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Diameter vulnerability of GC graphs

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Abstract

Concern over fault tolerance in the design of interconnection networks has stimulated interest in finding large graphs with maximum degree Δ and diameter D such that the subgraphs obtained by deleting any set of s vertices have diameter at most D', this value being close to D or even equal to it. This is the so-called (Δ, D, D', s)-problem. The purpose of this work has been to study this problem for $s=1$ on some families of generalized compound graphs. These graphs were designed by Gómez (Ars Combin. 29-B (1990) 33) as a contribution to the (Δ, D)-problem, that is, to the construction of graphs having maximum degree Δ, diameter D and an order large enough. When approaching the mentioned problem in these graphs, we realized that each of them could be redefined as a compound graph, the main graph being the underlying graph of a certain iterated line digraph. In fact, this new characterization has been the key point to prove in a suitable way that the graphs belonging to these families are solutions to the $(\Delta, D, D+1, 1)$-problem.

MSC: 05C40; 05C12; 05C20

Keywords: Compound graph; (Δ, D, D', s)-problem; Graph on alphabets; Large graph; Vulnerability

1. Introduction

The designer of interconnection networks must allow for the fact that machines and/or communication links may malfunction or cease to function. In this event, it is important that communication can still be achieved with reasonable efficiency. It may be required, for instance, that between any two nodes of the remaining network there still exists a path of length not exceeding some fixed value, see Kuhl and Reddy

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[20]. In terms of graphs, this problem is modelled in the literature as the *vulnerability of the diameter*, more specifically the (Δ, D, D', s)-*problem*. This problem asks for the largest graphs of maximum degree Δ and diameter D such that the subgraphs obtained by deleting any set of up to s vertices, $1 \leqslant s \leqslant \delta - 1$, where δ is the minimum degree of the graph, have diameter at most D'. The cases $s = 1$ and $D' - D = 0, 1, 2$ have been widely studied, see [2,4–6,14,21].

A technique that has proved to be useful for designing large (Δ, D)-graphs (i.e., graphs having maximum degree Δ and diameter D and an order large enough) is the so-called *compounding of graphs*. It was first introduced by Bermond et al. [3], and subsequently it has been used by several authors in order to give new constructions of (Δ, D, D', s) graphs. This method consists basically of joining together several copies of one or two graphs, according to the structure of another one, called the *main graph* of the construction. To be more precise, all these designs can be unified according to the following definitions:

Definition 1.1. Let $G_2 = (V_2, E_2)$, $G_1 = (V_1, E_1)$ be two graphs. Then, $G_2[G_1] = (V, E)$ denotes any graph obtained in the following way:

- Each vertex $x \in V_2$ is replaced by one copy of G_1 represented by G_1^x, that is,

$$V = V(G_2[G_1]) = \bigcup_{x \in V_2} V(G_1^x) = \bigcup_{x \in V_2} \{(x, x') \colon x' \in V_1\}.$$

- Each edge $xy \in E_2$ is replaced by at least one edge that joins one vertex of G_1^x with another of G_1^y, that is,

$$xy \in E_2 \Rightarrow \exists x', y' \in V_1 \quad \text{such that } (x, x')(y, y') \in E[G_2[G_1]] = E.$$

Definition 1.2. Let $G_2 = (U_2 \cup V_2, E_2)$ be a bipartite graph, and let $G_1 = (V_1, E_1)$, $G_1' = (V_1', E_1')$ be two graphs. Then, $G_2[G_1, G_1'] = (V, E)$ denotes any graph obtained in the following way:

- Each vertex $x \in U_2$ is replaced by one copy G_1^x of G_1, and each vertex $y \in V_2$ by one copy $G_{1'}^y$ of G_1', that is,

$$V = V(G_2[G_1, G_1']) = \bigcup_{x \in U_2} V(G_1^x) \cup \bigcup_{y \in V_2} V(G_{1'}^y)$$

- Each edge $xy \in E_2$ is replaced by at least one intercopy edge, that is,

$$xy \in E_2 \Rightarrow \exists x' \in V_1, y' \in V_1' \quad \text{such that } (x, x')(y, y') \in E[G_2[G_1, G_1']] = E,$$

  in such a way that every vertex of the new graph must be an endvertex of at least one intercopy edge.

The first definition corresponds to three known constructions. The first one was introduced by Bermond et al. [3]. They imposed exactly one edge between copies. The

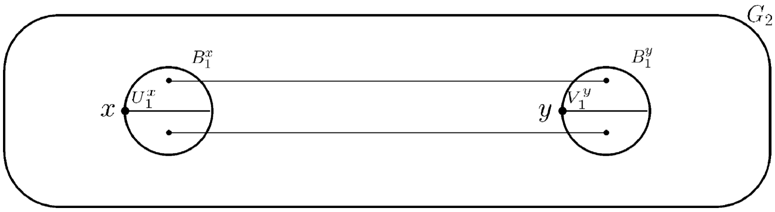

Fig. 1. Bipartite compound graphs with two intercopy edges.

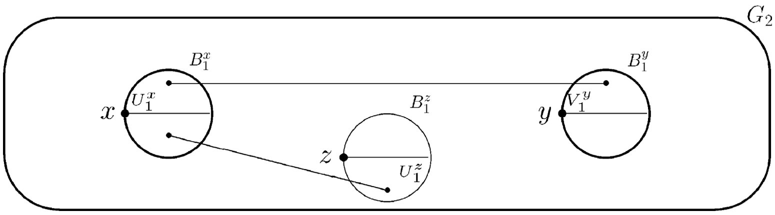

Fig. 2. Bipartite compound graphs with one intercopy edge.

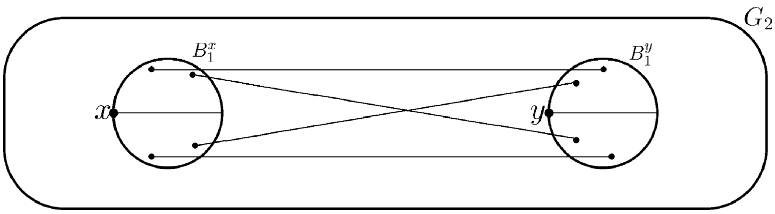

Fig. 3. Intercopy edges of compound graphs FF.

graphs obtained are the so-called *basic compound graphs*. The second construction yields the so-called *bipartite compound graphs*, which were considered by Delorme [9]. He took a bipartite graph B_1 as G_1 and replaced each edge of G_2 by one or two edges between copies (see Figs. 1 and 2). Finally, Fiol and Fàbrega presented in [12] the so-called *compound graphs FF* in a similar way as in the previous one, but replacing each edge of G_2 by four edges between copies, as is shown in Fig. 3.

As for the second definition, Delorme and Quisquater [11] introduced the *compound graphs* DQ_Λ, $DQ_\Lambda = G_2[B_0, G_1]$, B_0 being a bipartite graph (see Fig. 4). Another construction corresponding to this definition produced the *compound graphs* $B_0 \nabla B_1$, defined by Gómez and Fiol [16]. They are graphs $G_2[B_0, B_1]$, B_0 and B_1 being both bipartite, whose intercopy edges can be either two, resulting in a bipartite graph, or four, resulting in a non-bipartite graph (see Figs. 5 and 6). In both constructions, the authors considered the main graph G_2 as to be a complete bipartite graph.

The order of compound graphs follows directly from the order of the original ones. For instance: $N(G_2[G_1]) = N(G_2)N(G_1)$. Their maximum degree Δ and their diameter

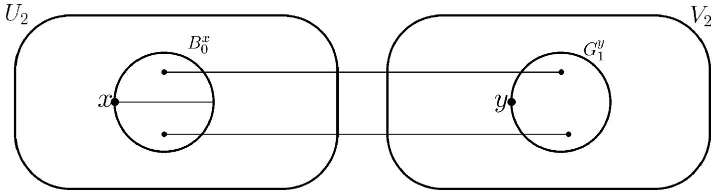

Fig. 4. Intercopy edges of compound graphs DQ_Λ.

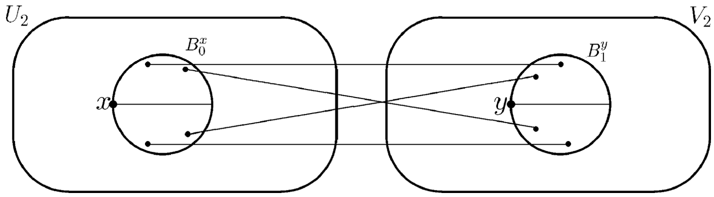

Fig. 5. Intercopy edges of compound graphs $B_0 \nabla B_1$ (non-bipartite case).

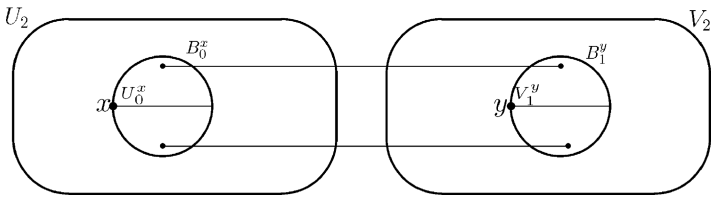

Fig. 6. Intercopy edges of compound graphs $B_0 \nabla B_1$ (bipartite case).

