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A simple algorithm for finding a maximum triangle-free 2-matching in subcubic graphs

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ABSTRACT

In this paper, we consider the problem of finding a maximum weight 2-matching containing no cycle of a length of at most three in a weighted simple graph, which we call the weighted triangle-free 2-matching problem. Although the polynomial solvability of this problem is still open in general graphs, a polynomial-time algorithm is given by Hartvigsen and Li for the problem in subcubic graphs, i.e., graphs with a maximum degree of at most three. Our contribution is to provide another polynomial-time algorithm for the weighted triangle-free 2-matching problem in subcubic graphs. Our algorithm consists of two basic algorithms: a steepest ascent algorithm and a classical maximum weight 2-matching algorithm, and is justified by fundamental results from the theory of discrete convex functions on jump systems.

1. Introduction

1.1. Previous work

In an undirected graph, an edge set M is said to be a 2-*matching* if each vertex is incident to at most two edges in M. We say that a 2-matching M is C_k-*free* if M contains no cycle of length k or less. The condition "C_3-free" is sometimes referred to as "triangle-free". The C_k-*free* 2-*matching problem* is to find a C_k-free 2-matching of maximum size in a given graph. This problem has been studied as a relaxation of the Hamiltonian cycle problem. The case $k \leq 2$ is exactly the classical simple 2-matching problem, which can be solved efficiently. Papadimitriou showed that the problem is NP-hard when $k \geq 5$ (see [1]), and Hartvigsen [2] gave an augmenting path algorithm for the case $k = 3$. The C_4-free 2-matching problem is left open. For the C_4-free 2-matching problem in bipartite graphs, a min–max formula [3] and polynomial-time algorithms [4,5] are proposed. If the given graph is simple and subcubic (i.e. each vertex has a degree of at most three), a polynomial-time algorithm for finding a maximum 2-matching containing no cycle of length four (which may contain triangles) is given in [6].

Weighted versions of the problems can naturally be considered. The *weighted* C_k-*free* 2-*matching problem* is to find a C_k-free 2-matching of maximum total weight when we are given a weight function on the edge set. This problem can be solved efficiently when $k \leq 2$. Vornberger [7] proved that the weighted C_4-free 2-matching problem is NP-hard, and stronger results on the NP-hardness are given in [8,6]. This problem is, however, polynomially solvable in bipartite graphs if the weight function satisfies a certain condition called "vertex-induced on every square" [9,10]. The case of $k = 3$, which we call the *weighted triangle-free* 2-*matching problem*, is still open. Hartvigsen and Li [11] gave a polyhedral description and a polynomial-time algorithm for the weighted triangle-free 2-matching problem in subcubic graphs.

Theorem 1.1 ([11]). *The weighted triangle-free 2-matching problem in subcubic graphs can be solved in polynomial time.*

Note that the running time of their algorithm is $O(n^3)$, where n is the number of vertices of the input graph [12].

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1.2. Our result

In this paper, we provide a simple polynomial time algorithm for the weighted triangle-free 2-matching problem in simple subcubic graphs, and give another proof of Theorem 1.1. Our algorithm is a steepest ascent algorithm in the space of degree sequences of the input graph $G = (V, E)$, whereas Hartvigsen and Li give a primal–dual algorithm based on the polyhedral description. The *degree sequence* of an edge set $F \subseteq E$ is the vector $d_F \in \mathbf{Z}^V$ such that $d_F(v)$ is the number of edges in F incident with v. Let $J_{\rm tri}(G) \subseteq \mathbf{Z}^V$ denote the set of all degree sequences of triangle-free 2-matchings in G, that is,

$$J_{\rm tri}(G) = \{d_M \mid M \text{ is a triangle-free 2-matching in } G\}.$$

For a weighted graph (G, w), define a function $f_{\rm tri}$ on $J_{\rm tri}(G)$ by

$$f_{\rm tri}(x) = \max \{w(M) \mid M \text{ is a triangle-free 2-matching}, d_M = x\},$$

where $w(F) = \sum_{e \in F} w(e)$ for an edge set $F \subseteq E$.

The objective of this paper is to show that the following simple algorithm solves the weighted triangle-free 2-matching problem in subcubic graphs. Note that since triangle-free 2-matchings contain neither parallel edges nor self-loops, we may assume that the input graph is simple.

