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# Properties of some statistics for AR-ARCH model with application to technical analysis

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ABSTRACT

In this paper, we investigate some popular technical analysis indexes for AR-ARCH model as real stock market. Under the given conditions, we show that the corresponding statistics are asymptotically stationary and the law of large numbers hold for frequencies of the stock prices falling out normal scope of these technical analysis indexes under AR-ARCH, and give the rate of convergence in the case of nonstationary initial values, which give a mathematical rationale for these methods of technical analysis in supervising the security trends.



## 1. Introduction

As long as financial markets have existed, people have tried to forecast them, in the hope that good forecasts would bring them great fortunes. Since Charles H. Dow first introduced the Dow theory in the late 1800s, technical analysis has been extensively used in stock market. The term "technical analysis" in general contains a large variety of trading techniques, which are based on past movements of asset price and a few other related variables. The use of trading rules to detect patterns in the time series of asset prices. There are numerous methods within technical analysis, which are principally independent from each other.

It is well-known that the efficiency of these indexes is 'proved' by the observed relative frequency of the occurrence of the corresponding behaviors of stock prices. In other words, the traders use the daily (hourly, weekly, …) stock prices as samples of certain statistics and use the observed relative frequency to show the validity of these well-known indexes. However, these samples are just the discrete observations of a realized path of a stochastic process, which are not independent, so the classical sample survey theory (especially the law of large numbers) does not apply to. But Liu et al. [7], and Zhu [11] found that some important technical analysis indexes are stationary process or the transformation of theirs. Liu et al. [7] discussed the Bollinger bands for Black–Scholes model as real stock market. Under Black–Scholes model, they introduced the statistics $U_t^{(n)}$ calculated according to the formula of the Bollinger Bands, which is stationary and $\{U_{t+kn}^{(n)}\}_{k=1,2,\ldots}$ are mutually independent for each fixed $t \geq 0$. Zhu [11] extended the above results to another index Relative Strength Index (RSI in short) for Black–Scholes model. Since we know that the frequency of the occurrence of stationary process can be computed, so these results have laid the theoretical foundation for statistical application of the technical analysis of stock price.

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Recently, much attention has been paid to the AR-ARCH models. Properties of various AR-ARCH type models are studied, especially the stationarity and geometric ergodicity are considered by Ango Nze [1], Lu [8], Lu and Jiang [9], Cline and Pu [4] and others. We consider log return process $\xi_t = \log S_t - \log S_{t-1}$. In the following we assume that the log return $\xi_t$ is generated by a nonlinear autoregressive(AR) model with an autoregressive conditional heteroscedastic (ARCH) term (denoted by AR-ARCH) as follows:

$$\xi_t = a(\xi_{t-1}, \ldots, \xi_{t-p}) + b(\xi_{t-1}, \ldots, \xi_{t-p})e_t, \tag{1.1}$$

where $a$ and $b$ are finitely piecewise continuous functions and $\{e_t\}$ is an independent identically distributed (i.i.d. in short) error sequence. The state vector for the time series is $X_t = (\xi_t, \ldots, \xi_{t-p+1})$. The nonlinear autoregression function $a(x)$ is continuous on individual, connected subregions of $R^p$; the boundaries of these regions are called thresholds, hence the nomenclature for the model. Frequently, $a(x)$ is assumed to be linear on each of these regions. Likewise, this model has a state dependent conditional variance, $b^2(x) = \operatorname{var}(\xi_t|X_{t-1} = x)$ if $\operatorname{var}(e_t) = 1$, which typically is of the order of magnitude of $\|x\|^2$. This provides the conditional heteroscedasticity (ARCH) behavior and for our purposes it is also assumed to have thresholds. Since the time series (1.1) is embedded into $\{X_t\}$, which is a Markov chain, it will have a stationary distribution when the Markov chain is ergodic.

In this paper, we aim at discussing the corresponding statistics' properties of some popular technical analysis indexes Bollinger bands, RSI and Rate of Change Index (ROC in short) for discrete-time AR-ARCH model as real stock market. Under the given conditions, we show that the corresponding statistics are asymptotical stationary and the law of large numbers hold for frequencies of the stock prices falling out normal scope of these technical analysis indexes under AR-ARCH, and give the rate of convergence in the case of nonstationary initial values.

