V(T')$, $h(x)=h(v)=1$, and $h(u)=h(w)=0$. Then, h is a WRDF, and so $\gamma_r(T) \leqslant w(h)=w(h')+2 \leqslant 2\gamma(T')+2 \leqslant 2(\gamma(T)-1)+2=2\gamma(T)$. Suppose $\gamma_r(T)=2\gamma(T)$. Then we have equality throughout the above inequality chain. In particular, $\gamma_r(T')=2\gamma(T')$ and $\gamma(T')=\gamma(T)-1$. By Corollary 13, T' has a unique $\gamma(T')$-set, namely $S' = S - \{v\}$. In particular, since $y \in S'$, it follows from Theorem 12 that $|\text{epn}(y, S')| \geqslant 2$. Thus, in the tree T', $\deg y \geqslant 2$ and therefore y is not a leaf in T'. Hence, every leaf in T' is also a leaf in T. Since T has no strong support vertex, neither too does T'. Consequently, T' is a tree with $\gamma(T') < k$ and with a unique $\gamma(T')$-set S' such that no vertex of S' is a strong support vertex. Applying the inductive hypothesis to T', $\gamma_r(T') < 2\gamma(T')$, a contradiction. Hence we must have $\gamma_r(T) < 2\gamma(T)$, as desired. □

As an immediate consequence of Lemma 16 we have the following result.

Corollary 17. *If F is a forest with a unique $\gamma(F)$-set S, and if F has a component T with no strong support vertex, then $\gamma_r(F) < 2\gamma(F)$.*

Proof. Let $T_1, \ldots, T_k$, $k \geqslant 1$, denote the components of F, where $T = T_1$. Since S is the unique $\gamma(F)$-set, $S \cap V(T_i)$ is the unique $\gamma(T_i)$-set for each i, $1 \leqslant i \leqslant k$. For $i=1, \ldots, k$, let f_i be a $\gamma_r(T_i)$-function and let $f : V(T) \to \{0, 1, 2\}$ be the $\gamma_r(F)$-function defined by $f(v) = f_i(v)$ if $v \in V(T_i)$. By Lemma 16, $w(f_1) = \gamma_r(T_1) < 2\gamma(T_1)$. Hence, $\gamma_r(T) = w(f) = \sum_{i=1}^{k} w(f_i) < 2\sum_{i=1}^{k} \gamma(T_i) = 2\gamma(T)$. □

In order to characterize the trees T for which $\gamma_r(T) = 2\gamma(T)$, we construct a family $\mathscr{F}$ of forests as follows.

Let F be a forest with a unique $\gamma(F)$-set S such that each component of F contains a strong support vertex. It follows from Observation 14 that $S(F) \subseteq S$. If $S(F) = S$, then we let $\tilde{F} = F$. Otherwise, if $S(F) \neq S$, then we define the subforest $\tilde{F}$ of F recursively by means of a sequence of subforests $F_0, F_1, \ldots, F_k$ of F, where $F_0 = F$, as follows: For $i = 0, \ldots, k-1$, let $S_i = S \cap V(F_i)$. If every component of F_i has a strong support vertex and if $S_i - S(F_i) \neq \emptyset$, then let

$$F_{i+1} = F_i - \left(\bigcup_{v \in S(F_i)} N[v] - (S_i - S(F_i)) \right).$$

Hence, F_{i+1} is obtained from F_i by deleting all vertices, except for possibly any vertices of $S_i - S(F_i)$, in the closed neighborhoods of every strong support vertex in F_i. Since $\gamma(F)$ is finite, there exists an integer $k \geqslant 1$ such that F_k has a component with no strong support vertex or $S_k = S(F_k)$. Then, $\tilde{F} = F_k$. For $i = 1, \ldots, k$, we call F_{i+1} the *pruning of* F_i and we define k to be the *number of prunings of the forest* F. Note that, if $k > i \geqslant 0$, then $S_{i+1} = S_i - S(F_i)$.