Steepest Ascent Algorithm for $f_{\rm tri}$
Input: A simple subcubic graph $G = (V, E)$ with a weight function $w : E \to \mathbf{R}_+$.
Output: A triangle-free 2-matching M maximizing $w(M)$.
Step 0. Set $x := (0, 0, \dots, 0) \in J_{\rm tri}(G)$.
Step 1. Compute $f_{\rm tri}(y)$ for every degree-sequence $y \in J_{\rm tri}(G)$ with $\sum_{v \in V} |y(v) - x(v)| \le 2$ (see Section 3), and let y^* be a maximizer. Go to Step 2.
Step 2. If $f_{\rm tri}(x) \ge f_{\rm tri}(y^*)$, then output a triangle-free 2-matching M such that $d_M = x$ and $w(M) = f_{\rm tri}(x)$, and stop the algorithm. Otherwise, update $x := y^*$ and go back to Step 1.

Let γ be the time to find a maximum weight b-matching for $b \in \{0, 1, 2\}^V$. That is, for a weighted (not necessarily simple) graph (G, w) and for a vector $b \in \{0, 1, 2\}^V$, we can find in γ time an edge set $F \subseteq E$ maximizing $w(F)$ subject to $d_F = b$. Note that γ is of the same order as the time to find a maximum weight matching, and it is bounded by $\mathrm{O}(n(m + n \log n))$ and $\mathrm{O}(m \log(nw(E))\sqrt{n\alpha(m, n) \log n})$, where $n = |V|$, $m = |E|$, and α is the inverse of the Ackermann function [13,14]. We also note that the running time is a bit better in the case of subcubic graphs, since $m = \mathrm{O}(n)$. Then, our main result is stated as follows.

Theorem 1.2. *In a weighted simple subcubic graph (G, w), the Steepest Ascent Algorithm for $f_{\rm tri}$ finds a triangle-free 2-matching with a maximum total weight in $\mathrm{O}(n^3\gamma)$ time.*

One can see that Theorem 1.1 immediately follows from this theorem, because we may assume that the input graph is simple when we consider the weighted triangle-free 2-matching problem. We mention here that in our proof for Theorem 1.2, we use the theory of M-concave functions on constant-parity jump systems introduced by Murota [15]. Jump systems and M-concave functions are understood as a natural framework of efficiently solvable problems. Besides studies of these structures themselves [16,17,15,18], their relation to efficiently solvable combinatorial optimization problems has been revealed (see [19,6,20–22,15,23]). In particular, on the basis of the theory of jump systems, an algorithm for the square-free 2-matching problem in subcubic graphs is proposed in [6]. This paper presents another such problem and enforces the importance of these structures.

This paper is organized as follows. In Section 2, we give definitions and previous work on triangle-free 2-matchings, jump systems, and M-concave functions. In Section 3, we give a polynomial-time algorithm to evaluate $f_{\rm tri}(x)$ for a given $x \in \mathbf{Z}^V$, which is used as a subroutine in our algorithm. Finally, in Section 4, we show that $f_{\rm tri}$ is an M-concave function on a jump system, and give a proof of Theorem 1.2.

2. Preliminaries

2.1. Triangle-free 2-matchings

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E, and n and m denote the number of vertices and the number of edges, respectively. An edge connecting $u, v \in V$ is denoted by (u, v). A cycle C, which is denoted as $C = (v_1, v_2, \dots, v_l)$, is a subgraph consisting of distinct vertices $v_1, v_2, \dots, v_l$ and edges $(v_1, v_2), \dots, (v_{l-1}, v_l), (v_l, v_1)$. For a subgraph H of G, the vertex set and the edge set of H are denoted by $V(H)$ and $E(H)$, respectively. Let $\delta(v)$ denote the set of edges incident to $v \in V$, and the *degree* of v is $|\delta(v)|$. The *degree sequence* of an edge set $F \subseteq E$ is the vector $d_F \in \mathbf{Z}^V$ defined by $d_F(v) = |\delta(v) \cap F|$. We say that a graph $G = (V, E)$ is *subcubic* if $|\delta(v)| \le 3$ for every $v \in V$. An edge set $M \subseteq E$ is said to be a 2-*matching* if $d_M(v) \le 2$ for every $v \in V$. In other words, a 2-matching is a vertex-disjoint collection of paths and cycles. An edge set $M \subseteq E$ is said to be *triangle-free* (or C_3-*free*) if M induces no cycle of a length of three or less as a subgraph. In a graph with a weight function w on the edge set, the *weighted triangle-free* 2-*matching problem* is to find a triangle-free 2-matching M maximizing $w(M)$.