The paper is organized as follows. Section 2 introduces the definition of some technical analysis indexes, Section 3 contains some Markov chain terminology and relevant results, in Section 4 the asymptotically stationary properties of corresponding statistics are investigated, in Section 5 presents the law of large numbers for frequencies of these statistics, in Section 6 the proofs of main results are given, and we conclude the paper in the last section.

## 2. Definitions of some technical analysis indexes

Let $S_t$ be observed stock price. Let us introduce the definitions of these technical analysis indexes as follows:

(1) Bollinger Bands Definition:

Denote by

$$\overline{S}_t^{(n)} = \frac{1}{n}\sum_{i=0}^{n-1} S_{t-i}, \qquad \hat{S}_t^{(n)} = \frac{1}{\sum_{i=1}^{n} i}\sum_{i=0}^{n-1}(n-i)S_{t-i}$$

and

$$\sigma_t^{(n)} = \sqrt{\frac{1}{n-1}\sum_{i=0}^{n-1}(S_{t-i} - \overline{S}_t^{(n)})^2}.$$

Bollinger bands consist of three curves draw in relation to securities prices, the curve $\gamma_t = \hat{S}_t^{(n)}$ is called the middle Bollinger band, the curve $\gamma_t^- = \hat{S}_t^{(n)} - 2\sigma_t^{(n)}$ is called the lower Bollinger band and $\gamma_t^+ = \hat{S}_t^{(n)} + 2\sigma_t^{(n)}$ is called the upper Bollinger band, where $n$ is the number of selected periods. Usually we take $n = 12$ or 20. Bollinger [2] introduced this pair of bands to provide a relative definition of high and low for a stock price in the early 1980s. By definition the stock price is "high" at the upper band and "low" at the lower band. The closer the prices move to the upper band, the more overbought the market, and the closer the prices move to the lower band, the more oversold the market.

(2) RSI (Relative Strength Index) Definition:

If we denote

$$\Delta S_t = S_{t+1} - S_t, \qquad \Delta S_t^+ = (S_{t+1} - S_t) \vee 0,$$

then $n$-day RSI is defined as

$$\Upsilon_t^{(n)} = 100 \times \frac{\sum_{i=1}^{n} \Delta S_{t-i}^+}{\sum_{i=1}^{n} |\Delta S_{t-i}|} \quad (\forall t > n),$$

where $n$ is the number of selected periods.

RSI was proposed by Welles Wilder Jr. in 1978. The market's condition is reflected by calculating the correlative value of the strength in buys and sells in a period of time. In a normal market, the price can be stabilized only when the both sides of the business strength obtain the balance. RSI takes its values in [0, 100]. In general, RSI value maintains above 50 for a strong trend market, and is lower than 50 for a weak trend market. RSI may be used in judging the ultra-buy and ultra-sell in market. Take 9-day RSI as the example, RSI above 80 may be regarded as ultra-buy area, and below 20 may regarded as ultra-sell area. It is a early warning signal of the price possibly reverse when the market enters the ultra-buy area or the ultra-sell area. Investors always pay closely attention on the market when this signal appears.

(3) ROC (Rate of Change Index) Definition:

If we denote closing price by $S_t$ and closing price of $n$-day before by $S_{t-n}$, then $n$-day ROC is defined as

$$W_t^{(n)} = 100 \times \frac{S_t - S_{t-n}}{S_{t-n}} \quad (\forall t > n),$$

where $n$ is the number of selected periods. Usually we take $n = 12$. The ROC must establish the antenna and the grounding also. But unlike other ultra-buy or ultra-sell indexes, its antennas and grounds are indefinite. When ROC undulates in normal scope, it is time to sell out while ROC rises to the first ultra-buy line (5) and it is time to buy in while ROC drops to the first ultra-sell line (−5). After ROC breaking through the first ultra-buy line upward, the rising trend mostly ends when it reaches the second ultra-buy line (10). And the dropping trend mostly ends when ROC reaches the second ultra-sell line (−10) after it breaks through the first ultra-sell line downward.

