A note on the extension of the null-additive set function on the algebra of subsets

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Abstract

In this work, we point out that the proof of Theorem 2 in [E. Pap, Extension of null-additive set functions on algebra of subsets, Novi Sad J. Math. 31 (2) (2001) 9–13] is incorrect and give a correct proof. Moreover, we also get a corresponding theorem on extension of the weakly null-additive set function.

Pap gave a theorem on extension of the null-additive set function on the algebra of subsets and its proof in [1], but the extension of the function which is given in the proof of the theorem is incorrect. In this work, we point out that the proof of Theorem 2 in [1] is incorrect by means of a counterexample and give a correct proof. In addition, we also investigate a corresponding theorem on the extension of the weakly null-additive set function.

Throughout this work, X always denotes a nonempty set. We will adopt the following terminology used in [2,3].

A collection 𝒓 of subsets of X is said to be a ring on X if ∅ ∈ 𝒓 and A ∪ B ∈ 𝒓, A \ B ∈ 𝒓 whenever A, B ∈ 𝒓, and an algebra on X if 𝒓 is a ring and X ∈ 𝒓. Let 𝒟, 𝒜 be two collections of subsets of X and 𝒟 ⊂ 𝒜; 𝒜 is called the smallest algebra on X containing 𝒟 if, for any algebra 𝒜∗ containing 𝒟, we have 𝒜 ⊂ 𝒜∗.

A non-negative extended real-valued set function μ : 𝒓 → [0, +∞] on ring 𝒓 with μ(∅) = 0 is said to be monotone if μ(A) ≤ μ(B) for any A, B ∈ 𝒓 whenever A ⊂ B; null-additive if μ(A ∪ B) = μ(A) for any A, B ∈ 𝒓 whenever A ∩ B = ∅ and μ(B) = 0; and weakly null-additive if μ(A ∪ B) = 0 for any A, B ∈ 𝒓 whenever μ(A) = μ(B) = 0.

Theorem 1 ([4]). Let 𝒓 be a ring on X. Let

\[ 𝒓_1 = \{ A : A ⊂ X, A' \in 𝒓 \}. \]

Then 𝒜 = 𝒓 ∪ 𝒓_1 is the smallest algebra on X containing 𝒓.
Remark 1. We also say that $\mathcal{A} = \mathcal{R} \cup \mathcal{R}_1$ is the algebra on $X$ generated by $\mathcal{R}$. Obviously, if $X \in \mathcal{R}$, then $\mathcal{R}_1 = \mathcal{R}$; otherwise $\mathcal{R} \cap \mathcal{R}_1 = \emptyset$.

Theorem 2 ([1]). Let $\mathcal{R}$ be a ring of subsets of a set $X$ such that $X \notin \mathcal{R}$. Let $\mu$ be a null-additive monotone set function on $\mathcal{R}$. Let $\mathcal{A}$ be the algebra on $X$ generated by $\mathcal{R}$. Then there exists a null-additive set function on $A$ possibly taking the value infinity which is an extension of $\mu$ from $\mathcal{R}$ to $\mathcal{A}$.

The method of the proof in [1]: Define $d = \sup_{R \in \mathcal{R}} \mu(R)$, and define the following set function $\bar{\mu}$:

1. \[\bar{\mu}(A) = \begin{cases} \mu(A) & A \in \mathcal{R} \\ \infty & A \in \mathcal{R}_1; \end{cases}\]
2. \[\bar{\mu}(A) = \begin{cases} \mu(A) & A \in \mathcal{R} \\ d - \mu(A') & A \in \mathcal{R}_1. \end{cases}\]

Then, $\bar{\mu}$ satisfies all requirement of Theorem 2.

But, when $d < \infty$, the definition of $\bar{\mu}$ does not agree with the requirement of Theorem 2 (see Example 1).