D depend on the number of intercopy edges and how they are placed. The following proposition provides an upper bound for the diameter D of each of the above constructions. The proofs of all these bounds are contained in the indicated references, although the proofs of the three last bounds were made by considering only the case when the main graph G_2 is a complete bipartite graph, that is, $D_2 = 2$. We have noticed that when G_2 is an arbitrary bipartite graph with diameter D_2, the proofs are very similar and for this reason we omit them.

Proposition 1.1. *Let G_2, G_1, G_1' be three graphs of diameters D_2, D_1 and D_1' respectively. If D is the diameter of a*

1. *basic compound graph, then $D \leqslant (D_1 + 1)D_2 + D_1$* [3].
2. *bipartite compound graph, then $D \leqslant D_1 D_2 + D_1$* [9].
3. *compound graph FF, then $D \leqslant D_1 D_2 + D_1 - 1$* [12].
4. *compound graph DQ_Λ and D_2 is even, then $D \leqslant D_2(D_1 + D_1' + 1)/2$* [11].

5. *compound graph* $B_0 \nabla B_1$ *with four intercopy edges and* D_2 *is even*, *then* $D \leqslant D_2 (D_1 + D_1')/2$ [16].
6. *bipartite compound graph* $B_0 \nabla B_1$ *with two intercopy edges and* D_2 *is even*, *then* $D \leqslant (D_2(D_1 + D_1') + 2)/2$ [16].

Another well-known way of obtaining large (Δ, D)-graphs is the design of *graphs on alphabets* (see, for instance, [17]). These graphs are constructed by labelling the vertices with a word on a given alphabet, together with a rule that relates pairs of different words to define the edges. For instance, the well-known *De Bruijn graph* $UB(d, D)$ is a graph on an alphabet defined as follows. It has vertex set X^D, $|X| = d$, and adjacency conditions:

$$\Gamma(x_1 x_2 \ldots x_D) = \{x_2 x_3 \ldots x_D x_{D+1}, x_{D+1} \in X\} \cup \{x_0 x_1 \ldots x_{D-1}, x_0 \in X\}.$$

The *Kautz graph* $UK(d, D)$ is the subgraph of the De Bruijn graph $UB(d+1, D)$ obtained by considering only the vertices represented by words whose consecutive letters (elements of X) are different, $x_{i+1} \neq x_i$, $1 \leqslant i \leqslant D-1$. The number of vertices of De Bruijn graphs and Kautz graphs in terms of their maximum degree $\Delta = 2d$ and diameter D are $(\Delta/2)^D$ and $(\Delta/2)^D + (\Delta/2)^{D-1}$, respectively.

The line digraph technique is a good general method for obtaining large digraphs with fixed degree and diameter. In the line digraph LG of a digraph G, each vertex represents an edge of G. Thus, $V(LG) = \{uv\colon (u, v) \in E(G)\}$; and a vertex uv is adjacent to a vertex wz if and only if $v = w$, that is, when the edge (u, v) is adjacent to the edge (w, z) in G. For any $k > 1$, the *k-iterated line digraph*, $L^k G$, is defined recursively by $L^k G = LL^{k-1} G$. From the definition, it is evident that the order of LG equals the size of G, $|V(LG)| = |E(G)|$, and that their maximum and minimum degrees coincide, $\Delta(LG) = \Delta(G) = \Delta$, $\delta(LG) = \delta(G) = \delta$. Moreover, if G is d-regular ($\delta^-(x) = \delta^+(x) = d > 1$, for any $x \in V$), has order n and diameter D, then $L^k G$ is also d-regular, has $d^k n$ vertices and diameter

$$D(L^k G) = D(G) + k. \tag{1}$$

See, for instance, [13,22]. In fact, (1) still holds for any (strongly) connected digraph other than a directed cycle (see [1]).

Two large families of digraphs obtained from the line digraph technique are the De Bruijn and Kautz digraphs. The *De Bruijn digraph* of degree d and diameter D is the $(D-1)$)-iterated line digraph of the complete graph K_d^*, $B(d, D) \cong L^{D-1} K_d^*$, whereas the *Kautz digraph* of degree d and diameter D is defined as the $(D-1)$)-iterated line digraph of the simple complete graph K_{d+1}, $K(d, D) \cong L^{D-1} K_{d+1}$ [19]. Let us denote by UG the underlying graph of a digraph G. Observe that the underlying graph of the De Bruijn digraph $B(d, D)$ (resp., Kautz digraph $K(d, D)$) is the De Bruijn graph (resp., Kautz graph) previously defined as a graph on an alphabet, and for this reason it is denoted $UB(d, D)$ (resp. $UK(d, D)$). Two well-known properties concerning these families are: $D(UG) = D(G) = D$, $\Delta(UG) = 2\Delta(G) = 2d$ if $D \geqslant 3$ (see [7]).

Generalized compound graphs, called *GC graphs* throughout this paper, were defined by Gómez [15]. They combine the advantages of compound graphs and graphs on alphabets. In general, the first method has been used to designing large graphs when the diameter is rather small, whereas graphs on alphabets has been so far the most usual way of constructing large graphs for large values of the diameter. As it was indicated in [15], among the different GC families it is possible, in a wide range of cases, to find graphs with an order significantly greater than that of any other known graph, both for small and large values of the diameter. This issue will be exhibited with some more detail in Section 3 (see Remark 3.1).

The main goal of this work is to study the diameter vulnerability, for the case $s=1$, of most of the families of GC graphs. For this, first of all, three families of iterated line digraphs are put forward in Section 2, the underlying graphs of which we will show as to be $(\Delta, D, D, 1)$-graphs. Section 3 is devoted to characterizing (i.e., to redefining) the GC graphs as compound graphs, by taking as the main graph some of the three line families just presented. Finally, these results are the starting point to studying the extent to which the diameter of some families of GC graphs increases when one vertex is deleted.

2. Three families of iterated line digraphs

Given any two (di)graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, denote by $G_1 \otimes G_2$ the *conjunction* of the two (di)graphs, which is defined in the following way: $V(G_1 \otimes G_2)=V_1 \times V_2$ and two vertices (v_1,v_2), (w_1,w_2) are adjacent if and only if v_i and w_i are adjacent in G_i, $i=1,2$. If G_i is a (Δ_i,D_i)-(di)graph on N_i vertices, then $G_1 \otimes G_2$ is a $(\Delta_1\Delta_2,D')$-(di)graph on N_1N_2 vertices. However, it could happen that $D'=+\infty$, even though each G_i is (strongly) connected. For instance, the conjunction of K_2 and the cycle C_4 gives two disjoint cycles with four vertices.

Proposition 2.1. *For any given two (di)graphs G_1 and G_2 it follows that*

$$L(G_1 \otimes G_2) \cong LG_1 \otimes LG_2.$$

Proof. Let u_1v_1 and u_2v_2 be edges of $E(G_1)$ and $E(G_2)$, respectively. Let us consider a one-to-one mapping ϕ from $V(L(G_1 \otimes G_2))$ onto $V(LG_1 \otimes LG_2)$, namely $\phi((u_1,u_2)(v_1,v_2))=(u_1v_1,u_2v_2)$. It follows that ϕ is an isomorphism, because it preserves adjacency. Indeed, $(u_1,u_2)(v_1,v_2)$ is adjacent to another vertex $(a_1,a_2)(b_1,b_2)$ of $L(G_1 \otimes G_2)$ if and only if $(v_1,v_2)=(a_1,a_2)$, which is equivalent to say that (u_1v_1,u_2v_2) is adjacent to (a_1b_1,a_2b_2) in $LG_1 \otimes LG_2$. □

Now, we are going to consider three families of digraphs on alphabets which are also families of iterated line digraphs. In the rest of this work, m,n,h denote integers greater than 1, and $J_m=\{1,2,\ldots,m\}$. The first family, which is actually a De Bruijn one, is defined as follows.

Definition 2.1. The vertex set of the so-called $G^{\mathrm{I}}(m,n,h)$ digraph is:

$$V=[J_m \times J_n]^h=\{(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_h,x_h), \beta_i \in J_m, x_i \in J_n\},$$

and its adjacency rule is:

$$\Gamma^+((\beta_1,x_1)(\beta_2,x_2)\dots(\beta_h,x_h)) = \{(\beta_0,x_0)(\beta_1,x_1)\dots(\beta_{h-1},x_{h-1}), \beta_0 \in J_m, x_0 \in J_n\}.$$

As a direct consequence of its definition, the following properties are obtained.

Proposition 2.2. *The digraph $G^{\mathrm{I}}(m,n,h)$ satisfies the following properties*:

1. *It is isomorphic to the De Bruijn digraph $B(mn,h) \cong L^{h-1}(K^*_{mn})$.*
2. *It is regular with degree and diameter*: $(\Delta,D)=(mn,h)$.
3. *Its underlying graph has diameter h and maximum degree $2mn$.*

The second family (which for $m=1$ is simply a family of Kautz digraphs), is defined next.

Definition 2.2. The vertex set of the so-called $G^{\mathrm{II}}(m,n,h)$ digraph is

$$V = \{(\beta_1,x_1)(\beta_2,x_2)\dots(\beta_h,x_h), \beta_i \in J_m, x_i \in J_{n+1}, x_i \neq x_{i+1}\}$$

and its adjacency rule is

$$\Gamma^+((\beta_1,x_1)(\beta_2,x_2)\dots(\beta_h,x_h))$$
$$=\{(\beta_0,x_0)(\beta_1,x_1)\dots(\beta_{h-1},x_{h-1}), \beta_0 \in J_m, x_0 \in J_{n+1} \setminus \{x_1\}\}.$$

As in the previous case, these digraphs can also be considered as iterated line digraphs obtained by conjunction of a De Bruijn digraph with a Kautz digraph, as is showed in the following proposition.