## 3. Some Markov chain terminology and relevant results

This section are drawn from the papers in [3,10,4]. Suppose $X := (X_k, k \in Z)$ is a Markov chain. For any $n \geq 1$, define the following dependent coefficients:

$$\alpha_X(n) := \sup_{k \in Z} \sup_{A \in \sigma(X_k), B \in \sigma(X_{n+k})} |P(A \cap B) - P(A)P(B)|; \tag{3.1}$$

$$\beta_X(n) := \sup_{k \in Z} \sup_{A_i \in \sigma(X_k), B_j \in \sigma(X_{n+k})} \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} |P(A_i \cap B_j) - P(A_i)P(B_j)|, \tag{3.2}$$

where the supremum is taken over all pairs of (finite) partitions $\{A_1, \ldots, A_I\}$ and $\{B_1, \ldots, B_J\}$ of $\Omega$ such that $A_i \in \sigma(X_k)$ for each $i$ and $B_j \in \sigma(X_{n+k})$ for each $j$. The Markov chain $X$ is said to be

'strongly mixing' (or '$\alpha$-mixing') if $\alpha_X(n) \to 0$ as $n \to \infty$;

'absolutely regular' (or '$\beta$-mixing') if $\beta_X(n) \to 0$ as $n \to \infty$.

The two mixing coefficients are ordered as follows:

$$\alpha_X(n) \leq \beta_X(n). \tag{3.3}$$

So that $\beta$-mixing implies $\alpha$-mixing.

We denote $P^n(x, A) = P(X_n \in A | X_0 = x)$. A Markov chain $\{X_t\}$ is V-uniformly ergodic, i.e. for some $\rho < 1$ and some fixed constant $M < \infty$,

$$\|P^n(x, \cdot) - \pi(\cdot)\|_V \leq MV(x)\rho^n, \quad x \in R, n \in N \tag{3.4}$$

where $\|\mu(\cdot)\|_V = \sup_{|f| \leq V} |\mu(f)|$ and $\int V(x)\pi(\mathrm{d}x) < \infty$. For any signed measure $\mu$ on $\mathcal{B}$, we define the total variation norm as $\|\mu\| = \sup_{|g| \leq 1} |\mu(g)|$. A Markov chain $\{X_t\}$ is said to be geometrically ergodic if there exist a probability measure $\pi$ on $(R, \mathcal{B})$, a constant $0 < \rho < 1$, and a $\pi$-integrable nonnegative measurable function $\gamma(x)$ such that

$$\|P^n(x, \cdot) - \pi(\cdot)\| \leq \rho^n \gamma(x), \quad \forall n \in Z^+, \ \forall x \in R. \tag{3.5}$$

If we restrict to functions $V$ on the left-hand side of (3.4) that satisfy $|V(x)| \leq 1$, we obtain the total variation norm $\|P^n(x, ) - \pi(\cdot)\|$. So the inequality (3.4) is a strong version of the condition of geometric ergodicity.

Cline and Pu [4] provided the following result:

**Proposition 3.1** (*Cline and Pu [4]*). *Suppose model (1.1) satisfy the following conditions:* (i) *The distribution of $e_t$ has Lebesgue density $f$ which is locally bounded away from 0. Also, $b$ is positive, locally bounded and locally bounded away from 0.* (ii) $\sup_u (1 + |u|) f(u) < \infty$ *and* $E(|e_1|^{r_0}) < \infty$ *for some* $r_0 > 0$. (iii) $a(x)/(1+\|x\|)$ *and* $b(x)/(1+\|x\|)$ *are bounded. If there exist nonnegative $c_i$ with $\sum_{i=1}^{p} c_i < 1$, $r > 0$ and $K < \infty$ such that $E(|\xi_1|^r | X_0 = x) \leq K + \sum_{i=1}^{p} c_i |x_i|^r$, for all $x = (x_1, \ldots, x_p) \in R^p$, then $\{X_t\}$ is V-uniformly ergodic with $V(x) = 1 + \sum_{i=1}^{p} d_i |x_i|^r$ for some positive $d_1, \ldots, d_p$. Furthermore, the stationary distribution has finite rth moment.*

## 4. Asymptotically stationary property

In the following we incorporate the nonstationary case into our considerations. Let $S_t$ be observed stock price by the model (1.1), then

$$S_t = S_0 \exp\left\{\sum_{k=1}^{t}(a(\xi_{k-1},\ldots,\xi_{k-p}) + b(\xi_{k-1},\ldots,\xi_{k-p})e_k)\right\}. \tag{4.1}$$

We denote

$$g(t,i,j) = \exp\left\{\sum_{k=t-i}^{t-j}(a(\xi_{k-1},\ldots,\xi_{k-p}) + b(\xi_{k-1},\ldots,\xi_{k-p})e_k)\right\}. \tag{4.2}$$

$$U_t^{(n)} = f(g(t,n-1,0),\ldots,g(t,n-1,n-1)), \tag{4.3}$$

where $f$ be a measurable function: $R^n \to R$.

