Complete Nevanlinna–Pick Property of
Dirichlet-Type Spaces

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Communicated by D. Sarason

Received January 2, 2001; accepted August 1, 2001

We prove that all Dirichlet-type spaces of functions analytic in the unit disk whose
derivatives are square area integrable with superharmonic weights have complete
Nevanlinna–Pick reproducing kernels. As a corollary, we obtain a commutant lifting
theorem for cyclic analytic two-isometries.

Key Words: complete Nevanlinna–Pick kernels, Dirichlet-type spaces.

1. INTRODUCTION

Recent progress in the theory of reproducing kernels is related to the
study of complete Nevanlinna–Pick kernels (in brief, NP kernels). These
are positive definite kernels \( k(z, \lambda) \) satisfying the identity

\[
k(z, \lambda) - \frac{k(z, \alpha) k(\alpha, \lambda)}{k(\alpha, \alpha)} = B_\alpha(z, \lambda) k(z, \lambda)
\]

for some point \( \alpha \) such that \( k(\alpha, \alpha) \neq 0 \) and some positive semidefinite function \( B_\alpha(z, \lambda) \) such that \( |B_\alpha(z, \lambda)| < 1 \). This condition for the kernel appears in the study of the Nevanlinna–Pick problem for multipliers of Hilbert
spaces of functions. Namely, if \( \mathcal{H} \) is a Hilbert space of functions defined
on a set \( \mathcal{X} \) such that the evaluation functionals \( f \mapsto f(x) \) are bounded for
any \( x \in \mathcal{X} \), and the reproducing kernel \( k \) for the space \( \mathcal{H} \) satisfies (1), then
the interpolation problem

\[
\phi(x_i) = w_i, \quad x_i \in \mathcal{X}, \ i = 1, \ldots, n
\]

has a solution \( \phi \) which is a multiplier of \( \mathcal{H} \) with the multiplier norm less
than or equal to one if and only if the matrix

\[
[(1 - w_i w_j^*) k(x_i, x_j)]_{i,j=1}^n
\]
is positive definite [9]. (If $w_j$ are scalar, $w_j^\ast$ denotes the complex conjugate $w_j$.) Moreover, if positivity of the matrix (3) implies the existence of a contractive multiplier $\phi$ solving (2) in matrix-valued context (i.e., where the interpolation data $w_j$ are matrices and we search for matrix-valued multipliers $\phi$ of the space $H \otimes \mathbb{C}^n$), then $k_H$ is an NP kernel [1, 7]. Of course, the Nevanlinna–Pick problem is a particular case of the commutant lifting problem. A general commutant lifting theorem for spaces with NP kernels is proved in [5].

A natural question related to NP kernels is to decide, given a particular Hilbert space of functions, whether or not the reproducing kernel for this space is an NP kernel. At present, the following examples of Hilbert spaces with NP kernels are known (see [5]).

1. Spaces $l^2(w_n)$ of functions $f(z) = \sum_{n \geq 0} f(n) z^n$ analytic in the unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$ and satisfying
   \[
   \|f\|^2 := \sum_{n \geq 0} |f(n)|^2 w_n < +\infty,
   \]
   where a weight sequence $(w_n)_{n \geq 0}$ is such that
   \[
   \left( \sum_{n \geq 0} w_n^{-1} u^n \right)^{-1} = w_0 - \sum_{n \geq 1} b_n u^n \quad \text{with} \quad b_n \geq 0, \; n \geq 1. \tag{4}
   \]
   A sufficient condition for the property (4) is that the sequence $(w_n)_{n \geq 0}$ is logarithmically concave; i.e., $w_{n+1}^2 \geq w_n w_{n+2}$, $n \geq 0$ (see [6, Chap. IV, Theorem 22]). In particular, the Dirichlet space $D$ corresponding to the weight sequence $w_n = n+1$ has an NP kernel.

2. Weighted Sobolev spaces on the interval. Given real-valued functions $w_0 \in C[x_0, x_1]$ and $w_1 \in C^1[x_0, x_1]$ with $w_1(x) > 0$, we assume that the weighted Sobolev form
   \[
   Q_{w_0, w_1}[f] := \int_{x_0}^{x_1} [f(x)]^2 w_0(x) \, dx + \int_{x_0}^{x_1} [f'(x)]^2 w_1(x) \, dx
   \]
   is strictly positive definite on $W^1_2 = \{ f : [x_0, x_1] \to \mathbb{C} : f \text{ is absolutely continuous with } |f'|^2 \text{ integrable } \}$; i.e., $Q_{w_0, w_1}[f] > 0$ for any $f \in W^1_2$, $f \neq 0$. Then the reproducing kernel of $W^1_2$ with respect to the norm
   \[
   \|f\|^2_{w_0, w_1} := Q_{w_0, w_1}[f]
   \]
   is an NP kernel.
A similar example is given by the space of functions $f$ defined and absolutely continuous on the interval $[a, b] \subset \mathbb{R}$ and satisfying $f(a) = 0$ and

$$
\|f\|_2^2 := \int_a^b \frac{|f'(x)|^2}{\rho(x)} \, dx < +\infty,
$$

where $\rho$ is some strictly positive weight function integrable and continuous on $[a, b]$ (see [9, Theorem 6.7]).

3. Arveson space $H^2_d$ [4]. This is the space of functions analytic in the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$ such that its reproducing kernel is

$$
k(z, \lambda) = \frac{1}{1 - \langle z, \lambda \rangle_{\mathbb{C}^d}}, \quad z, \lambda \in \mathbb{B}_d.
$$

(5)

This space can be identified with the symmetric Fock space associated with the Hilbert space $E = \mathbb{C}^d$. Kernels of the form (5) are universal NP kernels in the following sense: any NP kernel after a suitable normalization can be represented as a restriction of the kernel (5) for certain $d \geq 1$, possibly $d = +\infty$ [1]. The aim of the present paper is to prove that all Dirichlet-type spaces $D(\mu)$ have NP reproducing kernels. Given a finite positive measure $\mu$ supported in $\mathbb{D}$, the space $D(\mu)$ is defined as a Hilbert space of functions $f$ analytic in the unit disk $\mathbb{D}$ and having a finite norm

$$
\|f\|_\mu^2 := \|f\|_H^2 + \int_{\mathbb{D}} |f'(\zeta)|^2 U_\mu(\zeta) \, dm_2(\zeta).
$$

(6)

