Pioneers of Representation Theory: Frobenius, Burnside, Schur, and Brauer

The History of Mathematics Series of the American and London Mathematical Societies is a fine example of the increasing interest in and commitment to the history of their discipline on the part of professional mathematicians. The viewpoints and expertise of both historians and mathematicians are essential for a full understanding of the development of mathematics, and I have been delighted to see both types of authors contributing volumes to the series. The volume under review is written by a mathematician whose career has been primarily devoted to researching and expounding the representation theory of finite groups. Now he has used his expository skills to write the first full-scale history of the subject. Although the work of numerous mathematicians is discussed, the book focuses upon the contributions of the four mathematicians of the title, since they did indeed supply the driving force for the development of the theory during the first half of the 20th century.

In the Preface Curtis clearly states the approach he has followed in writing the book, namely to give “an account … presented through an analysis of the published work of the four principal contributors to the theory. … While many of the proofs of the main theorems are given today using new methods, the first proofs … are still interesting, and complete the historical picture by showing exactly how the creators of the theory used the mathematics that was available to them at the time. … By concentrating on the mathematics in the original papers, I have not given as complete a survey of the historical setting of the mathematical discoveries as a reader might desire. Fortunately, a full historical account … has been given in a series of articles by Thomas Hawkins, and frequent references to them are included.” Curtis’s approach of analyzing publications is unquestionably an essential part of historical analysis, and he is well equipped to carry it out. By virtue of the single-mindedness of his approach, however, he has admittedly produced a book with some weaknesses, mainly in the earlier chapters, in addition to its many strengths.

The first chapter is intended as a mathematical introduction to the ensuing chapters. It takes as its starting point a quotation from an 1896 letter Dedekind sent to Frobenius. The quotation contains Dedekind’s synopsis of the mathematical occurrences of group characters up to that time (when Frobenius was in the process of generalizing it). Curtis then gives a mathematical exposition, in modern notation, of all the mentioned topics so that the book will be accessible to students unfamiliar with these notions. This chapter is the least historical of all. For example, the reader will come away with no real appreciation for the key role played by Dedekind in abstracting the modern notion of character on an abelian group from what in retrospect can be seen as instances of its occurrence. On the other hand, for example, the reader will acquire a good understanding of how characters are related to arithmetical investigations involving \( L \)-series, which is important for a full appreciation of subsequent developments.

Chapter 2 is devoted to Frobenius, who ended up extending the notion of a character to nonabelian groups as a result of his work on a problem posed to him by Dedekind and involving the factorization of a certain group determinant. Whereas Curtis expounds Frobenius’s extension in accordance with the progression of his publications (p. 52), I have argued on the basis of Frobenius’s extensive letters to Dedekind that “the progression of ideas as published by Frobenius … is the reverse of that originally pursued by him” [Hawkins, 1974, p. 218]. In addition, Curtis tends to emphasize in his exposition only those aspects of Frobenius’s papers that can be interpreted as belonging to the representation theory of commutative semisimple algebras. This belies not only how Frobenius originally conceived of his characters, but also his low opinion of hypercomplex numbers as a mathematical tool.
and the emphasis he himself gave to reasoning based upon determinants and linear transformations. Consequently, Frobenius’s culminating paper on the factorization of the group determinant—the goal of his investigation—is hardly discussed at all. Despite this major shortcoming, the chapter as a whole has strong points as well. In particular, the discussion of Frobenius’s investigations that intertwine number theory and the structure of nonabelian groups in the 1880s is clear and thorough. This work is important for understanding Frobenius’s move into the relatively new area of abstract nonabelian groups, as well as for its connections with Artin’s generalized $L$-series, which utilize characters in Frobenius’s sense.

The third chapter, on Burnside, I found far more satisfying. It contains a wealth of biographical information about him, some of which is not readily accessible elsewhere. Extensive use is also made of Burnside’s unpublished letters to the Cambridge geometer H. F. Baker. Although these letters do not contain insights into his work on group representations, they document his interest in the related topic of group invariants as well as his sense of isolation as a specialist in finite group theory in England. Also commendable is the thorough discussion of the numerous applications of group characters to the structure of groups that Burnside made. Frobenius once quipped, partly in jest but partly in complaint, that he and Burnside seemed to share an “intellectual harmony” that led them to the same mathematical results. Thus Burnside had unwittingly republished some of Frobenius’s theorems on finite groups, and he came close to independently creating Frobenius’s representation theory. Furthermore, both had the idea to apply the new theory of characters to questions involving the structure of groups, but in this area Burnside not only beat Frobenius into print, but also discovered many more applications, as Curtis clearly shows.