Example 1. Let $X = \{a, b, c\}$, $\mathcal{R} = \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$. Define $\mu : \mathcal{R} \to [0, +\infty]$, $\mu(A) = \begin{cases} 1 & A = \{a, b, c\}, \{b, c\}, \{a, c\}, \{c\} \\ 1/4 & A = \{b\}, \{a, b\} \\ 0 & A = \emptyset, \{a\}. \end{cases}$

Obviously, $\mathcal{R}$ is a ring. By Theorem 1, we know $\mathcal{R}_1 = \{\{a, b, c\}, \{b, c\}, \{a, c\}, \{a, b\}, \{c\}, \{b, c\}, \{a, b, c\}\}$, $\mathcal{A} = \mathcal{R} \cup \mathcal{R}_1$ and $d = \sup_{R \in \mathcal{R}} \mu(R) = 1 < \infty$. According to the aforementioned method of proof of Theorem 2 in [1], the extension $\bar{\mu}$ which satisfies the requirement of Theorem 2 of $\mu$ ought to be

$\bar{\mu}(A) = \begin{cases} 1 & A = \{a, b, c\}, \{b, c\}, \{a, c\}, \{c\}, \{a, b, c\} \\ 3/4 & A = \{a, c\}, \{c\} \\ 1/4 & A = \{b\}, \{a, b\} \\ 0 & A = \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}. \end{cases}$

Take $A = \{c\}$, $B = \{e\}$; then $\bar{\mu}(A) = 1$, $\bar{\mu}(B) = 0$ and $A \cap B = \emptyset$, but $\bar{\mu}(A \cup B) = \bar{\mu}(\{c, e\}) = 3/4 \neq 1 = \bar{\mu}(A)$. Thus, $\bar{\mu}$ is not null-additive; namely, $\bar{\mu}$ does not satisfy the requirement of Theorem 2. This shows that the original proof is incorrect.

Next we give a correct proof of Theorem 2 and this proof is more concise than the original proof.

New proof. Take $\mathcal{R}_1 = \{A : A \subset X, A' \in \mathcal{R}\}$. By Theorem 1 we have $\mathcal{A} = \mathcal{R} \cup \mathcal{R}_1$. Define $d = \sup_{R \in \mathcal{R}} \mu(R)$. Define $\bar{\mu} : \mathcal{A} \to [0, +\infty]$ by

$\bar{\mu}(A) = \begin{cases} \mu(A) & A \in \mathcal{R} \\ d & A \in \mathcal{R}_1. \end{cases}$

By definition, $\bar{\mu}$ is an extension of $\mu$ from $\mathcal{R}$ to $\mathcal{A}$.

(1) If $d = 0$, then $\bar{\mu}(A) = 0$ for any $A \in \mathcal{A}$. Naturally, $\bar{\mu}$ is null-additive.

(2) If $d > 0$, then, since $[0, d]$ is order isomorphic to $[0, \infty]$ (in fact, we can define a function $f : [0, d] \to [0, \infty]$, $f(x) = \frac{1}{d-x}$, and it is obvious that $f(x)$ is an increasing isomorphic mapping; thus, $[0, d]$ is order isomorphic to $[0, \infty]$), the proof for $\sup_{R \in \mathcal{R}} \mu(R) = \infty$ in [1] applies to the proof for this case.

Thus, $\bar{\mu}$ is null-additive. This completes the proof of the theorem. \(\square\)
Note that for the above set function \( \bar{\mu} \) defined by formula (\( \ast \)), arbitrarily taking \( A, B \in \mathcal{A} \) and \( A \subset B \), by Remark 1, we know that there exist only two cases: (i) \( A, B \in \mathcal{R} \), or (ii) \( A, B \in \mathcal{R}_1 \). Obviously, in these two cases, we get \( \bar{\mu}(A) \leq \bar{\mu}(B) \), i.e., \( \bar{\mu} \) is monotone, and hence Theorem 2 can be rewritten in the following form.

**Theorem 2'.** Let \( \mathcal{R} \) be a ring of subsets of a set \( X \) such that \( X \notin \mathcal{R} \). Let \( \mu \) be a null-additive monotone set function on \( \mathcal{R} \). Let \( \mathcal{A} \) be the algebra on \( X \) generated by \( \mathcal{R} \). Then there exists a null-additive monotone set function on \( \mathcal{A} \) possibly taking the value infinity which is an extension of \( \mu \) from \( \mathcal{R} \) to \( \mathcal{A} \).

Now, we prove that the similar conclusion of Theorem 2 also holds for the weakly null-additive set function.