Observation 18. *If F is a forest with a unique $\gamma(F)$-set S such that each component of F contains a strong support vertex, then for $i = 0, \ldots, k$, the set S_i is the unique $\gamma(F_i)$-set.*

Proof. We proceed by induction on i. If $i = 0$, then $S_0 = S$ and $F_0 = F$, and so S_0 is the unique $\gamma(F_0)$-set. Thus the statement is true for $i = 0$. Suppose that the set S_m is the unique $\gamma(F_m)$-set, where $0 \leqslant m < k$. By construction, S_{m+1} is a dominating set of F_{m+1}, and so $\gamma(F_{m+1}) \leqslant |S_{m+1}|$. If $\gamma(F_{m+1}) < |S_{m+1}|$, then adding the set $S(F_m)$ to any $\gamma(F_{m+1})$-set produces a dominating set of F_m of cardinality $|S(F_m)| + \gamma(F_{m+1}) < |S(F_m)| + |S_{m+1}| = |S(F_m)| + |S_m - S(F_m)| = |S_m| = \gamma(F_m)$, which is impossible. Hence, $\gamma(F_{m+1}) = |S_{m+1}|$. If F_{m+1} has two distinct $\gamma(F_{m+1})$-sets X and Y, then $X \cup S(F_m)$ and $Y \cup S(F_m)$ are both $\gamma(F_m)$-sets, contradicting the inductive hypothesis that S_m is the unique $\gamma(F_m)$-set. Hence, S_m is the unique $\gamma(F_m)$-set. □

We define the family $\mathscr{F}$ to consist of all forests F, every component of which contains a strong support vertex, that have a unique $\gamma(F)$-set S such that $\tilde{F} = F_k$ and $S_k = S(F_k)$. Note that if $F \in \mathscr{F}$ and $\tilde{F} = F_k$, then each of the subgraphs $F_0, \ldots, F_k$ belong to the family $\mathscr{F}$.

Lemma 19. *If $F \in \mathscr{F}$, then $\gamma_r(F) = 2\gamma(F)$.*

Proof. We proceed by induction on the number k of prunings of the forest F. Let S be the unique $\gamma(F)$-set. We shall adopt the notation introduced in constructing the family $\mathscr{F}$. Suppose $k = 0$. Then, $\tilde{F} = F$ and $S = S(F)$. Thus every vertex of S is a strong support vertex. Hence it follows from Lemma 15 that $\gamma_r(F) = 2\gamma(F)$. Therefore the base case when $k = 0$ is true.

Suppose that all forests $F \in \mathscr{F}$ with $\tilde{F} = F_m$ where $0 \leqslant m < k$ satisfy $\gamma_r(F) = 2\gamma(F)$. Let $F \in \mathscr{F}$ satisfy $\tilde{F} = F_k$. Then, $S_k = S(F_k)$. Since $k \geqslant 1$, $S - S(F) \neq \emptyset$. We consider the forest $F_1 = F - (\bigcup_{v \in S(F)} N[v] - S_1)$. By Observation 18, S_1 is the unique $\gamma(F_1)$-set.

Since $F \in \mathscr{F}$, every component of F_1 has a strong support vertex (possibly, $S_1 = S(F_1)$). Now, $F_1 \in \mathscr{F}$ and $k-1$ prunings of the forest F_1 are needed to construct the forest $\widetilde{F}_1$. Applying the inductive hypothesis to F_1, $\gamma_r(F_1) = 2\gamma(F_1)$.