**Theorem 4.1.** *Under the conditions of Proposition 3.1, the process $\{U_t^{(n)}\}_{t\ge n}$ is asymptotically stationary.*

**Remark 4.1.** Using the above conclusion, we can draw a series of asymptotically stationary processes from AR-ARCH model so long as the desired conditions are satisfied. This fact can be applied to the technical analysis indexes which we have explained before. The conclusions list in the following corollaries.

**Corollary 4.1.** *Let $S_t$ be the stock price generated by the model (1.1), $\Lambda_t^{(n)} = \frac{S_t-\hat{S}_t^{(n)}}{\sigma_t^{(n)}}$ $(\forall t \ge n)$, then the process $\{\Lambda_t^{(n)}\}_{t\ge n}$ is asymptotically stationary under the conditions of Proposition 3.1.*

**Proof.** We can obtain immediately by using (4.2):

$$S_t = S_{t-n}g(t,n-1,0),$$

$$\hat{S}_t^{(n)} = S_{t-n}\frac{1}{\sum_{i=1}^{n} i}\sum_{i=0}^{n-1}(n-i)g(t,n-1,i),$$

$$\overline{S}_t^{(n)} = S_{t-n}\frac{1}{n}\sum_{i=0}^{n-1}g(t,n-1,i),$$

and

$$(n-1)\left[\frac{\sigma_t^{(n)}}{S_{t-n}}\right]^2 = \sum_{i=0}^{n-1}\left(g(t,n-1,i) - \frac{1}{n}\sum_{i=0}^{n-1}g(t,n-1,i)\right)^2.$$

So $\{\Lambda_t^{(n)}\}_{t\ge n}$ is a function of $(g(t,n-1,0),\ldots,g(t,n-1,n-2))$. So by Theorem 4.1 we have $\Lambda_t^{(n)}$ is asymptotically stationary. □

**Corollary 4.2.** *Let $S_t$ be the stock price generated by the model (1.1), then the process*

$$\Upsilon_t^{(n)} = 100 \times \frac{\sum_{i=1}^{n}\Delta S_{t-i}^{+}}{\sum_{i=1}^{n}|\Delta S_{t-i}|} \quad (\forall t > n)$$

*is asymptotically stationary under the conditions of Proposition 3.1.*

**Proof.** We can obtain immediately by using (4.2):

$$S_t = S_{t-n}g(t,n-1,0),$$

$$S_{t-i} = S_{t-n}g(t,n-1,i).$$

Then

$$\Upsilon_t^{(n)} = 100 \times \frac{\sum_{i=1}^{n}(g(t,n-1,i-1) - g(t,n-1,i)) \vee 0}{\sum_{i=1}^{n}|g(t,n-1,i-1) - g(t,n-1,i)|},$$

here we decree $g(t,n-1,n) = 1$. Then it is clearly that $\Upsilon_t^{(n)}$ is a measurable function of $(g(t,n-1,0),\ldots,g(t,n-1,n-1))$. So by Theorem 4.1 we have $\Upsilon_t^{(n)}$ is asymptotically stationary. □

Similarly, we have the following corollary.