**Theorem 3.** Let \( \mathcal{R} \) be a ring of subsets of a set \( X \) such that \( X \notin \mathcal{R} \) and \( \mu \) be a weakly null-additive monotone set function on \( \mathcal{R} \). Let \( \mathcal{A} \) be the algebra on \( X \) generated by \( \mathcal{R} \). Define \( \bar{\mu} : \mathcal{A} \rightarrow [0, +\infty] \) by

\[
\bar{\mu}(A) = \begin{cases} 
\mu(A) & A \in \mathcal{R} \\
\mu(A') & A \in \mathcal{R}_1.
\end{cases}
\]

(\( \ast \ast \))

Then \( \bar{\mu} \) is a weakly null-additive set function on \( \mathcal{A} \) which is an extension of \( \mu \) from \( \mathcal{R} \) to \( \mathcal{A} \) (here, \( \mathcal{A} = \mathcal{R} \cup \mathcal{R}_1 \) and \( \mathcal{R}_1 = \{ A : A \subset X, A' \in \mathcal{R} \} \)).

**Proof.** By definition, \( \bar{\mu} \) is an extension of \( \mu \) from \( \mathcal{R} \) to \( \mathcal{A} \). Arbitrarily take \( A, B \in \mathcal{A} \).

(a) \( A, B \in \mathcal{R} \). If \( \bar{\mu}(A) = \bar{\mu}(B) = 0 \), then also \( \mu(A) = \mu(B) = 0 \); by \( A \cup B \in \mathcal{R} \) and the weakly null-additivity of \( \mu \) we have

\[ \bar{\mu}(A \cup B) = \mu(A \cup B) = 0. \]

(b) \( A \in \mathcal{R}, B \in \mathcal{R}_1 \), then \( B' \in \mathcal{R} \), and since \( \mathcal{R} \) is a ring, we have

\[ (A \cup B)' = A' \cap B' = B' \setminus A \in \mathcal{R}. \]

and then \( A \cup B \in \mathcal{R}_1 \). Again for \( \bar{\mu}(A) = \bar{\mu}(B) = 0 \), we have \( \mu(A) = \mu(B') = 0 \). So, by the monotonicity of \( \mu \),

\[ \bar{\mu}(A \cup B) = \mu((A \cup B)') = \mu(A' \cap B') \leq \mu(B') = 0. \]

Naturally, \( \bar{\mu}(A \cup B) = 0 \).

(c) \( A \in \mathcal{R}_1, B \in \mathcal{R} \); then like in the discussion of (b) we can obtain \( \bar{\mu}(A \cup B) = 0 \) when \( \bar{\mu}(A) = \bar{\mu}(B) = 0 \).

(d) \( A, B \in \mathcal{R}_1; \) then \( A' \in \mathcal{R}, B' \in \mathcal{R} \). Again since \( \mathcal{R} \) is a ring, we have

\[ (A \cup B)' = A' \cap B' = A' \setminus (A' \cap B') \in \mathcal{R}. \]

and then \( A \cup B \in \mathcal{R}_1 \). For \( \bar{\mu}(A) = \bar{\mu}(B) = 0 \), we have \( \mu(A') = \mu(B') = 0 \). So, by the monotonicity of \( \mu \),

\[ \bar{\mu}(A \cup B) = \mu((A \cup B)') = \mu(A' \cap B') \leq \mu(A') = 0. \]

Naturally, \( \bar{\mu}(A \cup B) = 0 \).

Thus, \( \bar{\mu} \) is weakly null-additive. This completes the proof of the theorem. \( \square \)

Finally, like for the proof of Theorem 2', it is easy to see that the following theorem is true by formula (\( \ast \)) for the case of the weakly null-additive monotone set function \( \mu \).

**Theorem 3'.** Let \( \mathcal{R} \) be a ring of subsets of a set \( X \) such that \( X \notin \mathcal{R} \) and \( \mu \) be a weakly null-additive monotone set function on \( \mathcal{R} \). Let \( \mathcal{A} \) be the algebra on \( X \) generated by \( \mathcal{R} \). Then there exists a weakly null-additive monotone set function on \( \mathcal{A} \) possibly taking the value infinity which is an extension of \( \mu \) from \( \mathcal{R} \) to \( \mathcal{A} \).

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**References**