The next two chapters are devoted to Isaai Schur, who carried on the algebraic tradition established by his teacher Frobenius. They contain a good account of Schur’s principal contributions to Frobenius’s new theory—his reformulation of it (via Schur’s Lemma), the theory of the Schur index, and his theory of projective representations. The order of presentation, however, I found a bit historically disconcerting. Thus Schur’s Berlin dissertation is presented last and in a separate chapter. The dissertation is a remarkable study of polynomial representations of the general linear group. A separate chapter is presumably devoted to it because the underlying group is continuous and because over 20 years later Schur returned to the subject and reestablished his old results by new, more satisfying methods that have since been expounded and embellished by J. A. Green. In fact in presenting some of the dissertation results Curtis employs (as he is careful to indicate) notions introduced by Green. The reader thus comes away with a good understanding of the final form of Schur’s brilliant dissertation but at the expense of a historical sense of Schur’s development as a mathematician. Also omitted is some of the historical setting, namely that Schur’s work was motivated by the theory of invariants and especially by a paper by Hurwitz [Hawkins, 2000, Sects. 10.2–10.3]. Next came Schur’s equally remarkable Habilitationsschrift on projective representations, which regrettably is the penultimate topic in Curtis’s presentation. Once again we lose the historical sense of Schur’s mathematical development, including his motivation, which came from Felix Klein’s school. Klein and his students were interested in classifying groups of projective transformations and in the question of the minimal number of variables in which a group could be so represented, but their work was limited to specific examples pushed through with computations and not informed by a general underlying theory, which Schur first provided.

The final two chapters deal with the work of Richard Brauer, who was one of Schur’s students. In the first a good deal of attention is given to work by Emmy Noether. Besides discussing her work on Schur’s index theory as generalized to algebras, which involved collaborative work with Brauer, Curtis rightly stresses her role in establishing the module-theoretic approach to representation theory. Most of Brauer’s major contributions were made within the context of modular representations, i.e., representations over a
field of finite characteristic $p$, with the focus on the case in which $p$ divides the order of the group. The final chapter is devoted to this work and its ramifications for the classification of simple groups. Here, too, I would have preferred more attention given to the historical background, namely to the work of Dickson, who was the first to consider what happens to Frobenius’s theory over fields of characteristic $p$. By considering Dickson’s work we gain both an understanding of what goes wrong when $p$ divides the order of the group and an appreciation of the seeming hopelessness of obtaining satisfying results in this case. On the other hand, Curtis provides the reader with a lucid account of Brauer’s theory of modular characters and related work on nonsemisimple algebras, his theory of blocks, and his applications of these theories, e.g., to the classification of simple groups. Excellent general accounts of Brauer’s work have already been written by mathematicians, and Curtis draws upon them, but those essays are written for a readership already familiar with the underlying mathematics, whereas Curtis takes great pains to explain it.

I should mention that, contrary to his above-quoted prefatory remarks, on many occasions Curtis does not give the proof of the mathematician being discussed but rather a later one due to someone else or a generic “modern proof.” At all times, however, he is scrupulous about the provenance of the given proof. As for the amount of historical setting provided, I should make it clear that the book contains many biographical details of interest. Rather, is lacking in historical matters that are less obvious and perhaps more of interest to historians. Some of these can be found in my papers, as he says, but my papers deal with the early stages of the historical development. As I have suggested by my remarks on the chapters on Schur and Brauer, there are no doubt many interesting things left to say about the historical setting of later developments. Any historian interested in dealing with such matters, however, will derive considerable benefit and mathematical insight from a careful reading of Curtis’s mathematically rich and well-documented book. I see it as a fine example of how mathematicians and historians can cooperate productively in the challenging task of piecing together the history of mathematics.

References


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