Here, $H^2$ is the Hardy space in $\mathbb{D}$, $dm_2$ is the normalized area measure in $\mathbb{D}$, and the function $U_\mu$ is defined as

$$
U_\mu(\zeta) := \int_{\mathbb{D}} \log \left| \frac{1 - \zeta_2^2}{1 - |z|^2} \right|^2 \, d\mu(z) + \int_{\mathbb{T}} \frac{1 - |\xi|^2}{1 - |2\xi - 2\zeta|^2} \, d\mu(z), \quad \zeta \in \mathbb{D}.
$$

(7)

The spaces $D(\mu)$ were introduced by Richter [10] in the case of $\mu$ supported on $\mathbb{T} = \partial \mathbb{D}$ and Aleman [2] for general $\mu$. In fact, any positive superharmonic function in $\mathbb{D}$ can be represented in the form $U_\mu$ for certain $\mu$ [2, p. 79], so that spaces $D(\mu)$ can be considered as weighted Dirichlet spaces of functions analytic in $\mathbb{D}$ whose derivatives are square integrable with superharmonic weights. The spaces $D(\mu)$ were studied also in [12–15].
Our main result is

**Theorem 1.1.** If $\mu$ is a finite positive measure in $D$, then the reproducing kernel $k = k_\mu$ for the space $D(\mu)$ is an NP kernel. In particular, there exists a positive semidefinite function $B(z, \lambda)$, $z, \lambda \in D$, such that $B(\cdot, 0) = 0$ and

$$k(z, \lambda) = \frac{1}{1 - B(z, \lambda)}, \quad z, \lambda \in D. \tag{8}$$

A bounded operator $T$ in a Hilbert space $H$ is called a two-isometry if

$$T^*T^2 - 2T^*T + I = 0.$$ 

It has been proved in [10] that any two-isometry which is analytic (which means that $\int_{a > 1} T^*H = \{0\}$) and cyclic is unitarily equivalent to the shift operator

$$S: f(z) \mapsto zf(z)$$
in the space $D(\mu)$ for some measure $\mu$ supported on $\mathbb{T}$. This representation, together with the abstract commutant lifting theorem for Hilbert spaces with NP kernels [5] leads to the following corollary of Theorem 1.1.

**Theorem 1.2.** Assume that $T$ is a cyclic analytic two-isometry in a separable Hilbert space $H$. If $J$ is a closed $T^*$-invariant subspace of $H$ and $X: J \mapsto J$ is a bounded operator commuting with the restriction $T^*|_J$, then there exists a bounded operator $Y: H \mapsto H$ such that

(i) $\|Y\| = \|X\|$
(ii) $Y$ commutes with $T^*$
(iii) $J$ is invariant for $Y$ and $Y|_J = X$.

The proof of Theorem 1.1 uses the following arguments. First, we prove that if a Hilbert space $\mathcal{H}$ of functions defined on a set $X$ has an NP reproducing kernel $k_\mathcal{H}$ and a finitely atomic measure $\mu = \sum_{k=1}^n a_k \delta_{x_k}$, $a_k > 0$, $x_k \in X$ is such that the norm

$$\|f\|_1^2 := \|f\|_\mathcal{H}^2 - \int_X |f|^2 \, d\mu \tag{9}$$
is strictly positive for any $f \in \mathcal{H}$, $f \neq 0$ (and in this case it is equivalent to the original norm in $\mathcal{H}$), then the reproducing kernel $k_1$ of $\mathcal{H}$ with respect to the new norm $\|\cdot\|_1$ is also an NP kernel. Of course, in many concrete
situations it is possible to use approximation arguments and to obtain a similar result for arbitrary measures $\mu$ such that the norm ($9$) is positive. Below, we shall discuss the case $H = H^2(\mathbb{D})$ (see Theorem 2.2).

Then, we prove Theorem 1.1 in the case where $\mu$ is a finitely atomic measure supported in $\mathbb{D}$. It turns out that for such measures the space $D(\mu)$ coincides with $H^2$ as a set and the norm in $D(\mu)$ is given by the formula

$$
\|f\|_{D(\mu)}^2 = \|F_\mu f\|_{H^2}^2 - \int_\mathbb{D} |f|^2 d\nu_\mu,
$$

where $F_\mu$ is a certain function bounded and bounded away from zero in $\mathbb{D}$ and $\nu_\mu$ is a certain finitely atomic measure supported in $\mathbb{D}$.

Finally, we use approximation arguments to deduce the general case of Theorem 1.1.

2. GENERAL REPRODUCING KERNELS

Let $\mathcal{H}$ be a Hilbert space of functions defined on a set $\mathcal{X}$. Saying this we always mean that the evaluation functionals $f \mapsto f(\lambda)$ are bounded for all $\lambda \in \mathcal{X}$. Then the reproducing kernel for $\mathcal{H}$ is a function $k_\mathcal{H}(z, \lambda)$, $z, \lambda \in \mathcal{X}$ uniquely determined by the properties

(i) $k_\mathcal{H}(\cdot, \lambda) \in \mathcal{H}$ for any $\lambda \in \mathcal{X}$;

(ii) $f(\lambda) = \langle f, k_\mathcal{H}(\cdot, \lambda) \rangle$ for any $\lambda \in \mathcal{X}$ and $f \in \mathcal{H}$.

We shall assume that $k_\mathcal{H}$ is nondegenerate; i.e. $k_\mathcal{H}(\alpha, \alpha) \neq 0$ for any $\alpha \in \mathcal{X}$.

**Definition 2.1.** The reproducing kernel $k_\mathcal{H}$ is said to be an irreducible complete Nevanlinna–Pick kernel (in brief, NP kernel) if there exists $\alpha \in \mathcal{X}$ and a positive semidefinite function $B_\alpha(z, \lambda)$, $z, \lambda \in \mathcal{X}$ such that $|B_\alpha(z, \lambda)| < 1$ and

$$
k_\mathcal{H}(z, \lambda) = k_\mathcal{H}(z, \alpha) k_\mathcal{H}(\lambda, \lambda) - k_\mathcal{H}(\alpha, \lambda) k_\mathcal{H}(\alpha, \alpha) = B_\alpha(z, \lambda) k_\mathcal{H}(z, \lambda)
$$

(10)

for any $z, \lambda \in \mathcal{X}$.