2.2. Jump systems

Let V be a finite set. For $u \in V$, we denote by χ_u the *characteristic vector* of u, with $\chi_u(u) = 1$ and $\chi_u(v) = 0$ for $v \in V \setminus \{u\}$. For $x, y \in \mathbf{Z}^V$, a vector $s \in \mathbf{Z}^V$ is called an (x, y)-*increment* if $x(u) < y(u)$ and $s = \chi_u$ for some $u \in V$, or $x(u) > y(u)$ and $s = -\chi_u$ for some $u \in V$. We say that a nonempty set $J \subseteq \mathbf{Z}^V$ is a *jump system* if it satisfies the following [24]:

For any $x, y \in J$ and for any (x, y)-increment s with $x+s \notin J$, there exists an $(x+s, y)$-increment t such that $x+s+t \in J$.

A set $J \subseteq \mathbf{Z}^V$ is a *constant-parity system* if $x(V) - y(V)$ is even for any $x, y \in J$. Here $x(S) = \sum_{v\in S} x(v)$ for $x \in \mathbf{Z}^V$ and $S \subseteq V$. For constant-parity jump systems, J. F. Geelen showed that a nonempty set J is a constant-parity jump system if and only if it satisfies the following (see [15] for details):

(EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$ and $y - s - t \in J$.

A constant-parity jump system is a generalization of the base family of a matroid, an even delta-matroid [25,26], and a base polyhedron of an integral polymatroid (or a submodular system) [27].

The degree sequences of all subgraphs in an undirected graph are a typical example of a constant-parity jump system [24,17]. Cunningham [28] showed that the set of degree sequences of all C_k-free 2-matchings is a jump system for $k \le 3$, but not a jump system for $k \ge 5$. Szabó [23] showed that it is also a jump system when $k = 4$.

2.3. M-concave functions

An *M-concave function on a constant-parity jump system* is a quantitative extension of a jump system, which is a generalization of valuated matroids [29,30], valuated delta-matroids [20], and M-concave functions on base polyhedra [31,32].

Definition 2.1 (*M-concave Function on a Constant-Parity Jump System [15]*)**.** For $J \subseteq \mathbf{Z}^V$, we call $f : J \to \mathbf{R}$ an *M-concave function on a constant-parity jump system* if it satisfies the following exchange axiom:

(M-EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$, $y - s - t \in J$, and $f(x) + f(y) \le f(x + s + t) + f(y - s - t)$.

It directly follows from (M-EXC) that J satisfies (EXC), and hence J is a constant-parity jump system. Conversely, for a constant-parity jump system J, a constant function defined on J is an M-concave function on J. M-concave functions on constant-parity jump systems appear in many combinatorial optimization problems such as the weighted matching problem, the minsquare factor problem [19], the weighted even factor problem in odd-cycle-symmetric digraphs [21], and the weighted square-free 2-matching problem [22,23]. Some properties of M-concave functions are investigated in [16].

The following algorithm is known to efficiently find a maximizer of an M-concave function on a constant-parity jump system.

Steepest Ascent Algorithm
Input: An M-concave function f on a finite constant-parity jump system J and a vector $x_0 \in J$.
Output: A vector $x^* \in J$ maximizing $f(x^*)$.
Step 0. Set $x := x_0$.
Step 1. Compute $f(y)$ for every vector $y \in J$ with $\sum_{v\in V} |y(v) - x(v)| \le 2$, and let y^* be a maximizer. Go to Step 2.
Step 2. If $f(x) \ge f(y^*)$, then output x and stop the algorithm. Otherwise, update $x := y^*$ and go back to Step 1.