**Corollary 4.3.** *Let $S_t$ be the stock price generated by the model (1.1), then the process*

$$W_t^{(n)} = \frac{S_t - S_{t-n}}{S_{t-n}} \quad (\forall t > n)$$

*is asymptotically stationary under the conditions of Proposition 3.1.*

## 5. Law of large numbers

Denote for $i \geq n$, $K_{\Gamma,i}^{(n)} = I_{[U_i^{(n)} \in \Gamma]}$, where $\Gamma$ is a subset of $\mathbf{R}$. Let

$$V_{N,\Gamma}^{(n)} = \frac{1}{N+1} \sum_{i=0}^{N} K_{\Gamma,n+i}^{(n)}$$

which is the observed frequency of the events $[U_{n+i}^{(n)} \in \Gamma]$ $(i = 0, 1, \ldots, N)$, then we can obtain the law of large numbers:

**Theorem 5.1.** *Under the conditions of Proposition 3.1, if $\{X_t\}$ is initialized from nonstationary measure $\nu$ and satisfy $\int V(x)\nu(\mathrm{d}x) < \infty$, then there exist a constant $\widetilde{C}$ such that*

$$E|V_{N,\Gamma}^{(n)} - P_0[U_n^{(n)} \in \Gamma]|^2 \leq \frac{\widetilde{C}}{N+1}.$$

**Remark 5.1.** From the above theorem, it is reasonable to use the stationary distribution of $U_n^{(n)}$ to calculate the observed frequency $V_{N,\Gamma}^{(n)}$. We can apply this conclusion to $\Lambda_t^{(n)}$, $\Upsilon_t^{(n)}$ and $W_t^{(n)}$. All of the three processes change between a normal scope. Out this normal scope, investors may buy in or sell out stocks which can make the values of $\Lambda_t^{(n)}$, $\Upsilon_t^{(n)}$ and $W_t^{(n)}$ come back into the normal scope. For example, the normal scope of $\Lambda_t^{(n)}$ is $[-2, 2]$. $\Upsilon_t^{(n)}$ always changes between 20 and 80. $W_t^{(n)}$ also has indefinite antennas and grounds. For simplicity, we unify the lower value of the normal scope be $\alpha$ and the upper value be $\beta$. As a application, we have the following corollaries.

**Corollary 5.1.** *Denote for $i \geq n$, $H_i^{(n)} = I_{[|\Lambda_i^{(n)}| \geq 2]}$. Let*

$$J_N^{(n)} = \frac{1}{N+1} \sum_{i=0}^{N} H_{n+i}^{(n)}$$

*which is the observed frequency of the events $[|\Lambda_{n+i}^{(n)}| \geq 2]$ $(i = 0, 1, \ldots, N)$, i.e. the frequency of the stock falling out of the Bollinger Bands. Let $P_0(.)$ denote the stationary distribution of the asymptotic stationary process $\{\Lambda_t^{(n)}\}_{t \geq n}$, then there exist a constant $\widetilde{C}$ such that*

$$E|J_N^{(n)} - P_0[|\Lambda_n^{(n)}| \geq 2]|^2 \leq \frac{\widetilde{C}}{N+1}.$$

**Corollary 5.2.** *Denote for $H_i^{(n)} = I_{[\Upsilon_i^{(n)} \in \Gamma]}$, where $\Gamma = [0, 20] \cup [80, 100]$. Let $P_0(.)$ denote the stationary distribution of the asymptotic stationary process $\{\Upsilon_t^{(n)}\}_{t \geq n}$ and*

$$J_N^{(n)} = \frac{1}{N+1} \sum_{i=0}^{N} H_{n+i}^{(n)},$$

*then there exist a constant $\widetilde{C}$ such that*

$$E|J_N^{(n)} - P_0[\Upsilon_n^{(n)} \in \Gamma]|^2 \leq \frac{\widetilde{C}}{N+1}.$$

**Corollary 5.3.** *Denote for $H_i^{(n)} = I_{[W_i^{(n)} \in \Gamma]}$, where $\Gamma = [-\infty, \alpha] \cup [\beta, \infty]$. and $(\alpha, \beta)$ are the indefinite antenna and ground of ROC. Let $P_0(.)$ denote the stationary distribution of the asymptotic stationary process $\{W_t^{(n)}\}_{t \geq n}$ and*

$$J_N^{(n)} = \frac{1}{N+1} \sum_{i=0}^{N} H_{n+i}^{(n)},$$

*then there exist a constant* $\widetilde{C}$ *such that*

$$E|J_N^{(n)} - P_0[W_n^{(n)} \in \Gamma]|^2 \le \frac{\widetilde{C}}{N+1}.$$

## 6. The proofs of main results

**Proof of Theorem 4.1.** From the expression of (4.2), we can see clearly that $g(t,i,j)_{0\le j\le i\le n-1}$ is just a function of $(X_{k-1}, e_k)_{k=t-n+1,\dots,t}$. We know $X_{t-1}$ be independent of $e_t$ and $X_t$ have stationary distribution for all $t \ge 0$ by Proposition 3.1. Without loss of generality, we suppose $X_t$ is initialized from its stationary distribution, then $(X_{k-1}, e_k)_{k=t-n+1,\dots,t}$ is a two-dimensional stationary process. Then we have for any $m \ge 0$,