Solving (10) for $k_\mathcal{H}(z, \lambda)$, we find that

$$
k_\mathcal{H}(z, \lambda) = k_\mathcal{H}(z, \alpha) k_\mathcal{H}(\alpha, \lambda) \frac{1}{1 - B_\alpha(z, \lambda)}.
$$

(11)
Since \( k_{\mathcal{H}}(z, z) \neq 0 \), we obtain that \( k_{\mathcal{H}}(z, \alpha) \neq 0 \) for any \( z \in \mathcal{X} \), and therefore \( k_{\mathcal{H}}(z, \lambda) \neq 0 \) for any \( z, \lambda \in \mathcal{X} \). It has been shown in [8] that in this case Definition 2.1 does not depend on the choice of the base point \( \alpha \): if (10) is fulfilled for some \( \alpha \in \mathcal{X} \), then for any other \( \beta \in \mathcal{X} \) there exists another positive semidefinite function \( B_{\beta}(z, \lambda) \), \( z, \lambda \in \mathcal{X} \) such that \( |B_{\beta}(z, \lambda)| < 1 \) and

\[
k_{\mathcal{H}}(z, \lambda) - \frac{k_{\mathcal{H}}(z, \beta) k_{\mathcal{H}}(\beta, \lambda)}{k_{\mathcal{H}}(\beta, \beta)} = B_{\beta}(z, \lambda) k_{\mathcal{H}}(z, \lambda)
\]

for any \( z, \lambda \in \mathcal{X} \). Moreover, as shown in [1], any reproducing kernel in the form

\[
k_{\mathcal{H}}(z, \lambda) = \frac{\delta(z) \overline{\delta(\lambda)}}{1 - B(z, \lambda)}
\]

where \( \delta(z) \neq 0 \) for any \( z \) and \( B(z, \lambda) \) is positive semidefinite is an NP kernel.

The next lemma shows that NP kernels are stable under certain perturbations of the norm in the original space.

**Lemma 2.1.** Assume that \( \mathcal{H} \) is as above and the reproducing kernel \( k = k_{\mathcal{H}} \) is an NP kernel. Assume also that \( \alpha \in \mathcal{X} \) and \( a > 0 \) are such that for any \( f \in \mathcal{H} \), \( f \neq 0 \)

\[
\|f\|_1^2 := \|f\|^2 - a |f(\alpha)|^2 > 0.
\]

Here, \( \| \cdot \| \) is the original norm in \( \mathcal{H} \). Then the new norm \( \| \cdot \|_1 \) is equivalent to the original norm in \( \mathcal{H} \) and the reproducing kernel \( k^1(z, \lambda) \) for \( \mathcal{H} \) with respect to the new norm \( \| \cdot \|_1 \) is an NP kernel.

**Proof.** Substituting \( f = k(\cdot, \alpha) \) into (14), we obtain \( ak(\alpha, \alpha) < 1 \). Since \( k(\alpha, \alpha) \) is the square of the norm of the evaluation functional \( f \mapsto f(\alpha) \), we get

\[
(1 - ak(\alpha, \alpha)) \|f\|^2 \leq \|f\|_1^2 \leq \|f\|^2,
\]

which shows that the norms \( \| \cdot \|_1 \) and \( \| \cdot \| \) are equivalent. The reproducing kernel \( k^1 \) with respect to the norm \( \| \cdot \|_1 \) has the following explicit representation:

\[
k^1(z, \lambda) = k(z, \lambda) + C \frac{k(z, \alpha) k(\alpha, \lambda)}{k(\alpha, \alpha)},
\]
where
\[ C = \frac{ak(\alpha, \alpha)}{1-ak(\alpha, \alpha)}. \]

Indeed, if \( k^1 \) is defined by the above formula, then for any \( \lambda \in \mathcal{X} \) we have \( k^1(\cdot, \lambda) \in \mathcal{H} \) and for any \( f \in \mathcal{H} \) and \( \lambda \in \mathcal{X} \)

\[
\langle f, k^1(\cdot, \lambda) \rangle = \left( f, k(\cdot, \lambda) + \frac{Ck(\alpha, \lambda)k(\cdot, \alpha)}{k(\alpha, \alpha)} \right) - af(\alpha)(C+1)k(\lambda, \alpha) = f(\lambda).
\]

Now, we check the NP property in the form (10) for the kernel \( k^1 \). We have

\[ k^1(z, \alpha) = (C + 1)k(z, \alpha) \]

and if the function \( B_z(z, \lambda) \) is such that

\[ k(z, \lambda) - \frac{k(z, \alpha)k(\alpha, \lambda)}{k(\alpha, \alpha)} = B_z(z, \lambda)k(z, \lambda), \]

then we obtain by an explicit calculation that

\[ k^1(z, \lambda) - \frac{k^1(z, \alpha)k^1(\alpha, \lambda)}{k^1(\alpha, \alpha)} = B^1_z(z, \lambda)k^1(z, \lambda), \]

where

\[ B^1_z(z, \lambda) = \frac{B_z(z, \lambda)}{C+1-CB_z(z, \lambda)} = \sum_{n \geq 0} \frac{C^n}{(C+1)^{n+1}}(B_z(z, \lambda))^n. \]

If \( k \) is an NP kernel, then the function \( B_z \) is positive semi-definite and satisfies \( |B_z(z, \lambda)| < 1 \) which implies the same properties of \( B^1_z \), since any power \( (B_z(z, \lambda))^n \) is positive semidefinite by Schur’s theorem on elementwise product of positive definite matrices.

Iterating the preceding lemma, we immediately obtain the following theorem.
**Theorem 2.1.** Assume that \( H \) is as above and \( k_H \) is an NP kernel. If points \( a_1, \ldots, a_n \in X \) and positive numbers \( a_1, \ldots, a_n \) are such that for any \( f \in H, f \neq 0 \)
\[
\|f\|_1^2 := \|f\|^2 - \sum_{k=1}^{n} a_k |f(a_k)|^2 > 0,
\]
(15)
then the norm \( \|\cdot\|_1 \) is equivalent to the original norm in \( H \) and the reproducing kernel \( k_1(z, l) \) of \( H \) with respect to the norm \( \|\cdot\|_1 \) is an NP kernel.

In many concrete situations it is possible to use approximation arguments and to obtain a similar result for a new norm in the form
\[
\|f\|_1^2 := \|f\|^2 - \int_X |f(\alpha)|^2 \, d\mu(\alpha),
\]
where \( \mu \) is some measure on \( X \). In order to do this and also for the proof of Theorem 1.1 we need some preparation lemmas on approximation of reproducing kernels.