Theorem 2.2 (*[15,33]*)**.** *Let $J \subseteq \mathbf{Z}^V$ be a finite constant-parity jump system, and $f : J \to \mathbf{Z}$ be an M-concave function on J. Suppose that a vector $x_0 \in J$ is given, and we can check whether $x \in J$ or not and evaluate $f(x)$ in γ_0 time for $x \in \mathbf{Z}^V$. Then the Steepest Ascent Algorithm finds a vector $x \in J$ maximizing $f(x)$ in $\mathrm{O}(n^3\Phi(J)\gamma_0)$ time, where $\Phi(J) = \max_{v\in V}\{\max_{y\in J} y(v) - \min_{y\in J} y(v)\}$.*

Note that an algorithm that runs in $\mathrm{O}(n^4(\log \Phi(J))^2\gamma_0)$ time is given for the problem in [18]. However, in this paper we only deal with the case when $\Phi(J)$ is fixed. In this case, this algorithm is slower than the Steepest Ascent Algorithm.

3. Computing values of f_{tri}

In this section, we give a polynomial-time algorithm to evaluate $f_{\mathrm{tri}}(x)$ for a given $x \in \mathbf{Z}^V$. Recall that γ denotes the time to find an edge set $F \subseteq E$ maximizing $w(F)$ subject to $d_F = b$ for a given $b \in \{0, 1, 2\}^V$. We show that a maximum weight b-matching with an additional condition "triangle-free" can also be found efficiently.

Lemma 3.1. *Suppose that we are given a simple subcubic graph $G = (V, E)$ with a weight function $w : E \to \mathbf{R}_+$, and a vector $x \in \{0, 1, 2\}^V$. We can test whether $x \in J_{\mathrm{tri}}(G)$ or not, and find if $x \in J_{\mathrm{tri}}(G)$ a triangle-free 2-matching M such that $w(M) = f_{\mathrm{tri}}(x)$ and $d_M = x$ in $\mathrm{O}(\gamma)$ time.*

Proof. For a cycle C of length three, *shrinking* C means deleting $E(C)$ and identifying all the vertices in $V(C)$. Take a maximal family of vertex-disjoint cycles $C_1, C_2, \dots, C_q$ of length three such that $x(v) = 2$ for each $v \in V(C_1) \cup \cdots \cup V(C_q)$. Let $G^\circ = (V^\circ, E^\circ)$ denote the graph obtained from $G = (V, E)$ by shrinking $C_1, C_2, \dots, C_q$. Note that E° may contain parallel edges.

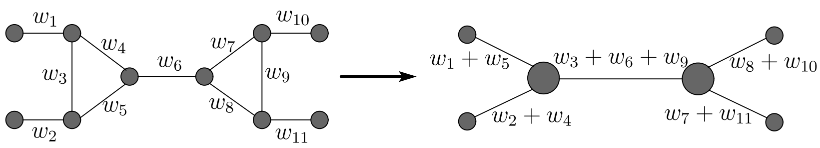

Fig. 1. Definition of the new weight function.

Define $E_1 \subseteq E$ as the set of all removed edges, that is, $E_1 = E(C_1) \cup \cdots \cup E(C_q)$, and let $E_0 = E \setminus E_1$. Similarly, define $V_1 \subseteq V$ as $V_1 = V(C_1) \cup \cdots \cup V(C_q)$, and let $V_0 = V \setminus V_1$. Then, V_0 is also a subset of V°, and E_0 and E° represent the same edge set.

Let $x^\circ \in \mathbf{Z}^{V^\circ}$ be the vector obtained from x by setting

$$x^\circ(v) = \begin{cases} x(v) & \text{if } v \in V_0, \\ 2 & \text{if } v \in V^\circ \setminus V_0. \end{cases}$$

For each cycle $C_i = (v_1^i, v_2^i, v_3^i)$, define $p : V_1 \to \mathbf{R}_+$ by $p(v_1^i) = w(v_2^i, v_3^i)$, $p(v_2^i) = w(v_3^i, v_1^i)$, and $p(v_3^i) = w(v_1^i, v_2^i)$. We define $w^\circ : E_0 \to \mathbf{R}_+$ by

$$w^\circ(e) = \begin{cases} w(e) & \text{if } e = (u, v) \text{ where } u, v \in V_0, \\ w(e) + p(v) & \text{if } e = (u, v) \text{ where } u \in V_0 \text{ and } v \in V_1, \\ w(e) + p(u) + p(v) & \text{if } e = (u, v) \text{ where } u, v \in V_1. \end{cases}$$