$$(g(t,n-1,0),\dots,g(t,n-1,n-1))(=)(g(t+m,n-1,0),\dots,g(t+m,n-1,n-1))$$

where $X(=)Y$ denote $X$ and $Y$ have the same distribution. So

$$\{U_t^{(n)}\}_{t\ge n}(=)\{U_{t+m}^{(n)}\}_{t\ge n}$$

and the proof is complete. □

In order to prove Theorem 5.1, we need the following lemmas.

**Lemma 6.1.** *Let $\mathcal{F}_i$ denote the $\sigma$-field generated by $(X_{k-1}, e_k)_{1\le k\le i}$, the following formula holds:*

$$E|V_{N,\Gamma}^{(n)} - \frac{1}{N+1}\sum_{i=0}^{N} P[U_{n+i}^{(n)} \in \Gamma|\mathcal{F}_i]\,|^2 \le \frac{n}{N+1}.$$

**Proof.** Since

$$V_{N,\Gamma}^{(n)} = \frac{1}{N+1}\sum_{j=0}^{n-1}\sum_{\{k;\ 0\le kn+j\le N\}} K_{\Gamma,(k+1)n+j}^{(n)}.$$

Denote for each fixed $j$,

$$X_j = \sum_{\{k;0\le kn+j\le N\}} [K_{\Gamma,(k+1)n+j}^{(n)} - P[U_{(k+1)n+j}^{(n)} \in \Gamma|\mathcal{F}_{kn+j}]].$$

Let

$$Z_{k,j} = K_{\Gamma,(k+1)n+j}^{(n)} - P[U_{(k+1)n+j}^{(n)} \in \Gamma|\mathcal{F}_{kn+j}].$$

Then we can obtain

$$EZ_{(k+1),j}Z_{k,j} = E[E(Z_{(k+1),j}Z_{k,j}|\mathcal{F}_{(k+1)n+j})] = E[Z_{k,j}E(Z_{(k+1),j}|\mathcal{F}_{(k+1)n+j})] = 0.$$

So

$$EX_j^2 = \sum_{\{k;0\le kn+j\le N\}} EZ_{kj}^2.$$

By $C_r$ inequality,

$$E\left|V_{N,\Gamma}^{(n)} - \frac{1}{N+1}\sum_{i=0}^{N} P[U_{n+i}^{(n)} \in \Gamma|\mathcal{F}_i]\right|^2 = E\left|\frac{1}{N+1}\sum_{j=0}^{n-1}X_j\right|^2 \le n\sum_{j=0}^{n-1}E\left[\frac{1}{N+1}X_j\right]^2$$
$$= \frac{n}{(N+1)^2}\sum_{j=0}^{n-1}\sum_{\{k;0\le kn+j\le N\}} EZ_{k,j}^2 = \frac{n}{N+1}EZ_{k,j}^2 \le \frac{n}{N+1}.$$

Thus, the proof is complete. □

**Lemma 6.2.** *Suppose the sequence $\{X_n, n\ge 1\}$ is $\alpha$-mixing, $X \in \mathcal{F}_{-\infty}^k$, $Y \in \mathcal{F}_{k+n}^{\infty}$ and $0 \le X \le c_1$, $0 \le Y \le c_2$, then*

$$|EXY - EXEY| \le c_1c_2\alpha(n).$$

**Proof.** Since $X, Y$ are nonnegative random variables,then we have

$$EXY = \int_0^\infty \int_0^\infty xy\mathrm{d}F(x,y) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \mathbf{I}_{\{s<x\}}\mathbf{I}_{\{t<y\}}\mathrm{d}s\mathrm{d}t\mathrm{d}F(x,y)$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \mathbf{I}_{\{s<x\}}\mathbf{I}_{\{t<y\}}\mathrm{d}F(x,y)\mathrm{d}s\mathrm{d}t$$
$$= \int_0^\infty \int_0^\infty P(X > s, Y > t)\mathrm{d}s\mathrm{d}t.$$