**Lemma 2.2.** Let \( k_\lambda(z, l), n \geq 0 \) be a sequence of reproducing kernels defined in \( X \times X \) such that
\[
\lim_{n \to \infty} k_\lambda(z, l) = k^0(z, l)
\]
pointwise in \( X \times X \). If all kernels \( k_\lambda \) for \( n \geq 1 \) are NP kernels and the limiting kernel \( k^0 \) is such that \( k^0(z, z) \neq 0 \) for any \( z \in X \) and there exists a point \( \alpha \in X \) such that \( k^0(\alpha, \alpha) \neq 0 \) for any \( \alpha \in X \), then \( k^0 \) is an NP kernel.

**Proof.** Since \( k_\lambda \) are NP kernels, we have
\[
k_\lambda(z, \lambda) - \frac{k_\lambda(z, \alpha) k_\lambda(\alpha, \lambda)}{k_\lambda(\alpha, \alpha)} = B_\lambda(z, \lambda) k_\lambda(\alpha, \lambda),
\]
(16)
for some positive semidefinite functions \( B_\lambda(z, \lambda) \) satisfying \( |B_\lambda(z, \lambda)| < 1 \) for \( z, \lambda \in X \). It follows in the limit from (16) that \( k^0(z, \lambda) \neq 0 \) for all \( z, \lambda \in X \).

Therefore, there exists a pointwise limit
\[
B_\lambda(z, \lambda) = \lim_{n \to \infty} B_\lambda(z, \lambda)
\]
which is a positive semidefinite function. We have then
\[
k^0(z, \lambda) - \frac{k^0(z, \alpha) k^0(\alpha, \lambda)}{k^0(\alpha, \alpha)} = B^0(z, \lambda) k^0(\alpha, \lambda).
\]
(17)
It remains to check that $|B_{0}^{\alpha}(z, \lambda)| < 1$. Substituting $\lambda = z$ into (17), we get $|B_{0}^{\alpha}(z, z)| < 1$ since $k^0(z, z) \neq 0$. Similarly, $|B_{0}^{\alpha}(\lambda, \lambda)| < 1$. The desired property follows now from the Cauchy–Bunyakovskii inequality.

If we have a sequence $\mathcal{H}$ of reproducing kernel Hilbert spaces on a set $\mathcal{X}$, then convergence of norms in $\mathcal{H}$ does not necessarily imply convergence of corresponding reproducing kernels, even in the case where all norms in $\mathcal{H}$ are uniformly equivalent to each other. For example, let $\mathcal{H}$ be the Hardy space $H^2$ supplied with the norm

$$\|f\|_n^2 := \frac{1}{2} |\hat{f}(0) + \hat{f}(n)|^2 + \frac{1}{2} |\hat{f}(0)|^2 + \sum_{k \neq 0} |\hat{f}(k)|^2.$$ 

Then the norms $\| \cdot \|_n$ are uniformly equivalent to the standard norm $\| \cdot \|_0$ in $H^2$ and for any $f \in H^2$

$$\lim_{n \to \infty} \|f\|_n^2 = \|f\|_0^2 = \sum_{k \neq 0} |\hat{f}(k)|^2.$$ 

On the other hand, for the function $f_n(z) := \sqrt{2} (1 - z^n)$ we have $\|f_n\|_n = 1$ and $f_n(0) = \sqrt{2}$, which shows that $k^\alpha(0, 0) \geq 2$, where $k^\alpha$ is the reproducing kernel for $\mathcal{H}$. Therefore, the sequence $k^\alpha(z, \lambda)$ cannot converge to the Szegö kernel $k^0(z, \lambda) = (1 - \lambda \bar{z})^{-1}$.

The following lemmas give some sufficient conditions for convergence of reproducing kernels.

**Lemma 2.3.** Assume that $\mathcal{H}$ is a Hilbert space of functions defined on a set $\mathcal{X}$ with the norm $\| \cdot \|$ and the reproducing kernel $k$. Assume also that the Hilbert norms $\| \cdot \|_n$, $n \geq 0$ in $\mathcal{H}$ are uniformly equivalent to the norm $\| \cdot \|$ and satisfy

$$\|x\|_n^2 - \|x\|_0^2 \leq e_n \|x\|_0^2, \quad x \in \mathcal{H}$$

for some sequence $e_n \to 0$. Then we have

$$\lim_{n \to \infty} k^\alpha(z, \lambda) = k^0(z, \lambda)$$

pointwise in $\mathcal{X} \times \mathcal{X}$, where $k^\alpha$, $n \geq 0$ is the reproducing kernel for $\mathcal{H}$ with respect to the norm $\| \cdot \|_n$. 
Proof. Let $H_n$ denote the space $H$ supplied with the norm $\|\cdot\|_n$ and $i_n$ denote the embedding $H_n \hookrightarrow H$. Then we have for any $x \in H$

$$\|x\|_n^2 = \|i_n^{-1}x\|_n^2 = \langle (i_n^{-1})^* i_n^{-1} x, x \rangle.$$ 

The condition of the lemma reads as

$$\langle ((i_n^{-1})^* i_n^{-1} - (i_0^{-1})^* i_0^{-1}) x, x \rangle \leq \varepsilon_n \|x\|^2,$$

which implies by polarization

$$\|(i_n^{-1})^* i_n^{-1} - (i_0^{-1})^* i_0^{-1}\| \leq 2\varepsilon_n.$$

Therefore, the sequence of operators $A_n = (i_n^{-1})^* i_n^{-1}$ is norm convergent to the operator $(i_0^{-1})^* i_0^{-1}$ and the operators $A_n$ have uniformly bounded inverses. This implies the norm convergence of the inverse operators $A_n^{-1} = i_n i_n^*$ which proves the lemma, since we have $k^n(\cdot, \lambda) = i_n^* k(\cdot, \lambda)$ and

$$k^n(z, \lambda) = \langle i_n^* i_n k(\cdot, \lambda), k(\cdot, z) \rangle.$$ 

The following two lemmas are particular cases of the classical Aronszajn description of limits of increasing and decreasing sequences of reproducing kernels (see [3, pp. 362–368]).