Let f_1 be a $\gamma_r(F_1)$-function, and let $f : V(F) \to \{0,1,2\}$ be defined by $f(v) = f_1(v)$ if $v \in V(F_1)$, $f(v) = 2$ if $v \in S(F)$, and $f(v) = 0$ otherwise. Then, f is a WRDF of F, and so $\gamma_r(F) \leqslant w(f) = w(f_1) + 2|S(F)| = \gamma_r(F_1) + 2|S(F)|$. On the other hand, let g be a $\gamma_r(F)$-function. Suppose $v \in S(F)$ and u is a leaf adjacent to v. If $g(u) = 1$, then we can reassign to v the value $g(v)+1$ and to u the value 0. Hence we may assume that $g(v) = 2$ for each $v \in S(F)$ and $g(u) = 0$ for each leaf u adjacent to v. Furthermore, if $u \in N[S(F)] - S$, then we may assume that $g(u) = 0$ for otherwise we can shift the positive weight on u to a neighbor of u that belongs to F_1. Let g' be the restriction of g to F_1. Then, g' is a WRDF of F_1, and so $\gamma_r(F_1) \leqslant w(g') = w(g) - 2|S(F)| = \gamma_r(F) - 2|S(F)|$. Consequently, $\gamma_r(F) = \gamma_r(F_1) + 2|S(F)|$.

Since S_1 is the unique $\gamma(F_1)$-set, $\gamma(F_1) = |S_1| = |S| - |S(F)| = \gamma(F) - |S(F)|$. Thus, since $\gamma_r(F_1) = 2\gamma(F_1)$, it follows that $\gamma_r(F) = \gamma_r(F_1) + 2|S(F)| = 2(\gamma(F_1) + |S(F)|) = 2\gamma(F)$. □

Lemma 20. *Let F be a forest. If $F \notin \mathscr{F}$, then $\gamma_r(F) < 2\gamma(F)$.*

Proof. Suppose $F \notin \mathscr{F}$. If the forest F does not have a unique $\gamma(F)$-set, then it follows from Corollary 13 that $\gamma_r(F) < 2\gamma(F)$. Hence we may assume that F has a unique $\gamma(F)$-set S. If F has a component with no strong support vertex, then, by Corollary 17, $\gamma_r(F) < 2\gamma(F)$. Hence we my assume that each component of F contains a strong support vertex. Now since $F \notin \mathscr{F}$, it follows that $\tilde{F} = F_k$ where F_k has a component with no strong support vertex. Let g be a $\gamma_r(F_k)$-function. Then, by Corollary 17, $w(g) = \gamma_r(F_k) < 2\gamma(F_k)$. By Observation 18, S_k is the unique $\gamma(F_k)$-set and, by construction, $S - S_k$ is a dominating set of $F - V(F_k)$. Let $f : V(F) \to \{0,1,2\}$ be defined by $f(v) = g(v)$ if $v \in V(F_k)$, $f(v) = 2$ if $v \in S - S_k$, and $f(v) = 0$ otherwise. Then, f is a WRDF of F, and so $\gamma_r(F) \leqslant w(f) = w(g) + 2|S - S_k| < 2\gamma(F_k) + 2(|S| - |S_k|) = 2|S_k| + 2(|S| - |S_k|) = 2|S| = 2\gamma(F)$. □

As an immediate consequence of Lemmas 19 and 20, we have the following characterization of forests F that satisfy $\gamma_r(F) = 2\gamma(F)$.

Theorem 21. *Let F be a forest. Then $\gamma_r(F) = 2\gamma(F)$ if and only if $F \in \mathscr{F}$.*

Note that each forest $F \in \mathscr{F}$ satisfies $\gamma_r(F) = \gamma_R(F) = 2\gamma(F)$. Hence the family $\mathscr{F}$ of forests is a subclass of Roman forests. A characterization of Roman trees, and hence of Roman forests, can be found in [10].

7. Complexity

The following decision problem for the domination number of a graph is known to be NP-complete, even when restricted to bipartite graphs (see Dewdney [5]) or chordal graphs (see Booth [1] and Booth and Johnson [2]).

Dominating set (DM)

Instance: A graph G and a positive integer $k \leqslant |V(G)|$.

Question: Does G have a dominating set of cardinality k or less?

We will demonstrate a polynomial time reduction of this problem to our weak Roman dominating function problem.

Weak roman dominating function (WRDF)

Instance: A graph H and a positive integer $j \leqslant |V(H)|$.

Question: Does H have a WRDF of weight j or less?