Proposition 2.3. *The digraph $G^{\mathrm{II}}(m,n,h)$, satisfies the following properties*:

1. *It is isomorphic to the digraph $B(m,h) \otimes K(n,h)$.*
2. *It is isomorphic to the iterated line digraph $L^{h-1}(K^*_m \otimes K_{n+1})$.*
3. *It is regular with degree and diameter*: $(\Delta,D)=(mn,h+1)$.
4. *The diameter of its underlying graph is $h+1$ and*, *if $h \geqslant 3$*, *then its maximum degree is $2mn$.*

Proof. The one-to-one mapping from $V(G^{\mathrm{II}}(m,n,h))$ onto $V(B(m,h) \otimes K(n,h))$ defined by

$$(\beta_1,x_1)(\beta_2,x_2)\dots(\beta_h,x_h) \mapsto (\beta_1\beta_2\dots\beta_h, x_1x_2\dots x_h)$$

induces clearly an isomorphism between the two digraphs.

Property (2) follows immediately from Proposition 2.1, together with the fact that the digraphs $B(m,h)$ and $K(n,h)$ are isomorphic to $L^{h-1}(K^*_m)$ and $L^{h-1}(K_{n+1})$, respectively.

From Definition 2.2, it follows immediately that $\Delta=mn$. Moreover, $K^*_m \otimes K_{n+1}$ has no loops and its diameter is 2, since $m \geqslant 2$. Hence, by the previous point and Eq. (1), $D=2+h-1=h+1$, and thus point 3, also holds.

Finally, from Definition 2.2 it follows immediately that the maximum degree of the underlying graph $UG^{\mathrm{II}}(m,n,h)$ is $2mn$, since $h \geqslant 3$. It is also obvious that the diameter of this graph is at most $h+1$. To see the equality it is enough to find two vertices at distance $h+1$ in $G^{\mathrm{II}}(m,n,h)$, which will remain at distance $h+1$ in $UG^{\mathrm{II}}(m,n,h)$. Let us consider the following two vertices:

$$u = (\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{h-1},x_{h-1})(\beta_h,x_h),$$

$$v = (\gamma_1,x_h)(\gamma_2,y_2)\ldots(\gamma_{h-1},y_{h-1})(\gamma_h,x_1).$$

Notice that, if $\beta_1 \neq \gamma_h$ and $\beta_h \neq \gamma_1$, then $d(u,v) = d(v,u) = h+1$ in the original digraph and so $d(u,v) = h+1$ in the underlying graph too. □

Finally, we put forward a family of bipartite digraphs on alphabets. In what follows, m_0, n_0, m_1, n_1 denote positive integers satisfying $m_0 n_0 \geqslant 2$, $m_1 n_1 \geqslant 2$, and h an odd integer greater than 1.

Definition 2.3. The partite vertex sets of the so-called $G^{\mathrm{III}}(m_0,n_0,m_1,n_1,h)$ bipartite digraph are:

$$U = \{(\gamma_1,y_1)(\beta_2,x_2)\ldots(\gamma_h,y_h), \beta_i \in J_{m_0}, \gamma_j \in J_{m_1}, x_i \in J_{n_0}, y_j \in J_{n_1}\},$$

$$V = \{(\beta_1,x_1)(\gamma_2,y_2)\ldots(\beta_h,x_h), \beta_i \in J_{m_0}, \gamma_j \in J_{m_1}, x_i \in J_{n_0}, y_j \in J_{n_1}\}$$

and its adjacency rules are:

$$\Gamma^+((\gamma_1,y_1)(\beta_2,x_2)\ldots(\gamma_h,y_h)) = \{(\beta_0,x_0)(\gamma_1,y_1)\ldots(\beta_{h-1},x_{h-1}), \beta_0 \in J_{m_0}, x_0 \in J_{n_0}\},$$

$$\Gamma^+((\beta_1,x_1)(\gamma_2,y_2)\ldots(\beta_h,x_h)) = \{(\gamma_0,y_0)(\beta_1,x_1)\ldots(\gamma_{h-1},y_{h-1}), \gamma_0 \in J_{m_1}, y_0 \in J_{n_1}\}.$$

As in the above two cases, this third family of bipartite digraphs is also a family of iterated line digraphs, as it is shown in the following proposition.

Proposition 2.4. *The digraph $G^{\mathrm{III}}(m_0,n_0,m_1,n_1,h)$ satisfies the following properties*:

1. *It is isomorphic to the bipartite line digraph $L^{h-1}(K_{m_0n_0,m_1n_1})$.*
2. *Its minimum degree is* $\min\{m_0n_0, m_1n_1\}$ *and its maximum degree and diameter are*: $(\Delta, D) = (\max\{m_0n_0, m_1n_1\}, h+1)$.
3. *Its underlying graph has diameter $h+1$ and maximum degree $m_0n_0 + m_1n_1$.*

Proof. The line digraph of a bipartite digraph is also bipartite. Furthermore, we can conclude that $G^{\mathrm{III}}(m_0,n_0,m_1,n_1,h) = L^{h-1}(K_{m_0n_0,m_1n_1})$. Hence, its maximum degree is equal to $\max\{m_0n_0, m_1n_1\}$ and by means of (1) we derive that the diameter is $h+1$. Finally, the proof of the third point is similar to that of Proposition 2.3.

Let us conclude this section by studying the extent to which the diameter of the underlying graphs of the different line digraphs just presented increases when one vertex is deleted. Some preliminary results must be reviewed first.

Line digraphs have been characterized by Heuchenne [18] by the following property: a digraph G is the line digraph of a digraph if and only if it has no multiple arcs, and for any pair of vertices u and v, either $\Gamma^+(u)\cap\Gamma^+(v)=\emptyset$ or $\Gamma^+(u)=\Gamma^+(v)$ (the same condition holds with Γ^- instead of Γ^+.) Bond and Peyrat proved [7] the following statement.

Proposition 2.5. *Let G be a (strongly) connected digraph such that for every x, $y\in V(G)$:*

$$|\Gamma^+(x)\cap\Gamma^+(y)|\neq 1 \quad \textit{and} \quad |\Gamma^-(x)\cap\Gamma^-(y)|\neq 1.$$

Then, for any $v\in V(UG)$: $D(UG-\{v\})\leqslant D(G)$.

As a consequence of the combination of the Heuchenne condition with this result, Bond and Peyrat proved in the same reference that if G is a Kautz digraph or a De Bruijn digraph (with D and Δ not equal both to 2), then $D(UG-\{v\})=D(G)$ for any vertex v of G. Following these ideas we get the following result.

Proposition 2.6. *Let $G=LH$ be a (strongly) connected line digraph, with minimum degree $\delta\geqslant 2$, for which $D(UG)=D(G)=D$. Then, UG is a $(\Delta,D,D,1)$-graph.*

Proof. From the Heuchenne condition, it follows that G satisfies the hypothesis of Proposition 2.5, since $\delta\geqslant 2$. Hence, for any vertex v of G, $D(UG-\{v\})\leqslant D(G)$. Moreover, there exists some vertex w for which $D(UG-\{w\})\geqslant D(UG)$ because $\delta\geqslant 2$, and since $D(G)=D(UG)$ we obtain $D(UG-\{w\})=D$, and hence we conclude that UG is a $(\Delta,D,D,1)$-graph. $\square$

As a direct consequence of the above results we have the following theorem.

Theorem 2.1. *Let m,n,m_0,n_0,m_1,n_1,h be integers greater than 1. Then, the graphs $UG^{\mathrm{I}}(m,n,h)$, $UG^{\mathrm{II}}(m,n,h)$ and $UG^{\mathrm{III}}(m_0,n_0,m_1,n_1,h)$ are solutions of the $(\Delta,D,D,1)$-problem.*

3. GC graphs: a new characterization

GC graphs were introduced by Gómez [15]. In order not to increase excessively this work, we refer to this paper for details about these families. When approaching the $(\Delta,D,D',1)$ problem in these graphs, we realized that they could be redefined as compound graphs by taking as the main graph some of the graphs belonging to the families presented in the previous section. Moreover, this new characterization was the key point to solve completely the mentioned problem in a more suitable way (see Section 4).

3.1. GC graphs of type I

Next, we are going to define from another point of view the two families of GC graphs that were denoted in [15] by $G_1\{m,k\}G$ and $B_1\{m,k\}G$, respectively. These families are put forward as compound graphs, the main graph belonging to the De Bruijn family introduced in Definition 2.1. In the rest of this work, k denotes an integer greater than 2.

Definition 3.1. Let $G_1=(J_n,E_1)$ be a (Δ_1,D_1)-graph. Then, the so-called $GC\{m,n,k;G_1\}$ graph is defined as the basic compound graph $G_2[G_1]$ where $G_2=UG^{\mathrm{I}}(m,n,k-1)$, and the adjacency rule which yields an intercopy edge is:

$$(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})x_k\sim(\beta_0,x_k)(\beta_1,x_1)\ldots(\beta_{k-2},x_{k-2})x_{k-1},$$

where

$$(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})\in[J_m\times J_n]^{k-1}, x_k\in J_n \text{ and } \beta_0\in J_m.$$

As an immediate consequence of this definition, we can assure that the maximum degree and order of this graph are: $(\Delta,N)=(\Delta_1+2m,\ m^{k-1}n^k)$. Furthermore, it was proved in [15] that its diameter is $D=kD_1+k-1$.