**Lemma 2.4.** Let $H_n, n \geq 0$ be a sequence of Hilbert spaces of functions defined on a set $X$, and let $\|\cdot\|_n$ and $k^n$ denote the norm in $H_n$ and the reproducing kernel for $H_n$. Assume that these spaces satisfy the following conditions:

(i)

$$H_1 \subset \cdots \subset H_n \subset H_{n+1} \subset \cdots \subset H_0,$$

and all inclusions are contractive;

(ii) $H_1$ is dense in $H_0$;

(iii) for any $x \in H_1$

$$\lim_{n \to \infty} \|x\|_n = \|x\|_0.$$

Then we have $k^n(z, \lambda) \to k^0(z, \lambda)$ pointwise in $X \times X$.

**Lemma 2.5.** Let $H_n, n \geq 0$ be a sequence of Hilbert spaces of functions defined on a set $X$, and let $\|\cdot\|_n$ and $k^n$ denote the norm in $H_n$ and the
reproducing kernel for $H_n$. Assume that these spaces satisfy the following conditions:

(i) 
$$H_0 \subset \cdots \subset H_{n+1} \subset H_n \subset \cdots \subset H_1,$$
and all inclusions are contractive;

(ii) for any $x \in H_0$
$$\lim_{n \to \infty} \|x\|_n = \|x\|_0;$$

(iii) if $x \in H_n$ for all $n \geq 1$ and $\sup_{n \geq 1} \|x\|_n < +\infty$, then $x \in H_0$.

Then we have $k^n(z, \lambda) \to k^q(z, \lambda)$ pointwise in $X \times X$.

As an application, we prove the following theorem.

Theorem 2.2. Assume that $\mu$ is a finite positive Borel measure in $D$ such that for any $\lambda \in D$ there is a constant $C(\lambda) > 0$ such that for any polynomial $p$

$$\frac{|p(\lambda)|^2}{C(\lambda)} \leq \left( \|p\|_H^2 - \int_D |p(z)|^2 \, d\mu(z) \right) =: \|p\|_{\mu}^2. \quad (19)$$

Let $H_\mu$ be the closure of polynomials with respect to the norm $\|\cdot\|_{\mu}$. Then $H_\mu$ is a reproducing kernel Hilbert space with an NP kernel.

Proof. Inequality (19) shows that the evaluation functionals are bounded in $H_\mu$, therefore $H_\mu$ possesses a reproducing kernel $k_\mu$. We check first that this reproducing kernel nowhere vanishes. This easily follows from standard division arguments. For $\lambda \in D$ we denote

$$\phi_\lambda(z) := \frac{\lambda - z}{1 - \lambda z}. \quad (20)$$

Then, clearly, $\|p/\phi_\lambda\|_{\mu} \leq \|p\|_{\mu}$ for any polynomial $p$ such that $p(\lambda) = 0$. By approximation, we obtain that $\|f/\phi_\lambda\|_{\mu} \leq \|f\|_{\mu}$ for any $f \in H_\mu$ with $f(\lambda) = 0$. If we had $k_\mu(\lambda, \alpha) = 0$ for some $\lambda, \alpha \in D$, then the function $g(z) = k_\mu(z, \alpha)/\phi_\lambda(z)$ would satisfy $\|g\|_{\mu} \leq \|k_\mu(\cdot, \alpha)\|_{\mu} = k_\mu(\lambda, \alpha)^{1/2}$ and $|g(\alpha)| > k_\mu(\lambda, \lambda)$, which is impossible since the evaluation functional $f \mapsto f(\alpha)$ in $H_\mu$ has the norm $k_\mu(\alpha, \alpha)^{1/2}$. 

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If $\mu$ is a finitely atomic measure, then the theorem is a particular case of Theorem 2.1, since the Szegö kernel

$$k_{\mu'}(z, \lambda) = \frac{1}{1 - \lambda z}$$

is an NP kernel. Assume now that $\mu$ is supported in $r\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq r\}$ for some $r \in (0, 1)$ and the norm $\| \cdot \|_\mu$ is equivalent to the standard norm in $H^2$. We shall approximate the reproducing kernel $k_\mu$ by reproducing kernels $k_{\mu_n}$ corresponding to finitely atomic measures $\mu_n$. For some sequence $\delta_n \to 0$, let

$$r\overline{\mathbb{D}} = \bigcup_{k=1}^{m_n} D^*_k$$

be a partition of $r\overline{\mathbb{D}}$ into the union of non-empty disjoint Borel subsets $D^*_k \subset r\overline{\mathbb{D}}$, $k = 1, \ldots, m_n$ satisfying $\text{diam} (D^*_k) \leq \delta_n$. Let $z^*_n$ be some point in $D^*_k$. For each $n$, we define a measure $\mu_n$ as

$$\mu_n = \sum_{k=1}^{m_n} \mu(D^*_k) \delta_{z^*_k}.$$

We have then, for some sequence $e_n \to 0$,

$$\| |f(z)|^2 - |f(z^*_n)|^2| \leq e_n \|f\|_{H^2}^2, \quad \forall z \in \overline{\mathbb{D}}$$

for any $f \in H^2$ which implies

$$\| |f|_\mu^2 - |f|_{\mu_n}^2| \leq e_n \|f\|_{H^2}^2.$$

In particular, the quadratic form $\| \cdot \|_{\mu_n}^2$ is positive definite and represents a norm equivalent to the standard norm in $H^2$ for sufficiently big $n$ (since the norm $\| \cdot \|_\mu$ is equivalent to the norm in $H^2$). By Lemma 2.3, we obtain that

$$\lim_{n \to \infty} k_{\mu_n}(z, \lambda) = k_{\mu}(z, \lambda)$$

pointwise in $\mathbb{D} \times \mathbb{D}$, and Lemma 2.2 implies that $k_{\mu}$ is an NP kernel.

Finally, let $\mu$ be an arbitrary measure satisfying (19). For $n \geq 1$, let $\mu_n$ be the measure

$$d\mu_n(z) = \left(1 - \frac{1}{n}\right) \chi_{(1-\frac{1}{n})\mathbb{D}}(z) \, d\mu(z).$$
If \( \mathcal{H}_n = \mathcal{H}_m, \ n \geq 1, \) and \( \mathcal{H}_0 = \mathcal{H}_m, \) then the Hilbert spaces \( \mathcal{H}_n, \ n \geq 0 \) satisfy the conditions of Lemma 2.4, and we obtain

\[
k_m(z, \lambda) = \lim_{n \to \infty} k_{\mathcal{H}_n}(z, \lambda)
\]

pointwise in \( \mathbb{D}. \) Since we have already proved that \( k_{\mathcal{H}_n} \) are NP kernels (the factor \( 1 - \frac{1}{n} \) guarantees that each norm \( \| \cdot \|_{\mathcal{H}_n} \) is equivalent to the norm in \( H^2 \)), Lemma 2.2 again implies the desired result.