Theorem 22. WRDF *is* NP-*complete*, *even when restricted to bipartite or chordal graphs*.

Proof. It is obvious that WRDF is a member of NP since we can, in polynomial time, guess at a function $f: V(H) \to \{0,1,2\}$ and verify that f has weight at most j and is a WRDF. We next show how a polynomial time algorithm for WRDF could be used to solve DM in polynomial time. Given a graph G and a positive integer k construct the graph H by adding to each vertex of G a path of length 4. It is easy to see that the construction of the graph H can be accomplished in polynomial time. Note that if G is a bipartite or chordal graph, then so too is H.

Lemma 23. $\gamma_r(H) = \gamma(G) + 2|V(G)|$.

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_r(H)$-function. Let $v \in V(G) \subset V(H)$, and let P_v: v, w, x, y, z be the path of length 4 added to v. Now, $f[u] \geqslant 1$ for every vertex u of H. In particular, $f[w] \geqslant 1$ and $f[z] \geqslant 1$, and so $f(V(P_v)) = f[w] + f[z] \geqslant 2$. We may assume that $f(z) = 0$ and $f(y) \geqslant 1$ (for otherwise we can simply shift any positive weight from z to its neighbor y). Let $S = (V_1 \cup V_2) \cap V(G)$.

If $f(V(P_v)) \geqslant 3$, then we may assume that $f(v) \geqslant 1$, $f(w) = f(y) = 1$ and $f(x) = f(z) = 0$ (for otherwise we can simply shift any additional positive weight on the path to v). Hence, if $f(V(P_v)) \geqslant 3$, then $v \in S$.

Suppose that $f(V(P_v)) = 2$. Then, $f[w] = 1$ and $f[z] = 1$. Thus, $f(z) = 0$ and $f(y) = 1$. If $f(x) = 1$, then $f(v) = f(w) = 0$. In particular, $v \in V_0$, and so v must be adjacent to a vertex $u \in V_1 \cup V_2$. Since $w \in V_0$, $u \in V(G)$. Hence, v is adjacent to a vertex of S. On the other hand, suppose $f(x) = 0$. Since the movement of a legion from y to x will create an undefended vertex, namely z, it follows that $f(w) = 1$ and $f(v) = 0$ and that the movement of a legion from w to x will not create an undefended vertex. But this implies that the vertex v must be adjacent to a vertex of S. Hence, if $f(V(P_v)) = 2$, then v is dominated by S.

Thus, S is a dominating set of G, and so $\gamma(G) \leqslant |S|$. Furthermore, if $v \in S$, then $f(V(P_v)) \geqslant 3$, while if $v \notin S$, then $f(V(P_v)) = 2$. Hence, $\gamma_r(H) = w(f) \geqslant 3|S| + 2(|V(G)| - |S|) = |S| + 2|V(G)| \geqslant \gamma(G) + 2|V(G)|$.

On the other hand, let D be a $\gamma(G)$-set. Let $g: V(H) \to \{0,1,2\}$ be the function defined as follows: if $v \in D$, then let $g(v) = g(w) = g(y) = 1$ and $g(x) = g(z) = 0$, while if $v \notin D$, then let $g(v)=g(x)=g(z)=0$ and $g(w)=g(y)=1$. Then, g is a WRDF of H, and so $\gamma_r(H) \leqslant w(g) = 3|D| + 2(|V(G)| - |D|) = |D| + 2|V(G)| = \gamma(G) + 2|V(G)|$. Consequently, $\gamma_r(H) = \gamma(G) + 2|V(G)|$, as desired. □

Lemma 23 implies that if we let $j = k + 2|V(G)|$, then $\gamma(G) \leqslant k$ if and only if $\gamma_r(H) \leqslant j$. This completes the proof of Theorem 22. □

Using the Wimer Technique, it is straightforward to show that if T is a tree, then there exists a linear-time algorithm for finding $\gamma_r(T)$. The proof is routine and similar to that presented in [4] and is therefore omitted.

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