Definition 3.2. Let B_1 be a bipartite (Δ_1,D_1)-graph with partite sets $U_1=\{0\}\times J_n$, and $V_1=\{1\}\times J_n$. Then, the so-called $GC\{m,n,k;B_1\}$ graph is defined as the bipartite compound graph $G_2[G_1]$ where $G_2=UG^{\mathrm{I}}(m,n,k-1)$ and, according to the parity of D_1, the intercopy adjacency rules are:

- If D_1 is odd, then the adjacency rule producing two intercopy edges is (see Fig. 1):

  $$(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})(\alpha,x_k)\sim(\beta_0,x_k)(\beta_1,x_1)\ldots(\beta_{k-2},x_{k-2})(\bar{\alpha},x_{k-1}),$$

  where $(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},\ x_{k-1})\in[J_m\times J_n]^{k-1}$, $\bar{\alpha}=\alpha+1\in\mathbb{Z}_2$, $(\bar{\alpha},x_{k-1}),(\alpha,x_k)\in U_1\cup V_1$ and $\beta_0\in J_m$.
- If D_1 is even, then the adjacency rule producing one intercopy edge (see Fig. 2):

  $$(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})(0,x_k)\sim(\beta_0,x_k)(\beta_1,x_1)\ldots(\beta_{k-2},x_{k-2})(1,x_{k-1}),$$

  $$(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})(1,x_k)\sim(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})(\beta_k,x_k)(0,x_1)$$

  where $(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})\in[J_m\times J_n]^{k-1}$, $(0,x_k)\in U_1,(1,x_k)\in V_1$ and $\beta_0,\beta_k\in J_m$.

Certainly, the order of this graph is $N=2m^{k-1}n^k$. It is also easy to see that, if D_1 is odd, then $\Delta=\Delta_1+2m$, whereas if D_1 is even, $\Delta=\Delta_1+m$. In addition, it was proved in [15] that in either case the diameter of this graph is: $D=kD_1$.

3.2. GC graphs of type II

GC graphs of type II were denoted $G_1(m,k)G$, $B_1(m,k)G$ and $FF(m,k)G$, respectively. These constructions were inspired by Kautz graphs. In fact, these families can be introduced as compound graphs where the main graph is the underlying graph of $G^{\mathrm{II}}(m,n,k-1) \cong L^{k-2}(K_m^* \otimes K_{n+1})$ (see Proposition 2.3). The main differences between them lie, on the one hand, in the type of copies used, and on the other, in the adjacency rules to produce intercopy edges.

First of all, let us introduce a family of bijections which was used to present the different adjacency rules in an appropriate way.

Definition 3.3. Given $l \in J_{n+1}$, let f_l denotes the only increasing bijection from $J_{n+1} \setminus \{l\}$ onto J_n.

Definition 3.4. Let $G_1 = (J_n, E_1)$ be a (Δ_1, D_1)-graph. Then, the so-called $GC(m,n,k;G_1)$ graph is defined as the basic compound graph $G_2[G_1]$, where $G_2 = UG^{\mathrm{II}}(m,n,k-1)$, and the adjacency rule producing one intercopy edge is

$$(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})x_k \sim (\beta_0,x_k')(\beta_1,x_1)\ldots(\beta_{k-2},x_{k-2})x_{k-1},$$

where $(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1}) \in V(G_2)$, $x_k \in f_{x_{k-1}}^{-1}(J_n)$, x_k' satisfies $f_{x_1}(x_k') = f_{x_{k-1}}(x_k)$ and $\beta_0 \in J_m$.

It is not difficult to see that $(\Delta, N) = (\Delta_1 + 2m, (n+1)(mn)^{k-1})$. Moreover, $D = kD_1 + k - 1$ (see [15]).

Definition 3.5. Let B_1 be a bipartite (Δ_1, D_1)-graph on $2n$ vertices, with partite sets $U_1 = \{0\} \times J_n$, $V_1 = \{1\} \times J_n$. Then, the so-called graph $GC(m,n,k;B_1)$ graph is defined as the bipartite compound graph $G_2[B_1]$, where $G_2 = UG^{\mathrm{II}}(m,n,k-1)$, and according to the parity of D_1 the intercopy adjacency rules are:

- If D_1 is odd, the adjacency rule produces two intercopy edges (see Fig. 1):

$$(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})(\alpha,x_k) \sim (\beta_0,x_k')(\beta_1,x_1)\ldots(\beta_{k-2},x_{k-2})(\bar{\alpha},x_{k-1}),$$

  where $(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1}) \in V(G_2)$, $(\alpha,x_k) \in \{0,1\} \times f_{x_{k-1}}^{-1}(J_n)$, $\bar{\alpha} = \alpha + 1 \in \mathbb{Z}_2$, $f_{x_1}(x_k') = f_{x_{k-1}}(x_k)$ and $\beta_0 \in J_m$.
- If D_1 is even, the adjacency rule yields one intercopy edge (see Fig. 2):

$$(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})(0,x_k) \sim (\beta_0,x_k')(\beta_1,x_1)\ldots(\beta_{k-2},x_{k-2})(1,x_{k-1}),$$

$$(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})(1,x_k) \sim (\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})(\beta_k,x_k)(0,x_1'),$$

  where $(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1}) \in V(G_2)$, $x_k \in f_{x_{k-1}}^{-1}(J_n)$, $f_{x_1}(x_k') = f_{x_{k-1}}(x_k)$, $f_{x_k}(x_1') = f_{x_2}(x_1)$ and $\beta_0, \beta_k \in J_m$.

Clearly, the order of this graph is $N = 2(n+1)(mn)^{k-1}$. It is also easy to see that, if D_1 is odd, then $\Delta = \Delta_1 + 2m$, whereas if D_1 is even, $\Delta = \Delta_1 + m$. Finally, it was proved in [15] that in either case the diameter of this graph is: $D = kD_1$.

Definition 3.6. Let B_1 be a bipartite (Δ_1, D_1)-graph on $2n$ vertices, with partite sets $U_1 = \{0\} \times J_n$, $V_1 = \{1\} \times J_n$. Then, the so-called graph $FGC(m,n,k;B_1)$ graph is defined as the FF compound graph $G_2[B_1]$, where $G_2 = UG^{\mathrm{II}}(m,n,k-1)$, and the adjacency rule producing four intercopy edges is (see Fig. 3):

$$(\beta_1,x_1)(\beta_2,x_2)\dots(\beta_{k-1},x_{k-1})(\alpha,x_k) \sim (\beta_0,x_k')(\beta_1,x_1)\dots(\beta_{k-2},x_{k-2})(\alpha',x_{k-1}),$$

where $(\beta_1,x_1)(\beta_2,x_2)\dots(\beta_{k-1},x_{k-1}) \in V(G_2)$, $(\alpha,x_k) \in \{0,1\} \times f^{-1}_{x_{k-1}}(J_n), \beta_0 \in J_m$, $f_{x_1}(x_k') = f_{x_{k-1}}(x_k)$ and $\alpha' \in \{0,1\}$.

Its maximum degree, diameter and order are $(\Delta, D, N) = (\Delta_1 + 4m, kD_1 - 1, 2(n+1)(mn)^{k-1})$ (see [15]).

3.3. GC graphs of type III

Let us finalize this list of redefinitions by presenting the so-called GC graphs of type III, which correspond to the families denoted by $DQ_\Delta\{m_1, m_0, k\}G$ and $B_0\nabla B_1\{m_1, m_0, k\}G$, respectively (see [15]). We show that all of these constructions can be seen as compound graphs $G_2[G_0, G_1]$, where the main graph G_2 is the underlying graph of $G^{\mathrm{III}}(m_0,n_0,m_1,n_1,k-1) \cong L^{k-2}(K_{m_0n_0,m_1n_1})$ (See Proposition 2.4).

Definition 3.7. Let B_0 be a bipartite (Δ_0, D_0)-graph of order $2n_0$, whose partite sets are $\{0\} \times J_{n_0}$, $\{1\} \times J_{n_0}$, and let $G_1 = (J_{n_1}, E_1)$ be a (Δ_1, D_1)-graph. Then, the so-called (non-bipartite) $GC\{m_0,n_0,m_1,n_1,k;B_0,G_1\}$ graph is defined as the DQ_Δ compound graph $G_2[B_0,G_1]$, where the main graph $G_2 = (U_2 \cup V_2, E_2)$ is the bipartite graph $UG^{\mathrm{III}}(m_0,n_0,m_1,n_1,k-1)$, and the adjacency rules which produce two intercopy edges are (see Fig. 4):

$$(\gamma_1,y_1)(\beta_2,x_2)\dots(\gamma_{k-1},y_{k-1})(\alpha,x_k) \sim (\beta_0,x_k)(\gamma_1,y_1)\dots(\beta_{k-2},x_{k-2})y_{k-1},$$

where $(\beta_1,x_1)(\gamma_2,y_2)\dots(\beta_{k-1},x_{k-1}) \in U_2$, $(\alpha,x_k) \in \{0,1\} \times J_{n_0}, \beta_0 \in J_{m_0}$;

$$(\beta_1,x_1)(\gamma_2,y_2)\dots(\beta_{k-1},x_{k-1})y_k \sim (\gamma_0,y_k)(\beta_1,x_1)\dots(\gamma_{k-2},y_{k-2})(\alpha,x_{k-1}),$$

where $(\beta_1,x_1)(\gamma_2,y_2)\dots(\beta_{k-1},x_{k-1}) \in V_2$, $y_k \in J_{n_1}, \gamma_0 \in J_{m_1}, \alpha \in \{0,1\}$.