3. REPRODUCING KERNELS FOR \( D(\mu) \)

In this section we prove Theorem 1.1. First, we recall an alternative description of spaces \( D(\mu) \) and some of their properties. For details, see [2] and [10]. The notations \( \| \cdot \|_{\mu} \) and \( k_{\mu} \) are used in this section to denote the norm in \( D(\mu) \) given by (6) and the corresponding reproducing kernel for \( D(\mu). \)

Given a function \( f \in H^2 \) and \( \zeta \in \overline{\mathbb{D}}, \) the local Dirichlet integral \( D_\zeta(f) \) of \( f \) at the point \( \zeta \) (see [11]) is defined as

\[
D_\zeta(f) := \int_{\mathbb{T}} \left| \frac{f(z) - f(\zeta)}{z - \zeta} \right|^2 \, dm_1(z).
\]

Here, \( dm_1 \) is the normalized arc measure on \( \mathbb{T}. \) In the case \( \zeta \in \mathbb{T} \) the local Dirichlet integral \( D_\zeta(f) \) is finite if and only if \( f \) has a nontangential limit \( f(\zeta) \) at the point \( \zeta \) and \( f(z) = f(\zeta) + (\zeta - z) g(z) \) for some \( g \in H^2; \) otherwise \( D_\zeta(f) \) is infinite.

The space \( D(\mu) \) can be described as the space of those functions \( f \in H^2 \) that the function \( \zeta \mapsto D_\zeta(f) \) is in \( L^1(\mu). \) The norm \( \| \cdot \|_{\mu} \) admits a representation

\[
\| f \|_{\mu}^2 = \| f \|_{H^2}^2 + \int_{\mathbb{T}} D_\zeta(f) \, d\mu(\zeta). \tag{21}
\]

If \( \theta \) is an inner function, then [11, Lemma 3.4]

\[
D_\zeta(\theta f) = D_\zeta(f) + D_\zeta(\theta) \cdot |f(\zeta)|^2
\]

(with the convention that \( \infty \cdot 0 = 0 \) for the second term) and hence

\[
\| \theta f \|_{\mu}^2 = \| f \|_{\mu}^2 + \int_{\mathbb{T}} |f(\zeta)|^2 D_\zeta(\theta) \, d\mu(\zeta). \tag{22}
\]
If $I \subset D(\mu)$ is a closed subspace invariant with respect to the shift operator $S$, then $\dim(I \odot SI) = 1$ and any function $\psi \in I \odot SI$ with $\|\psi\|_\mu = 1$ has the properties

(i) $|\psi(z)| \leq 1$ for any $z \in \mathbb{D}$;

(ii) $\|\psi f\|_\mu = \|f\|_{\mu_\psi}$ for any $f \in D(\mu)$. Here, $d\mu_\psi(\zeta) = |\psi(\zeta)|^2 \, d\mu(\zeta)$.

(iii) Any $g \in I$ has the form $g = \psi f$ for some $f \in D(\mu_\psi)$.

Now, we turn to the proof of Theorem 1.1. We note first that the formula (8) follows from the NP property of kernels $k_\mu$ if we choose $\alpha = 0$ in the identity (10) and take into account that $k_\mu(z, 0) = 1$, which follows from the definition of the norm in $D(\mu)$. Therefore, it suffices to prove the NP property of $k_\mu$.

We prove this first in the case where $\mu$ is a finitely atomic measure in $\mathbb{D}$, i.e. $\mu = \sum_{k=1}^{n} a_k \delta_{\lambda_k}$, with $a_k > 0$, $\lambda_k \in \mathbb{D}$. For such measures, as one can see from (21), the space $D(\mu)$ coincides with $H^2$ as a set and the norm $\|\cdot\|_\mu$ is equivalent to the standard norm in $H^2$. The next proposition gives one more alternative representation of the norm in $D(\mu)$ for finitely atomic $\mu$ in $\mathbb{D}$.

**Proposition 3.1.** Assume that

$$\mu = \sum_{k=1}^{n} a_k \delta_{\lambda_k}, \quad a_k > 0, \quad \lambda_k \in \mathbb{D} \tag{23}$$

is a finitely atomic measure in $\mathbb{D}$. Then there exists a function $F = F_\mu$ bounded and bounded away from zero in $\mathbb{D}$ and a measure $\nu = \nu_\mu$ in the form $\nu = \sum_{k=1}^{n} b_k \delta_{\lambda_k}$ with $b_k > 0$ such that for any $f \in H^2$

$$\|f\|_\mu^2 = \|F f\|_{H^2}^2 - \int_{\mathbb{D}} |f|^2 \, dv. \tag{24}$$

**Proof.** We prove the proposition by induction on $n$. For $n = 0$, we have $\mu = 0$ and $D(\mu)$ coincides with $H^2$ with equality of norms; therefore (24) holds with $F = 1$ and $\nu = 0$.

Now, let $\mu$ be given by (23). Consider the invariant subspace

$$I_{\lambda_n} = \{ f \in D(\mu) : f(\lambda_n) = 0 \}$$

in $D(\mu)$ and let $\psi$ be an element of unit norm in $I_{\lambda_n} \odot SI_{\lambda_n}$. Let

$$d\mu_{\lambda_n}(\zeta) := |\psi(\zeta)|^2 \, d\mu(\zeta) = \sum_{k=1}^{n-1} |\psi(\lambda_k)|^2 \, a_k \, d\delta_{\lambda_k}(\zeta).$$
Since norms in both spaces $D(\mu)$ and $D(\mu_\lambda)$ are equivalent to the norm in $H^2$ and $I_\lambda$ coincides as a set with the subspace $\phi_\lambda H^2$ of $H^2$, it follows from the above properties of the function $\psi$ that

$$c \|f\|_{H^2} \leq \left\| f \frac{\psi}{\phi_\lambda} \right\|_{H^2} \leq C \|f\|_{H^2}$$

for any $f \in H^2$ with some constants $c$ and $C$ and that any $g \in H^2$ has the form $g = (\psi / \phi_\lambda) f$ for some $f \in H^2$ (Here, $\phi_\lambda$ is given by (20)). Therefore, the function $G := \psi / \phi_\lambda$ is bounded and bounded away from zero in $D$.