So by the property of $\alpha$-mixing,

$$|EXY - EXEY| = \left|\int_0^\infty \int_0^\infty (P(X > s, Y > t) - P(X > s)P(Y > t))\mathrm{d}s\mathrm{d}t\right|$$
$$= \left|\int_0^{c_1} \int_0^{c_2} (P(X > s, Y > t) - P(X > s)P(Y > t))\mathrm{d}s\mathrm{d}t\right|$$
$$\le c_1c_2\alpha(n). \quad \square$$

Let $\pi$ be the unique stationary measure of Markov chain $\{X_n\}$. Define

$$\zeta_n(\tau) := \int \|P^n(x,\cdot) - \pi(\cdot)\|\tau(\mathrm{d}x) \tag{6.1}$$

for any probability measure $\tau$ on $\mathcal{B}$. $\zeta_n$ may be regard as a measure of dependence of the random variables $X_1, X_2, \ldots$. The following Lemma gives a close relationship between quantities $\zeta_n$ and $\beta$-mixing coefficients:

**Lemma 6.3** (*Liebscher [6]*). *The following inequality holds for a Markov sequence $\{X_n\}$ with initial distribution $\nu$:*

$$\beta_n \le 3\zeta_m(\nu) + \zeta_m(\pi) \quad \text{with } m := [n/2]. \tag{6.2}$$

*Here $[x]$ is the greatest integer $l \le x$. In the case of a stationary Markov chain we have even $\beta_n = \zeta_n(\pi)$ (see [5]).*

By Proposition 3.1, there exist a $\rho, \rho \in (0, 1)$ and $M < \infty$ such that

$$\|P^n(x,\cdot) - \pi(\cdot)\| \le MV(x)\rho^n, \quad \forall n \in Z^+, \ \forall x \in R$$

where $\rho, M$ are constants. If $\{X_t\}$ is initialized from its invariant measure $\pi$, then $\beta_X(n) \le c\rho^n$, where $c = \int MV(x)\pi(\mathrm{d}x)$. By the proof of Theorem 4.1, we know that $\{U_t^{(n)}\}_{t\ge n}$ is a function of $(g(t, n-1, 0), \ldots, g(t, n-1, n-2))$, $\{g(t, i, j)_{0\le j<i\le n-1}\}$ are just functions of $\{(X_{k-1}, e_k)_{k=t-n+1,\ldots,t}\}$. Thus, we can obtain

$$P[U_{n+i}^{(n)} \in \Gamma | \mathcal{F}_i] = P[U_{n+i}^{(n)} \in \Gamma | X_i] \triangleq f(X_i), \tag{6.3}$$

where $f$ is a deterministic measurable function and $0 \le f \le 1$.

**Lemma 6.4.** *Under the conditions of Proposition 3.1, if $\{X_t\}$ is initialized from nonstationary measure $\nu$ and satisfy $\int V(x)\nu(\mathrm{d}x) < \infty$, then there exist a constant $C_0$ such that*

$$E\left|\frac{1}{N+1}\sum_{i=0}^{N} f(X_i) - P_0[U_n^{(n)} \in \Gamma]\right|^2 \le \frac{C_0}{N+1},$$

*where $P_0[U_n^{(n)} \in \Gamma] = E(P[U_{n+i}^{(n)} \in \Gamma | X_i \text{ is stationary}])$.*

**Proof.** Since

$$E\left|\frac{1}{N+1}\sum_{i=0}^{N} f(X_i) - P_0[U_n^{(n)} \in \Gamma]\right|^2 \le 2E\left|\frac{1}{N+1}\sum_{i=0}^{N} f(X_i) - \frac{1}{N+1}\sum_{i=0}^{N} Ef(X_i)\right|^2$$
$$+ 2\left|\frac{1}{N+1}\sum_{i=0}^{N} Ef(X_i) - P_0[U_n^{(n)} \in \Gamma]\right|^2.$$