Then, w° can be regarded as a weight function on E°, because E_0 and E° represent the same edge set (see Fig. 1). We will show that $f_{\text{tri}}(x) = f(x^\circ)$ where

$$f(x^\circ) = \max\{w^\circ(M^\circ) \mid M^\circ \text{ is a 2-matching in } G^\circ, d_{M^\circ} = x^\circ\}.$$

Let M be a triangle-free 2-matching in $G = (V, E)$ such that $d_M = x$ and $w(M) = f_{\text{tri}}(x)$. Since G is subcubic, we can see that $|E(C_i) \cap M| = 2$ for $i = 1, 2, \ldots, q$. Then, $M^\circ = M \cap E_0$ is a 2-matching in G° such that $d_{M^\circ} = x^\circ$ and $w^\circ(M^\circ) = f_{\text{tri}}(x)$. This shows that $f(x^\circ) \geq f_{\text{tri}}(x)$.

Conversely, let M° be a 2-matching in $G^\circ = (V^\circ, E^\circ)$ such that $d_{M^\circ} = x^\circ$ and $w^\circ(M^\circ) = f(x^\circ)$. For each shrunk cycle C_i, two properly chosen edges of C_i can be added to M° so that the obtained edge set M forms a 2-matching, because G is subcubic. Then M is triangle-free, $d_M = x$, and $w(M) = f(x^\circ)$, which shows that $f_{\text{tri}}(x) \geq f(x^\circ)$.

Hence, $f_{\text{tri}}(x) = f(x^\circ)$. Since we can find a 2-matching M° in G° such that $d_{M^\circ} = x^\circ$ and $w^\circ(M^\circ) = f(x^\circ)$ in $O(\gamma)$ time, the desired triangle-free 2-matching M can be found in $O(\gamma)$ time. □

4. *M*-concavity of f_{tri}

In this section, we show the M-concavity of f_{tri}.

Theorem 4.1. *For a simple subcubic graph $G = (V, E)$ with a weight function $w : E \to \mathbf{R}_+$, f_{tri} is an M-concave function on the constant-parity jump system $J_{\text{tri}}(G)$.*

Before proving this theorem, we give a proof of Theorem 1.2 on the basis of Theorem 4.1.

Proof of Theorem 1.2. By combining Theorem 4.1 with Lemma 3.1 and Theorem 2.2, we see that the Steepest Ascent Algorithm for f_{tri} correctly finds a triangle-free 2-matching with maximum total weight in $O(n^3 \Phi(J_{\text{tri}}(G))\gamma)$ time. Since $\Phi(J_{\text{tri}}(G)) \leq 2$, the running time is $O(n^3\gamma)$. □

The rest of this section is devoted to a proof of Theorem 4.1. We prove Theorem 4.1 by giving an algorithm for finding an $(x + s, y)$-increment t satisfying (M-EXC). Without loss of generality, we assume that $s = -\chi_u$ for some $u \in V$.

4.1. Preliminaries for the proof

In this subsection, we give some preliminaries for the proof.

Definition 4.2. Let $C = (v_1, v_2, v_3)$ be a cycle of length three in $G = (V, E)$. *Contracting* $e = (v_1, v_2)$ in C consists of the following operations:

- Delete e and identify v_1 with v_2 to obtain a new vertex u ("contract" e in the standard meaning).
- Identify two edges between the new vertex u and v_3.

In the obtained graph, the edge between u and v_3 corresponding to $E(C)$ is called a *triangle-edge*.

Let M and N be edge sets in an undirected (not necessarily simple) graph. We say that a path $P = (v_0, v_1, v_2, \ldots, v_l)$ is an (M, N)-*alternating path* if

- $(v_i, v_{i+1}) \in M \setminus N$ if i is even,

- $(v_i, v_{i+1}) \in N \setminus M$ if i is odd, and
- $(v_i, v_{i+1}) \neq (v_j, v_{j+1})$ for $i \neq j$.

Obviously, $d_{M\Delta E(P)} = d_M - \chi_{v_0} + (-1)^l \chi_{v_l}$ and $d_{N\Delta E(P)} = d_N + \chi_{v_0} - (-1)^l \chi_{v_l}$, where Δ denotes the symmetric difference. By taking the longest (M, N)-alternating path, we can see the following.