Using (22) with $\theta = \phi_\lambda$, we obtain for any $f \in H^2$

$$\|f\|^2_\mu = \|\phi_\lambda f\|^2_\mu - \int_D |f(\xi)|^2 D_\lambda(\phi_\lambda) \mu(\xi)$$

$$= \|\psi G^{-1} f\|^2_\mu - \int_D |f(\xi)|^2 D_\lambda(\phi_\lambda) \mu(\xi)$$

$$= \|G^{-1} f\|^2_{\mu_\lambda} - \int_D |f(\xi)|^2 D_\lambda(\phi_\lambda) \mu(\xi)$$

(the last equality follows from (ii) above). Using the induction hypothesis on representation of the norm $\|\cdot\|_{\mu_\lambda}$, we obtain from this formula the desired result.

**Remark.** An alternative proof of representation (24) is as follows. An explicit calculation shows that for $\zeta \in D$ and $f \in H^2$

$$D_\lambda(f) = \int_T \frac{|f(\zeta)|^2}{|z-\zeta|^2} \nu(z) - \frac{1}{1-|\zeta|^2} |f(\zeta)|^2.$$

Therefore, if $\mu$ is given by (23), then using representation (21), we get

$$\|f\|^2_\mu = \int_T \left(1 + \sum_{k=1}^n \frac{a_k}{|z-\lambda_k|^2}\right) |f(z)|^2 \nu(z) - \sum_{k=1}^n \frac{a_k}{1-|\lambda_k|^2} |f(\lambda_k)|^2.$$

If $F$ is such an outer function that

$$|F(z)|^2 = 1 + \sum_{k=1}^n \frac{a_k}{|z-\lambda_k|^2}, \quad z \in T,$$

then (24) follows.
If a function $F$ is bounded and bounded away from zero in $D$, then the reproducing kernel for $H^2$ with respect to the norm $\|f\| = \|Ff\|_{H^2}$ is
\[
    k(z, \lambda) = \frac{(F(z))^{-1} (F(\lambda))^{-1}}{1 - \bar{\lambda}z},
\]
which is an NP kernel. Therefore, the representation (24) together with Theorem 2.1 proves Theorem 1.1 in the case where $\mu$ is a finitely atomic measure in $D$.

The next step of the proof of Theorem 1.1 is the case where $\mu$ is supported in the disk $rD$ for some $r \in (0, 1)$. In this case we can approximate $k_\mu$ by kernels $k_{\mu_n}$ corresponding to finitely atomic measures $\mu_n$ essentially in the same way as we did in the proof of the corresponding case of Theorem 2.2. The fact that for any $\mu$ the kernel $k_\mu(z, \lambda)$ is nowhere vanishing follows from the division arguments explained in the same proof (the division by $\phi_k$ is a contractive operation in $D(\mu)$ by (22)). The details are left to the reader.

If $\mu$ is supported in $D$ (and it has no part on $T$), then we define $\mu_0 := \mu$ and
\[
    d\mu_n(z) := \chi_{(1 - \frac{1}{n})B}(z) \, d\mu(z)
\]
for $n \geq 1$. The representation (21) of the norm in $D(\mu)$ shows that the spaces $H_n = D(\mu_n)$, $n \geq 0$ satisfy the conditions of Lemma 2.5. We obtain an approximation
\[
    \lim_{n \to \infty} k_{\mu_n}(z, \lambda) = k_\mu(z, \lambda),
\]
which proves Theorem 1.1 for measures $\mu$ having no part on $T$.

Finally, let $\mu$ be an arbitrary measure supported in $D$. Then $\mu = \nu + \mu'$, where $\nu$ is supported on $T$ and $\mu'$ has no part on $T$. Consider the family of functions
\[
    V_r(\zeta) := \int_{T} r^2 \frac{1 - |\zeta|^2}{|z - r\zeta|^2} \, d\nu(z)
    = \frac{r^2}{1 - r^2} \int_{T} (1 - |\phi_r(\zeta)|^2) \, d\nu(z), \quad \zeta \in D, \ r \in (0, 1).
\]
For any $r \in (0, 1)$, the function $V_r$ is superharmonic in $D$ and real analytic in $D$; it vanishes on $T$ and therefore it admits a representation in the form
\[
    V_r(\zeta) = U_r(\zeta) = \int_{D} \log \frac{|1 - \bar{\zeta}|^2}{|\zeta - z|^2} \, d\nu(z).
\]
where
\[ dv_r(z) = -(1 - |z|^2) \, dV_r(z) \, dm(z). \]

We set \( \mu_r := \nu_r + \mu' \). Since for \( z \in \mathbb{T} \)
\[ \frac{\partial}{\partial r} \left( \frac{r^2}{|z - r|^2} \right) = 2r \frac{1 - r \, \text{Re}(\bar{z}z)}{|z - r|^2} \geq 0, \]
we obtain that \( V_r(\zeta) \leq V_r'(\zeta) \) for \( r \leq r' \), and therefore \( U_{\mu_r}(\xi) \leq U_{\mu_r}(\zeta) \) for \( r \leq r' \). Moreover,
\[ \lim_{r \to 1-0} V_r(\zeta) = \int_{\mathbb{T}} \frac{1 - |\zeta|^2}{|z - \zeta|^2} \, dv(z) = U_r(\zeta) \]
pointwise in \( \mathbb{D} \) and hence
\[ \lim_{r \to 1-0} U_{\mu_r}(\zeta) = U_{\mu}(\zeta) \]
pointwise in \( \mathbb{D} \). If we define now \( H_0 := D(\mu) \) and \( H_n := D(\mu_n) \) for some increasing sequence \( r_n \) with \( r_n \to 1 \), then the above properties of functions \( U_{\mu_r} \) show that these spaces \( H_n, n \geq 0 \) satisfy the conditions of Lemma 2.5. We obtain the approximation
\[ \lim_{r \to 1-0} k_{\mu_r}(z, \lambda) = k_\mu(z, \lambda), \]
and since we have already proved that \( k_\mu \) are NP kernels, this accomplishes the proof of Theorem 1.1.