It was proved in [15] that its maximum degree, diameter and order are:

$$(\Delta, D, N) = \left(\max\{\Delta_0 + 2m_0, \Delta_1 + 4m_1\}, \frac{k(D_0+D_1+1)}{2},\right.$$
$$\left.(m_0+2m_1)(m_0m_1)^{k/2-1}(n_0n_1)^{k/2}\right).$$

Definition 3.8. Let B_i, $i=0,1$, be two bipartite (Δ_i, D_i)-graphs of order $2n_i$, with partite sets $\{0\}\times J_{n_i}$, $\{1\}\times J_{n_i}$. Then, the so-called (non-bipartite) $GC\{m_0, n_0, m_1, n_1, k; B_0, B_1\}$ graph is defined as the $B_0\nabla B_1$ compound graph $G_2[B_0, B_1]$, where the main graph $G_2=(U_2\cup V_2, E_2)$ is the bipartite graph $G_2 = UG^{\mathrm{III}}(m_0, n_0, m_1, n_1, k-1)$, and the adjacency rules producing four intercopy edges are (see Fig. 5):

$$(\gamma_1, y_1)(\beta_2, x_2)\ldots(\gamma_{k-1}, y_{k-1})(\alpha, x_k) \sim (\beta_0, x_k)(\gamma_1, y_1)\ldots(\beta_{k-2}, x_{k-2})(\alpha', y_{k-1}),$$

where $(\gamma_1, y_1)(\beta_2, x_2)\ldots(\gamma_{k-1}, y_{k-1})\in U_2, (\alpha, x_k)\in\{0,1\}\times J_{n_0}, \beta_0\in J_{m_0}, \alpha'\in\{0,1\}$;

$$(\beta_1, x_1)(\gamma_2, y_2)\ldots(\beta_{k-1}, x_{k-1})(\alpha, y_k) \sim (\gamma_0, y_k)(\beta_1, x_1)\ldots(\gamma_{k-2}, y_{k-2})(\alpha', x_{k-1}),$$

where $(\beta_1, x_1)(\gamma_2, y_2)\ldots(\beta_{k-1}, x_{k-1})\in V_2, (\alpha, y_k)\in\{0,1\}\times J_{n_1}, \gamma_0\in J_{m_1}, \alpha'\in\{0,1\}$.

Its maximum degree, diameter and order are (see [15]):

$$(\Delta, D, N) = \left(\max\{\Delta_0+4m_0, \Delta_1+4m_1\}, \frac{k(D_0+D_1)}{2},\right.$$
$$\left. 2(m_0+m_1)(m_0m_1)^{k/2-1}(n_0n_1)^{k/2}\right).$$

Definition 3.9. Let B_i, $i=0,1$, be two bipartite (Δ_i, D_i)-graphs of order $2n_i$, with partite sets $\{0\}\times J_{n_i}$ and $\{1\}\times J_{n_i}$. Then, the so-called bipartite $BGC\{m_0, n_0, m_1, n_1, k; B_0, B_1\}$ graph is defined as the $B_0\nabla B_1$ compound graph $G_2[B_0, B_1]$, where the main graph $G_2=(U_2\cup V_2, E_2)$ is the bipartite graph $G_2 = UG^{\mathrm{III}}(m_0, n_0, m_1, n_1, k-1)$, and the adjacency rules producing two intercopy edges are (see Fig. 6):

$$(\gamma_1, y_1)(\beta_2, x_2)\ldots(\gamma_{k-1}, y_{k-1})(\alpha, x_k) \sim (\beta_0, x_k)(\gamma_1, y_1)\ldots(\beta_{k-2}, x_{k-2})(\bar{\alpha}, y_{k-1}),$$

where $(\gamma_1, y_1)(\beta_2, x_2)\ldots(\gamma_{k-1}, y_{k-1})\in U_2, (\alpha, x_k)\in\{0,1\}\times J_{n_0}, \beta_0\in J_{m_0}, \bar{\alpha}=\alpha+1\in\mathbb{Z}_2$;

$$(\beta_1, x_1)(\gamma_2, y_2)\ldots(\beta_{k-1}, x_{k-1})(\alpha, y_k) \sim (\gamma_0, y_k)(\beta_1, x_1)\ldots(\gamma_{k-2}, y_{k-2})(\bar{\alpha}, x_{k-1}),$$

where $(\beta_1, x_1)(\gamma_2, y_2)\ldots(\beta_{k-1}, x_{k-1})\in V_2, (\alpha, y_k)\in\{0,1\}\times J_{n_1}, \gamma_0\in J_{m_1}, \bar{\alpha}=\alpha+1\in\mathbb{Z}_2$.

As in the previous case, it was proved in [15] that its maximum degree, diameter and order are:

$$(\Delta, D, N) = \left(\max\{\Delta_0+2m_0, \Delta_1+2m_1\}, \frac{k(D_0+D_1)+2}{2},\right.$$
$$\left. 2(m_0+m_1)(m_0m_1)^{k/2-1}(n_0n_1)^{k/2}\right).$$

Remark 3.1. One of the main motivations for approaching the study of the $(\Delta, D, D', 1)$-problem in the GC families has been the fact that they contain, for a wide range of both the maximum degree and the diameter, graphs with an order significantly greater than that of any other known graph. Although we do not intend to treat this issue

Table 1
Parameters (Δ, D, N) of the GC graphs

G	Δ	D	N
$GC\{m,n,k;G_1\}$	Δ_1+2m	kD_1+k-1	$m^{k-1}n^k$
$GC\{m,n,k;B_1\}(D_1$ odd$)$	Δ_1+2m	kD_1	$2m^{k-1}n^k$
$GC\{m,n,k;B_1\}(D_1$ even$)$	Δ_1+m	kD_1	$2m^{k-1}n^k$
$GC(m,n,k;G_1)$	Δ_1+2m	kD_1+k-1	$(n+1)(mn)^{k-1}$
$GC(m,n,k;B_1)(D_1$ odd$)$	Δ_1+2m	kD_1	$2(n+1)(mn)^{k-1}$
$GC(m,n,k;B_1)(D_1$ even$)$	Δ_1+m	kD_1	$2(n+1)(mn)^{k-1}$
$FGC(m,n,k;B_1)$	Δ_1+4m	kD_1-1	$2(n+1)(mn)^{k-1}$
$GC\{m_0,n_0,m_1, n_1,k;B_0,G_1\}$	$\max\{\Delta_0+2m_0, \Delta_1+4m_1\}$	$k(D_0+D_1+1)/2$	$(m_0+2m_1)(m_0m_1)^{k/2-1}(n_0n_1)^{k/2}$
$GC\{m_0,n_0,m_1, n_1,k;B_0,B_1\}$	$\max\{\Delta_0+4m_0, \Delta_1+4m_1\}$	$k(D_0+D_1)/2$	$2(m_0+m_1)(m_0m_1)^{k/2-1}(n_0n_1)^{k/2}$
$BGC\{m_0,n_0,m_1, n_1,k;B_0,B_1\}$	$\max\{\Delta_0+2m_0, \Delta_1+2m_1\}$	$(k(D_0+D_1)+2)/2$	$2(m_0+m_1)(m_0m_1)^{k/2-1}(n_0n_1)^{k/2}$

in this work, we want to illustrate the previous statement by showing two particular cases.

1. Let $G_1 = H'_q$ the *quotient graph* of the generalized hexagon H_q, q being an odd power of 3 (see [8]). It was proved in the mentioned paper that H'_q is a regular graph with degree $\Delta_1 = q+1$, diameter $D_1 = 5$ and order $n = (q+1)(q^4+q^2+1)$. Consider the GC graph of type I: $GC\{m,n,k;H'_q\}$ (see Definition 3.1 and Table 1), where $q \geqslant 27$, $m = q/9$. We proceed to compare the order N of this graph with respect to that of the De Bruijn graph $UB(\Delta/2, D)$:
$$\frac{N}{(\frac{\Delta}{2})^D} = \frac{m^{k-1}\,n^k}{(\frac{q+1+2m}{2})^{6k-1}} \geqslant \frac{m^{k-1}\,q^{5k}\,2^{6k-1}}{(q+1+2m)^{6k-1}} = \frac{q^{6k-1}\,2^{6k-1}}{9^{k-1}\,(q+1+\frac{2q}{9})^{6k-1}}$$
$$= \frac{2^{6k-1}}{9^{k-1}\,(1+\frac{1}{q}+\frac{2}{9})^{6k-1}} = \frac{9}{\sqrt[6]{9}}\left[\frac{2}{\sqrt[6]{9}(1+\frac{1}{q}+\frac{2}{9})}\right]^{6k-1}$$
$$\geqslant 6,2[1,1]^{6k-1} \geqslant 1.$$

2. Let G_1 be the generalized quadrangle Q_q, q being a prime power, which is a Moore bipartite graph with degree $\Delta_1 = q+1$, diameter $D_1 = 4$ and order $n = 2(q+1)(q^2+1)$ (see [3,8]). Consider the GC graph of type I: $GC\{m,n,k;Q_q\}$, where $q \geqslant 9$, $\frac{1}{4} \leqslant m/q \leqslant \frac{3}{10}$. We proceed to compare the order N of this graph with respect to that of the graph $K_2 \otimes UB(\frac{\Delta}{2}, D-1)$, which is known as to be a large bipartite graph with degree $\Delta = q+m+1$, diameter $D = 4k$ and order $2(\Delta/2)^{D-1}$

(see [10]):

$$\frac{N}{2(\frac{\Delta}{2})^{D-1}} \geqslant \frac{m^{k-1}q^{3k}}{(\frac{\Delta}{2})^{4k-1}} = \frac{\Delta}{2m}\left[\frac{16mq^3}{\Delta^4}\right]^k = \frac{\Delta}{2m}\left[\frac{16m}{q}\right]^k\left[1+\frac{1}{q}+\frac{m}{q}\right]^{-4k}$$
$$\geqslant \frac{\Delta}{2m}4^k\left[1+\frac{1}{q}+\frac{m}{q}\right]^{-4k} \geqslant \frac{\Delta}{2m}4^k\left[1+\frac{1}{9}+\frac{3}{10}\right]^{-4k}$$
$$\geqslant \frac{\Delta}{2m}\left[\frac{4}{3,97}\right]^k \geqslant 1.$$

Observe that GC graphs of type II [resp. III] are significantly larger than those of type I and notice also that these families provide a wider variety of degrees [resp. diameters].