Lemma 4.3. *For 2-matchings M, N in an undirected graph and for a (d_M, d_N)-increment $s = -\chi_u$, there exists an (M, N)-alternating path P beginning with $v_0 = u$ such that both $M\Delta E(P)$ and $N\Delta E(P)$ are 2-matchings (not necessarily triangle-free), $d_{M\Delta E(P)} = d_M + s + t$, and $d_{N\Delta E(P)} = d_N - s - t$ for some $(x + s, y)$-increment t.*

4.2. Constructive proof for Theorem 4.1

Suppose that $s = -\chi_u$ for some $u \in V$. In this subsection, we give a procedure to find an $(x+s, y)$-increment t satisfying (M-EXC).

For given degree sequences $x, y \in J_{\text{tri}}(G)$, take triangle-free 2-matchings $M, N \subseteq E$ such that $d_M = x$, $d_N = y$, $f_{\text{tri}}(x) = w(M)$, and $f_{\text{tri}}(y) = w(N)$. Let $s = -\chi_u$ be an (x, y)-increment for some $u \in V$. Let $C_1, C_2, \ldots, C_q$ be a maximal family of vertex-disjoint cycles of a length of three in G such that $E(C_i) \subseteq M \cup N$ and $|E(C_i) \cap M| = |E(C_i) \cap N| = 2$ for $i = 1, 2, \ldots, q$. Then, there exists exactly one edge $e_i \in E(C_i) \cap M \cap N$ for each i. We contract e_i in C_i to obtain a new vertex u_i for $i = 1, 2, \ldots, q$. Let $G^\circ = (V^\circ, E^\circ)$ be the obtained graph, and let $M^\circ, N^\circ, x^\circ, y^\circ, u^\circ$ and s° be counterparts in G° to M, N, x, y, u and s, respectively.

In G°, a *triangle* denotes a cycle of length three whose vertices are not incident to a triangle-edge. In other words, a cycle in G° is a triangle if its corresponding edges in G form a cycle of length three. We say that an edge set in G° is *triangle-free* if it contains no triangle. Then, G° satisfies the following condition.

(A) Both edge sets M° and N° contain all triangle-edges in G°, and G° has no triangle C such that $E(C) \subseteq M^\circ \cup N^\circ$ and $|E(C) \cap M^\circ| = |E(C) \cap N^\circ| = 2$.

In order to obtain an $(x + s, y)$-increment t, we find an (M°, N°)-alternating path P in G°.

Lemma 4.4. *Suppose that $M^\circ, N^\circ, x^\circ, y^\circ, u^\circ$ and s° are defined as above. There exists an (M°, N°)-alternating path P beginning with u° such that both $M^\circ\Delta E(P)$ and $N^\circ\Delta E(P)$ are triangle-free 2-matchings, $d_{M^\circ\Delta E(P)} = x^\circ + s^\circ + t^\circ$, and $d_{N^\circ\Delta E(P)} = y^\circ - s^\circ - t^\circ$ for some $(x^\circ + s^\circ, y^\circ)$-increment t°.*

Proof. Let $P = (v_0, v_1, v_2, \ldots, v_l)$ be an (M°, N°)-alternating path beginning with $v_0 = u^\circ$ such that both $M^\circ\Delta E(P)$ and $N^\circ\Delta E(P)$ are (not necessarily triangle-free) 2-matchings, $d_{M^\circ\Delta E(P)} = d_{M^\circ} + s^\circ + t^\circ$, and $d_{N^\circ\Delta E(P)} = d_{N^\circ} - s^\circ - t^\circ$ for some $(x^\circ + s^\circ, y^\circ)$-increment t°. The existence of such a path is guaranteed by Lemma 4.3. We choose v_1 such that $N \cup \{(v_0, v_1)\}$ is triangle-free if possible. Furthermore, we assume the minimality of P, that is, no subpath $(v_0, v_1, v_2, \ldots, v_p)$ satisfies the above conditions for $1 \leq p \leq l - 1$. Let $P^{(p)}$ be the subpath $(v_0, v_1, v_2, \ldots, v_p)$ of P, and define $M^{(p)} = M^\circ\Delta P^{(p)}$ and $N^{(p)} = N^\circ\Delta P^{(p)}$.