4. COMMUTANT LIFTING FOR CYCLIC ANALYTIC TWO-ISOMETRIES

We turn now to the proof of Theorem 1.2. According to Theorem 5.1 of [10], a cyclic analytic two-isometry \( T \) is unitarily equivalent to the shift operator \( S \) in the space \( D(\mu) \) for some measure \( \mu \) supported on \( \mathbb{T} \). We assume thus that \( H = D(\mu) \) and \( T = S \).

We shall say that a function \( \phi(z), z \in \mathbb{D} \) is a multiplier of \( D(\mu) \), if the multiplication operator \( M_\phi: f(z) \mapsto \phi(z) \, f(z) \) is a bounded operator in \( D(\mu) \). Any multiplication operator \( M_\phi \) can be approximated in the weak
operator topology by operators $p(S)$, where $p$ are polynomials (see [11, Lemma 5.4]). Therefore, if $J$ is a closed and $S^*$-invariant subspace of $D(\mu)$ then $J$ is also invariant for any operator $M^*_k$ and if $X: J \mapsto J$ commutes with the restriction $S^*|_J$, then $X$ commutes also with any restriction $M^*_k|_J$.

Now, the commutant lifting theorem for NP kernel Hilbert spaces (Theorem 5.1 of [5]) implies that $X$ has the form $X = M^*_k|_J$ for some multiplier $w$ of $D(\mu)$ whose multiplier norm is less than or equal to $\|X\|$.

Therefore, the operator $M^*_k$ is the desired operator $Y$ from Theorem 1.2. A theorem similar to Theorem 1.2 also holds for operators $T$ of the form $T = S \otimes I$, where $S$ is the shift operator in some space $D(\mu)$ with $\mu$ supported on $\mathbb{T}$ and $I$ is the identity operator in a Hilbert space $E$. But for general analytic noncyclic two-isometries $T$ an analog of Theorem 1.2 fails. Consider, for example, two measures $\mu_1$ and $\mu_2$ supported on $\mathbb{T}$ such that $\mu_1 \geq \mu_2$ but $\mu_1 \neq \mu_2$. Let $H := D(\mu_1) \oplus D(\mu_2)$ and $T$ be the direct sum of the shift operators $S_1$ and $S_2$ in $D(\mu_1)$ and $D(\mu_2)$. Let

$$J := E_1 \oplus E_2,$$

where $E_k$ is the subspace consisting of constant functions in $D(\mu_k)$, $k = 1, 2$, and let $X: J \mapsto J$ be defined as

$$X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(with respect to the decomposition $J = E_1 \oplus E_2$). Obviously, all conditions except cyclicity of $T$ of Theorem 1.2 are fulfilled and $\|X\| = 1$. Assume that $X$ extends to a contractive operator $Y: H \mapsto H$ commuting with $T^*$. If

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

then the operator $Y_{21}: D(\mu_1) \mapsto D(\mu_2)$ is a contraction satisfying $Y_{21}S^*_1 = S^*_2Y_{21}$. Therefore (since polynomials are dense in all $D(\mu)$, see [2, 10]),

$$Y_{21} = M^*_\psi,$$

where $\psi$ is a contractive multiplier from $D(\mu_2)$ to $D(\mu_1)$. Since $\mu_1 \geq \mu_2$, $\psi$ is also a contractive multiplier from $D(\mu_2)$ into itself and hence $\|\psi\|_{L^\infty} \leq 1$. Now, the condition $Y|_J = X$ implies $\psi(0) = 1$, whence $\psi(z) \equiv 1$. But then $M^*_\psi$ cannot be a contractive operator from $D(\mu_2)$ to $D(\mu_1)$ since $\mu_2 \neq \mu_1$, and we obtain a contradiction.
5. SOME OPEN PROBLEMS

Let $X$ be a Hilbert space of functions analytic in the unit disk $\mathbb{D}$ such that $X$ is invariant with respect to the operators $S$ and $L$ of forward and backward shifts:

$$S: f(z) \mapsto zf(z); \quad L: f(z) \mapsto \frac{f(z) - f(0)}{z}.$$ 

A natural question is whether certain properties of these operators imply the complete NP property of the reproducing kernel $k_X$.

In the example $X = l^2(w_n)$ from the introduction a sufficient condition for the complete NP property of $k_X$ is

$$\frac{w_{n+1}}{w_n} \geq \frac{w_{n+2}}{w_{n+1}}, \quad n \geq 0.$$ 

An easy calculation shows that this condition is equivalent to each of the operator inequalities

$$S^*SS^*S \geq S^*S \quad (25)$$

or

$$LL^* \geq L^*L. \quad (26)$$

It is natural to ask whether these operator inequalities are sufficient for the complete NP property of $k_X$ for general spaces $X$. In the normalized case where $k_X(z, 0) = 1$, conditions (25) and (26) are equivalent to each other. Indeed, in this case the operators $S$ and $L^*$ are Cauchy dual to each other (see [16]), therefore $L = (S^*S)^{-1}S^*$ and $S = L^*(LL)^{-1}$. Now, we have the following chain of equivalences:

$$S^*SS^*S \geq S^*S \iff (LL^*)^{-1} \geq (LL^*)^{-1} L(LL)^{-1} L^*(LL)^{-1} \iff L(LL)^{-1} L^* \leq I \iff \|L(LL)^{-1/2}\| \leq 1 \iff (LL^*)^{-1/2} L^*L(LL^*)^{-1/2} \leq I \iff L^*L \leq LL^*.$$ 

The shift operator $S$ in spaces $D(\mu)$ satisfies the concavity inequality

$$S^*S^2 - 2S^*S + I \leq 0, \quad (27)$$
or equivalently
\[ \|S^2 f\|^2_m + \|f\|^2_m \leq 2 \|Sf\|^2_m, \quad \forall f \in D(\mu). \]

This last inequality follows easily from (22). In the general case the inequality (27) is stronger than (25). Indeed, if we have \( \|S^2 f\|^2 + \|f\|^2 \leq 2 \|Sf\|^2 \), then
\[ \|Sf\|^2 \geq \|f\| \cdot \|S^2 f\|, \]
which implies
\[ \|S^* S f\| = \sup_{\varepsilon \neq 0} \frac{\|S^* S f, g\|}{\|g\|} \geq \frac{\|S^* S f, f\|}{\|f\|} \geq \|S^2 f\|, \]
which gives us (25). One may ask whether this stronger condition (27) implies the complete NP property of \( k_\mu \) in the abstract context.

REFERENCES