4. The (Δ, D, D', 1)-problem in the GC graphs

This section is devoted to study the extent to which the diameter D of a GC graph G increases when one vertex is deleted. Let us begin by showing the theorem which contains the obtained result.

Theorem 4.1. *Let w be a vertex belonging to a GC graph G of diameter D. Then, the diameter of $G-w$ is at most $D+1$.*

The proof of this statement has been obtained as a corollary of three lemmas, which are put forward next.

Lemma 4.1. *Let u,v,w be three vertices belonging to the same copy G_1^x of a GC graph G. If D_1 is the diameter of G_1, then there exists an $u-v$ path avoiding w, whose length is at most D_1+4.*

Lemma 4.2. *Let u,v,w be three vertices belonging to three different copies of a GC graph G with diameter D. Then, there exists an $u-v$ path avoiding w, whose length is at most $D+1$.*

Lemma 4.3. *Let v,w be two vertices belonging to the same copy G_1^y of a GC graph G. If u is a vertex not belonging to G_1^y, then there exists an $u-v$ path avoiding w, whose length is at most $D+1$.*

To prove each of these lemmas in a complete and accurate way, we have considered each of the ten possibilities displayed in Table 1. As a result of this work, we have noticed that all of these proofs are very similar. For this reason, we present in this paper the proofs of the above lemmas highlighting the key points of each of them and considering a few particular cases.

Proof of Lemma 4.1. Consider the bipartite GC graph $G=GC(m,n,k;B_1)$, B_1 being a bipartite graph of even diameter D_1 (see Definition 3.5 and Fig. 2).

Since u,v,w belong to the same copy B_1^x in G, we can denote $u=x(0,x_k)$, $v=x(\alpha,y_k)$ and $w=x(j,z_k)$, where $\alpha,j\in\{0,1\}$, and $x=(\beta_1,x_1)(\beta_2,x_2)\dots(\beta_{k-1},x_{k-1})$ is a vertex of G_2. Let $\mu=(0,x_k)\dots(\bar{\alpha},\bar{x}_k)(\alpha,y_k)$ be a shortest path in B_1 joining $(0,x_k)$ to (α,y_k) (notice that $d((0,x_k),(\bar{\alpha},\bar{x}_k))\leqslant D_1-1$). Let us consider in G the following path η joining u and v:

$$
\begin{aligned}
u=\;&(\beta_1,x_1)(\beta_2,x_2)\dots(\beta_{k-1},x_{k-1})(0,x_k)\\
&(\varepsilon_k,x_k')(\beta_1,x_1)\dots(\beta_{k-2},x_{k-2})(1,x_{k-1})\\
&\left.\begin{aligned}
&(\beta_1,x_1)(\beta_2,x_2)\dots(\varepsilon_{k-1},x_{k-1})(0,x_k)\\
&\qquad\vdots\\
&(\beta_1,x_1)(\beta_2,x_2)\dots(\varepsilon_{k-1},x_{k-1})(0,\tilde{x}_k)
\end{aligned}\right\}\leqslant\tilde{D}_1\\
&(\varepsilon_k,\tilde{x}_k')(\beta_1,x_1)\dots(\beta_{k-2},x_{k-2})(1,x_{k-1})\\
&(\beta_1,x_1)(\beta_2,x_2)\dots(\beta_{k-1},x_{k-1})(0,\tilde{x}_k)\\
&\\
v=\;&(\beta_1,x_1)(\beta_2,x_2)\dots(\beta_{k-1},x_{k-1})(\alpha,y_k)
\end{aligned}
$$

where $\tilde{x}_k=y_k$, $\tilde{D}_1=D_1$ if $\alpha=0$ and $\tilde{x}_k=\bar{x}_k$, $\tilde{D}_1=D_1-1$ if $\alpha=1$. Notice that, if $\varepsilon_k\neq\beta_1$ and $\varepsilon_{k-1}\neq\beta_{k-1}$, then the path η avoids w and its length is at most D_1+4, since $m\geqslant 2$. $\square$

Proof of Lemma 4.2. This proof is based on the fact that in every GC graph, the main graph is an iterated line digraph which moreover is a solution to the $(\Delta,D,D,1)$-problem (see Theorem 2.1). Let us distinguish the following three cases, in which Λ denotes the set of bipartite GC graphs $G_2[B_1]$, the diameter of B_1 being even:

1. G is GC graph of type either I or III, not belonging to Λ: This proof is based on the following key points:
   (i) The diameter of G attains the upper bound given in Proposition 1.1.
   (ii) Between two copies corresponding to adjacent vertices in G_2, if at least one of them is bipartite, then there are two or more intercopy edges (see Figs. 1, 4–6).

   Consider firstly the GC graph $G=GC\{m,n,k;G_1\}$. Every path ρ in G_2 induces in G a path η_ρ, whose length depends on the length of ρ, on the diameter of G_1 and on the intercopy edges (see Fig. 7).

   If we denote $u=xx_k$, $v=yy_k$ and $w=zz_k$, then, from Theorem 2.1, it follows that there exists in G_2 a path ρ joining x and y, which does not go through z, whose length is at most $D_2=k-1$. Therefore, and keeping in mind (i), we conclude that η_ρ is a path between u and v avoiding w, whose length is at most D.

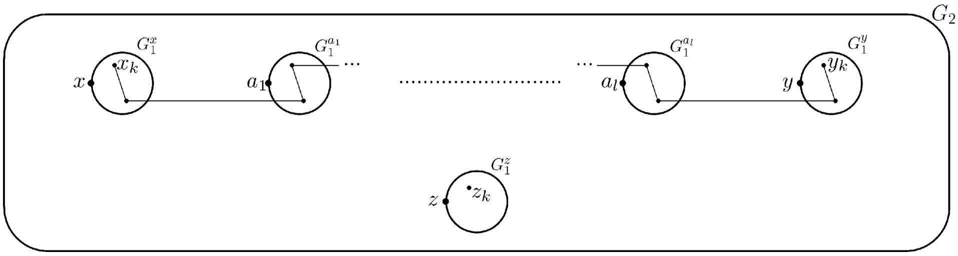

Fig. 7. Path η_ρ in G, induced by a path ρ: $xa_1 \dots a_l y$ of G_2.

Finally, notice that if G is a GC graph with bipartite copies, then there exists at least an intercopy edge from each partite set (see Figs. 1, 4–6). From this fact, it follows that the contribution of the subpath contained in any bipartite copy to the length of η_ρ is at most $D_1 - 1$. Hence, in each of these cases, we can also assure that the length of the path η_ρ cannot be greater than D.

2. G is GC graph of type II, not belonging to Λ: In this case, condition (ii) of the above case still holds, but not (i). For instance, the diameter of the GC graph $FGC(m,n,k;B_1)$ is $D = kD_1 - 1$, which is less than the upper bound obtained in Proposition 1.1 for this case, namely, $D_1D_2 + D_1 - 1 = (k+1)D_1 - 1$. Since the proofs for the three families of type II presented in the previous section are very similar, only one of them is showed. Consider the GC graph $G = GC(m,n,k;G_1)G$ (see Definitions 2.2 and 3.4). Let $u = xx_k$, $v = yy_k$ and $w = zz_k$ be three vertices placed in three different copies, where:

$$x = (\beta_1, x_1)(\beta_2, x_2) \dots (\beta_{k-1}, x_{k-1}),$$

$$y = (\gamma_1, y_1)(\gamma_2, y_2) \dots (\gamma_{k-1}, y_{k-1})$$

are two vertices of G_2. We distinguish four cases:

(a) If $x_1 \neq y_{k-1}$, $x_{k-1} \neq y_1$ then the digraph $G^{\mathrm{II}}(m,n,k-1)$ must contain two paths of length at most $k-1$, one from x to y and the other from y to x:

$$P_{xy}\colon xa_1 \dots a_{r}y, \quad P_{yx}\colon yb_1 \dots b_s x,$$

which induce two paths in G joining u and v, of length at most $kD_1 + k - 1 = D$. Suppose that the vertex z lies in both P_{xy} and P_{yx}:

$$a_i = b_j = z \Rightarrow z \in \Gamma^+(a_{i-1}) \cap \Gamma^+(b_{j-1}), z \in \Gamma^-(a_{i+1}) \cap \Gamma^-(b_{j+1}).$$

Since $G^{\mathrm{II}}(m,n,k-1)$ is a Δ_2-regular line digraph with $\Delta_2 \geqslant 2$, by applying the Heuchenne condition, we obtain that $\Gamma^+(a_{i-1}) = \Gamma^+(b_{j-1})$, $\Gamma^-(a_{i+1}) = \Gamma^-(b_{j+1})$. So, we can consider $z_1 \in \Gamma^+(a_{i-1}) \setminus \{z\}$, $z_2 \in \Gamma^-(a_{i+1}) \setminus \{z\}$. Hence, we have found two paths in G_2:

$$xa_1 \dots a_{i-1}z_1 b_{j-1} \dots b_1 y, \quad xb_s \dots b_{j+1}z_2 a_{i+1} \dots a_r y,$$

which do not contain the vertex z, and at least one of them is of length less than or equal to $k-1$, because $r + s \leqslant 2k-2$. Therefore, its corresponding induced path in G avoids w and its length is at most D.