In what follows, we show that $M^{(l)}$ and $N^{(l)}$ are triangle-free, which implies that P is the desired path. Assume, in order to obtain a contradiction, that $M^{(l)}$ or $N^{(l)}$ contains a triangle. Then, there exists an integer p such that $M^{(0)}, M^{(1)}, \ldots, M^{(p)}$ and $N^{(0)}, N^{(1)}, \ldots, N^{(p)}$ are triangle-free, and $M^{(p+1)}$ or $N^{(p+1)}$ contains a triangle.

We consider the case when p is even, that is, $M^{(p+1)}$ is triangle-free and $N^{(p+1)}$ has a triangle containing (v_p, v_{p+1}). The case when p is odd can be dealt with in the same way. Let $C = (v_{p+1}, v_p, w)$ be the triangle in $N^{(p+1)}$.

When $p \geq 1$, by the minimality of l, $M^{(p)}$ is not a 2-matching, that is, $d_{M^{(p)}}(v_p) = 3$. Therefore $\{(v_p, v_{p+1}), (v_p, w)\} \subseteq M^{(p)}$, because G° is subcubic. This means that

$$|E(C) \cap M^\circ| + |E(C) \cap N^\circ| = |E(C) \cap M^{(p)}| + |E(C) \cap N^{(p)}| = 4,$$

which contradicts condition (A).

When $p = 0$, by condition (A), $M \cap E(C) = \{(v_0, v_1)\}$ and $N \cap E(C) = \{(v_0, w), (w, v_1)\}$. Since $d_{M^\circ}(v_0) > d_{N^\circ}(v_0)$, there exists an edge $e \in \delta(v_0) \cap (M^\circ \setminus N^\circ)$ such that $N^\circ \cup \{e\}$ is triangle-free, which contradicts the definition of P. $\square$

Let C_i and e_i be as defined in the beginning of this subsection, and P be a path satisfying the conditions in Lemma 4.4. Define $E_1 = E(C_1) \cup \cdots \cup E(C_q)$, and let $E_0 = E \setminus E_1$. Since $E(P) \subseteq E^\circ$ contains no triangle-edges, $E(P)$ can be regarded as the edge set of E_0. Let P' be the path in G whose edge set $E(P') \subseteq E$ is defined by

$$E(P') = E(P) \cup \bigcup_{i:u_i \in V(P)} (E(C_i) \setminus \{e_i\}).$$

Then, P' is an (M, N)-alternating path beginning with u (see Fig. 2). Hence, the edge sets M' and N' in G defined by $M' = M\Delta E(P')$ and $N' = N\Delta E(P')$ satisfy $d_{M'} = x+s+t$ and $d_{N'} = y-s-t$, where t is an $(x+s, y)$-increment corresponding to t°. Furthermore, since $M^\circ\Delta E(P)$ and $N^\circ\Delta E(P)$ are obtained from M' and N' by contracting e_i in C_i, M' and N' have no cycle of a length of three.

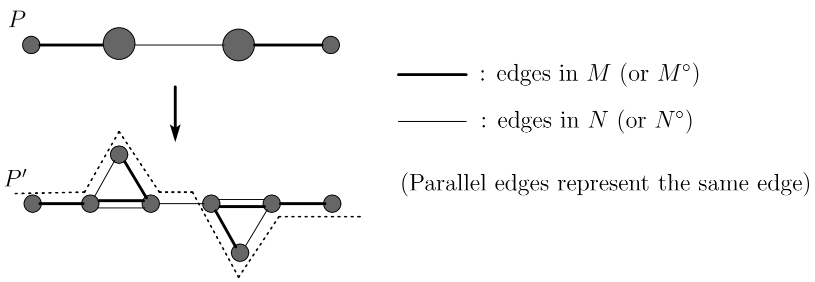

Fig. 2. Construction of P'.

Then, we have

$$\begin{aligned} f_{\mathrm{tri}}(x) + f_{\mathrm{tri}}(y) &= w(M) + w(N) \\ &= w(M') + w(N') \\ &\le f_{\mathrm{tri}}(x+s+t) + f_{\mathrm{tri}}(y-s-t). \end{aligned}$$

Hence f_{tri} is an M-concave function on J_{tri}, which completes the proof of Theorem 4.1.

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