(b) If $x_1 = y_{k-1}$ and $x_{k-1} \neq y_1$, then $G^{\mathrm{II}}(m,n,k-1)$ contains a path P_{yx} of length $k-1$. Moreover, if $\beta_1 = \gamma_{k-1}$ then there exists a path P_{xy} in $G^{\mathrm{II}}(m,n,k-1)$ of length at most $k-2$ and the reasoning is the same as in (a). If $\beta_1 \neq \gamma_{k-1}$, then we consider the following path of G:

$$u = xx_k = (\beta_1, y_{k-1})(\beta_2, x_2)\dots(\beta_{k-1}, x_{k-1})x_k,$$

$$\tilde{u} = \tilde{x}x_1' = (\beta_2, x_2)(\beta_3, x_3)\dots(\varepsilon_k, x_k)x_1',$$

$$\hat{u} = \hat{x}x_k = (\gamma_{k-1}, y_{k-1})(\beta_2, x_2)\dots(\beta_{k-1}, x_{k-1})x_k,$$

where we choose ε_k in such a way that $\tilde{x} \neq z$. Therefore, we can find in the digraph $G^{\mathrm{II}}(m,n,k-1)$ another path $P_{\hat{x}y}$ of length at most $k-2$. These two directed paths, P_{yx} and $P_{\hat{x}y}$, induce paths in the graph G between u and v, of length at most D. If the vertex z lies in both of them, then reasoning as in case (a), we find two paths in G_2:

$$x\tilde{x}\hat{x}a_3\dots a_{i-1}z_1b_{j-1}\dots b_1y, \quad xb_s\dots b_{j+1}z_2a_{i+1}\dots a_{ry}$$

the length of one of them being at most $k-1$, this one inducing a path in G avoiding w, of length at most D.

(c) If $x_1 \neq y_{k-1}$, $x_{k-1} = y_1$, then the reasoning is the same as that of (b).

(d) Finally, suppose that $x_1 = y_{k-1}$, $x_{k-1} = y_1$. If $\beta_1 = \gamma_{k-1}$ or $\beta_{k-1} = \gamma_1$, then the reasoning is as in (b). If $\beta_1 \neq \gamma_{k-1}$ and $\beta_{k-1} \neq \gamma_1$, consider in G the following $u - \hat{u}$ and $v - \hat{v}$ paths:

$$u = xx_k = (\beta_1, y_{k-1})(\beta_2, x_2)\dots(\beta_{k-1}, x_{k-1})x_k,$$

$$\tilde{u} = \tilde{x}x_1' = (\beta_2, x_2)(\beta_3, x_3)\dots(\varepsilon_k, x_k)x_1',$$

$$\hat{u} = \hat{x}x_k = (\gamma_{k-1}, y_{k-1})(\beta_2, x_2)\dots(\beta_{k-1}, x_{k-1})x_k,$$

$$v = yy_k = (\gamma_1, x_{k-1})(\gamma_2, y_2)\dots(\gamma_{k-1}, y_{k-1})y_k,$$

$$\tilde{v}' = \tilde{y}y_1' = (\gamma_2, y_2)(\gamma_3, y_3)\dots(\varepsilon_0, y_k)y_1',$$

$$\hat{v} = \hat{y}y_k = (\beta_{k-1}, x_{k-1})(\gamma_2, y_2)\dots(\gamma_{k-1}, y_{k-1})y_k,$$

where ε_k and ε_0 are chosen in such a way that $z \notin \{\tilde{x}, \tilde{y}\}$. Thus, $G^{\mathrm{II}}(m,n,k-1)$ must contain two paths of length at most $k-2$, one from $\hat{x}$ to y and the other from $\hat{y}$ to x:

$$P_{\hat{x}y}\colon \hat{x}a_3\dots a_{ry}, \quad P_{\hat{y}x}\colon \hat{y}b_3\dots b_{sx},$$

which induce two paths in G of length at most D joining u and v. If $P_{\hat{x}y}$ and $P_{\hat{y}x}$ have in common the vertex z (see Fig. 8),
then reasoning as in (a), two paths of G_2 are obtained:

$$\rho_1\colon x\tilde{x}\hat{x}a_3\dots a_{i-1}z_1b_{j-1}\dots b_3\hat{y}\tilde{y}y, \quad \rho_2\colon xb_s\dots b_{j+1}z_2a_{i+1}\dots a_{ry}.$$

If the length of ρ_2 is $\leqslant k-1$, then it induces in G a path avoiding $w = zz_k$ of length at most D. Finally, if the length of ρ_2 is $\geqslant k$, then the length of

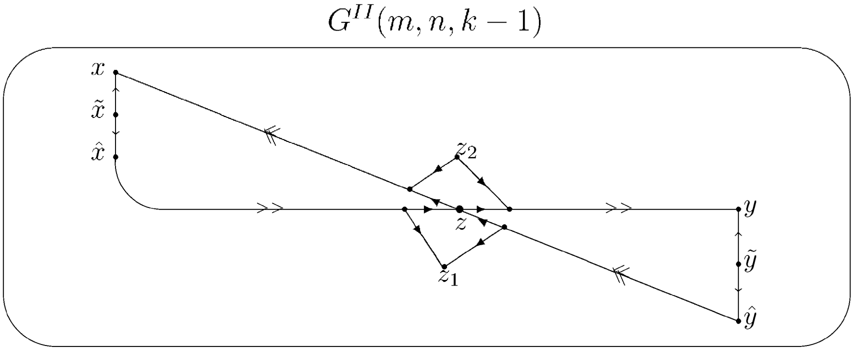

Fig. 8. Two paths in $G^{\mathrm{II}}(m,n,k-1)$ intersecting in z.

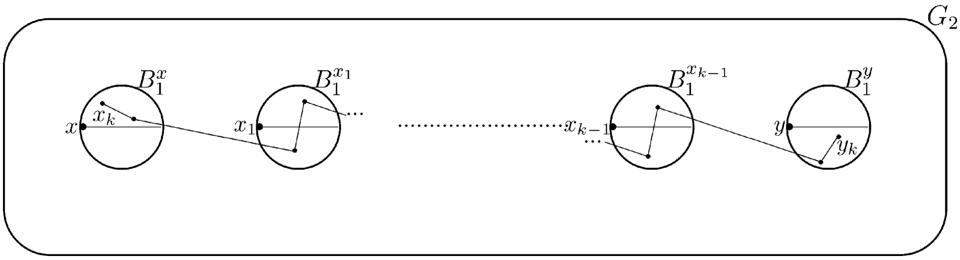

Fig. 9. Path η_ρ between $u=x(0,x_k)$ and $v=y(1,y_k)$, induced by ρ: $xx_1\ldots x_{k-2}y$ in G_2.

the subpath ρ_1 between $\hat{x}$ and $\hat{y}$ must be less than or equal to $k-4$, and therefore ρ_1 induces in G a path avoiding w, of length less than or equal to:

$$2+(D_1+1)(k-4)+D_1+2=(D_1+1)k-3D_1<(D_1+1)k-1=D.$$

3. G is a GC graph belonging to Λ: The two families of GC graphs belonging to Λ have an essential difference from the rest of families studied before: none of them satisfy the condition (ii) (see Fig. 2). Apart from that, the proof for $G=GC\{m,n,k\colon B_1\}$ is similar to that of case (a), whereas for $G=GC(m,n,k\colon B_1)$ it is similar to that exposed in (b). But now D_1 is even, and so the following happens: let $u=x(0,x_k)$, $v=y(1,y_k)$ and $w=z(i,z_k)$ be vertices of G such that $d(x,y)=k-1$ in the graph $G_2=UG^{\mathrm{II}}(m,n,k-1)$. Let ρ: $xx_1\ldots x_{k-2}y$ be a shortest path avoiding z, and consider its induced path η_ρ in G (see Fig. 9). Notice that the length of η_ρ is at most:

$$D_1+1+D_1(k-2)+D_1=kD_1+1=D+1,$$

since vertices belonging to different partite sets are at distance at most D_1-1 because D_1 is even. $\square$

Proof of Lemma 4.3. To prove this statement, we distinguish the following two cases, in which Λ again denotes the set of bipartite GC graphs $G_2[B_1]$, the diameter of B_1 being even:

1. Assume that G is a GC graph not belonging to Λ. Let $\tilde{v}$ be a vertex adjacent to $v = yy_k$ and not belonging to the copy G_1^y . From Lemma 4.2, it follows that there exists a path between u and $\tilde{v}$ of length less than or equal to D, avoiding w. Hence, by adding the edge $\tilde{v}v$, we obtain a path linking u and v of length at most $D+1$, which does not contain w.
2. Suppose that G is a GC graph belonging to one of the two families of Λ. Since the proofs are similar for both families, we show only one of them. Consider the GC graph $G = GC(m,n,k;B_1)$, the diameter D_1 of G_1 being even. Let $u = x(1,x_k)$, $v = y(i,y_k)$ and $w = y(j,z_k)$. If $i = 0$, then the proof is the same as in case (a). If $i = 1$, then consider the following path p_{uv}^{ε} joining u and v:

$$
\begin{array}{ll}
u = & (\beta_1,x_1)(\beta_2,x_2) \quad \dots \quad (\beta_{k-1},x_{k-1})(1,x_k) \\
 & \dots\dots\dots\dots \quad \dots \quad \dots\dots\dots\dots\dots\dots \\
 & (\beta_1,x_1)(\beta_2,x_2) \quad \dots \quad (\beta_{k-1},x_{k-1})(0,y_k') \\
 & (\varepsilon,y_k)(\beta_1,x_1) \quad \dots \quad (\beta_{k-2},x_{k-2})(1,x_{k-1}) \\
 & \dots\dots\dots\dots \quad \dots \quad \dots\dots\dots\dots\dots\dots \\
 & (\varepsilon,y_k)(\beta_1,x_1) \quad \dots \quad (\beta_{k-2},x_{k-2})(0,y_{k-1}') \\
 & (\gamma_{k-1},y_{k-1})(\varepsilon,y_k) \quad \dots \quad (\beta_{k-3},x_{k-3})(1,x_{k-2}) \\
 & \dots\dots\dots\dots \quad \dots \quad \dots\dots\dots\dots\dots\dots \\
 & \dots\dots\dots\dots \quad \dots \quad \dots\dots\dots\dots\dots\dots \\
 & (\gamma_2,y_2)(\gamma_3,y_3) \quad \dots \quad (\varepsilon,y_k)(1,x_1) \\
 & \dots\dots\dots\dots \quad \dots \quad \dots\dots\dots\dots\dots\dots \\
 & (\gamma_2,y_2)(\gamma_3,y_3) \quad \dots \quad (\varepsilon,y_k)(0,y_1') \\
v = & (\gamma_1,y_1)(\gamma_2,y_2) \quad \dots \quad (\gamma_{k-1},y_{k-1})(1,y_k)
\end{array}
$$

   Observe that the length of p_{uv}^{ε} is at most $kD_1 = D$, because D_1 is even. If $m \geqslant k$, then taking $\varepsilon \notin \{\gamma_1,\dots,\gamma_{k-1}\}$, we can conclude that p_{uv}^{ε} is a path joining u and v and avoiding w. If $m < k$, then we may consider in the digraph $G^{\mathrm{II}}(m,n,k-1)$ the path ρ: $xa_1\dots a_{k-1}y$, which induces in G the path p_{uv}^{ε}. If $y \notin \{a_1,\dots,a_{k-1}\}$, then p_{uv}^{ε} joins u and v avoiding w. Finally, if $y \in \{a_1,\dots,a_{k-1}\}$, then taking $\varepsilon \neq \gamma_{k-2}$, we can assure that $y \in \{a_1,\dots,a_{k-3}\}$. Assume for instance that $y = a_{k-3}$ (the proof in any other case is identical). Thus, it follows that $y \in \Gamma^+(a_{k-4}) \cap \Gamma^+(a_{k-1})$ (see Fig. 10), and from Heuchenne conditions we obtain that there must exist a vertex $b \in \Gamma^+(a_{k-4}) \cap \Gamma^+(a_{k-1})$. Let us consider the path η_μ induced by the path

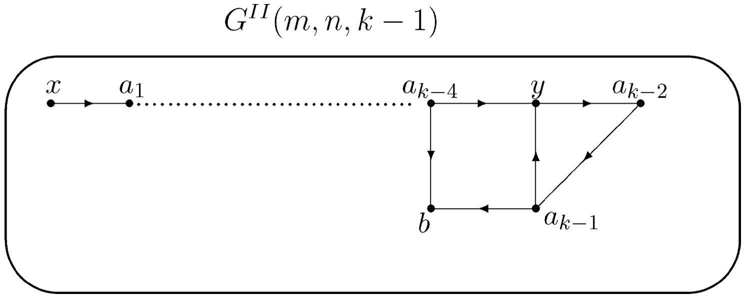

Fig. 10. Paths ρ and μ in $G^{\mathrm{II}}(m,n,k-1)$.

μ: $xa_1 \ldots a_{k-4}ba_{k-1}y$ of $G_2 = UG^{\mathrm{II}}(m,n,k-1)$. Clearly η_μ is a path joining u and v avoiding w, whose length is at most $k(D_1-1)+D_1+1 = D+1$. $\square$

Summarizing, it has been proved that all of the GC graphs belonging to the different families put forward in Section 3 are, each of them, a solution to the $(\Delta, D, D+1, 1)$ problem, even though some of the copies used in its construction be a $(\Delta_1, D_1, +\infty, 1)$ graph.

To finalize, notice that in every case, the compound graph $G_2[G_1]$ has been designed by taking an arbitrary connected graph G_1, a very particular graph G_2 and a certain adjacency rule. Certainly, an interesting work to be done would be to find some additional restrictions on the graph G_1, in such a way that the resulting compound graph be a $(\Delta, D, D, 1)$-graph. Furthermore, it seems plausible that a similar work to ours might be approached by interchanging the constraints, for example, by considering an arbitrary graph satisfying Proposition 2.6, a $(\Delta, D, D, 1)$-graph G_1 and some suitable adjacency rule.

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References

[1] M. Aigner, On the linegraph of a directed graph, Math. Z. 102 (1967) 56–61.
[2] J.C. Bermond, J. Bond, M. Paoli, C. Peyrat, Graphs and interconnection networks: diameter and vulnerability, London Math. Soc. Lecture Notes Ser. 82 (1983) 1–29.
[3] J.C. Bermond, C. Delorme, J.J. Quisquater, Grands graphes non dirigés de degré et diamètre fixés, Ann. Discrete Math. 17 (1982) 65–73.
[4] J-C Bermond, N. Homobono, C. Peyrat, Large fault-tolerant interconnection networks, Graphs Combin. 5 (1989) 107–123.
[5] J. Bond, Grands réseaux d'interconnexions, Thesis, Université de Paris-Sud, 1987.

[6] J. Bond, C. Peyrat, Diameter vulnerability in networks, in: Graph Theory with Application to Algorithms and Computer Science (Kalamazoo, Michigan, 1984), Wiley–Interscience, New York, 1985, pp. 123–149.
[7] J. Bond, C. Peyrat, Diameter vulnerability of some large interconnection networks, Congr. Numer. 66 (1988) 267–282.
[8] C. Delorme, Grands graphes de degré et diamètre donnés, European J. Combin. 6 (1985) 291–302.
[9] C. Delorme, Large bipartite graphs with given degree and diameter, J. Graph Theory 8 (1985) 325–334.
[10] C. Delorme, G. Farhi, Large graphs with given degree and diameter-part I, IEEE Trans. Comput. 33 (9) (1984) 857–860.
[11] C. Delorme, J.J. Quisquater, Some new constructions of large graphs, LRI Research Report. Univ. Paris-Sud, Orsay, 1986, p. 317.
[12] M.A. Fiol, J. Fàbrega, Algunos grafos compuestos, Stochastica VII (2) (1983) 137–143.
[13] M.A. Fiol, J.L.A. Yebra, I. Alegre, Line digraph iterations and the (d,k) problem, IEEE Trans. Comput. 33 (5) (1984) 400–403.
[14] J. Gómez, Diametro y vulnerabilidad en redes de interconexión, Thesis, Universitat Politècnica de Catalunya, 1986.
[15] J. Gómez, Generalized compound graphs, Ars Combin. 29-B (1990) 33–53.
[16] J. Gómez, M.A. Fiol, Dense compound graphs, Ars Combin. 20-A (1985) 211–235.
[17] J. Gómez, M.A. Fiol, J.L.A. Yebra, Graphs on alphabets as models for large interconnection networks, Discrete Appl. Math. 37/38 (1992) 227–243.
[18] C. Heuchenne, Sur une certaine correspondance entre graphes, Bull. Soc. Roy. Sci. Liège 33 (1964) 174–177.
[19] W.H. Kautz, Design of optimal interconnection networks for multiprocessors, Architecture and Design of Digital Computer, NATO Adv. Summer Inst. (1969) 249–272.
[20] J.G. Kuhl, S.M. Reddy, Fault-tolerance considerations in large multiprocessor systems, IEEE Comput. (1986) 56–67.
[21] C. Peyrat, Diameter vulnerability of graphs, Discrete Appl. Math. 9 (1984) 245–250.
[22] S.M. Reddy, J.G. Kuhl, S.H. Hosseini, H. Lee, On digraphs with minimum diameter and maximum connectivity, Proceedings of the 20th Annual Allerton Conference, 1982, pp. 1018–1026.