We write $d = \int MV(x)\nu(\mathrm{d}x)$. By Lemmas 6.2 and 6.3, we have

$$
\begin{aligned}
E\left|\frac{1}{N+1}\sum_{i=0}^{N} f(X_i) - \frac{1}{N+1}\sum_{i=0}^{N} Ef(X_i)\right|^2 &\le \frac{1}{(N+1)^2}\sum_{i=0}^{N} D(f(X_i)) + \frac{2}{(N+1)^2}\sum_{0\le i<j\le N} |Ef(X_i)f(X_j) - Ef(X_i)Ef(X_j)| \\
&\le \frac{1}{N+1} + \frac{2}{(N+1)^2}\sum_{0\le i<j\le N} \alpha_X(j-i) \\
&\le \frac{1}{N+1} + \frac{2}{(N+1)^2}\sum_{k=1}^{N}(N+1-k)\beta_X(k) \\
&\le \frac{1}{N+1} + \frac{2}{(N+1)^2}\sum_{k=1}^{N}(N+1-k)(3d+c)\rho^{\frac{k}{2}-1} \\
&\le \frac{1+2(3d+c)\frac{1}{\rho^{\frac{1}{2}}-\rho}}{N+1}.
\end{aligned}
$$

Note that $0 \le f \le 1$ and $X_t$ is exponential ergodic, we have

$$
\begin{aligned}
\left|\frac{1}{N+1}\sum_{i=0}^{N} Ef(X_i) - P_0[U_n^{(n)} \in \Gamma]\right|^2 &= \frac{1}{(N+1)^2}\left|\sum_{i=0}^{N}(Ef(X_i) - P_0[U_n^{(n)} \in \Gamma])\right|^2 \\
&\le \frac{1}{(N+1)^2}\sum_{0\le i,j\le N}\left|(Ef(X_i) - P_0[U_n^{(n)} \in \Gamma])(Ef(X_j) - P_0[U_n^{(n)} \in \Gamma])\right| \\
&\le \frac{1}{N+1}\sum_{i=0}^{N}|Ef(X_i) - P_0[U_n^{(n)} \in \Gamma]| \\
&\le \frac{1}{N+1}\sum_{i=0}^{N}\left|\int\int f(y)P^i(x,\mathrm{d}y)\nu(\mathrm{d}x) - \int\int f(y)\pi(\mathrm{d}y)\nu(\mathrm{d}x)\right| \\
&\le \frac{1}{N+1}\sum_{i=0}^{N}\int\left|\int f(y)(P^i(x,\mathrm{d}y) - \pi(\mathrm{d}y))\right|\nu(\mathrm{d}x) \\
&\le \frac{1}{N+1}\sum_{i=0}^{N}\int \|P^i(x,\cdot) - \pi(\cdot)\|\nu(\mathrm{d}x) \\
&\le \frac{1}{N+1}\frac{d}{1-\rho}.
\end{aligned}
$$

So we take $C_0 = \max\{2(1+2(3d+c)\frac{1}{\rho^{\frac{1}{2}}-\rho}), 2(\frac{d}{1-\rho})\}$, then

$$
E\left|\frac{1}{N+1}\sum_{i=0}^{N} f(X_i) - P_0[U_n^{(n)} \in \Gamma]\right|^2 \le \frac{C_0}{N+1}.
$$

Thus, we complete the proof. □

**Proof of Theorem 5.1.** Since

$$
E|V_{N,\Gamma}^{(n)} - P_0[U_n^{(n)} \in \Gamma]|^2 \le 2E\left|V_{N,\Gamma}^{(n)} - \frac{1}{N+1}\sum_{i=0}^{N} P[U_{n+i}^{(n)} \in \Gamma|\mathcal{F}_i]\right|^2 + 2E\left|\frac{1}{N+1}\sum_{i=0}^{N} f(X_i) - P_0[U_n^{(n)} \in \Gamma]\right|^2.
$$

We take $\widetilde{C} = 2\max\{n, C_0\}$, then the proof is complete by Lemmas 6.1 and 6.4. □

## 7. Conclusions

It is well known that technical analysis has been a part of financial practice for many decades, but this discipline has not received the same level of academic scrutiny and acceptance as more traditional approaches such as fundamental analysis, but we show here that if we recognize the current popular stock price models, then we can do statistics based on relative frequency of occurrence for some technical analysis indexes.

In this paper, we get some asymptotically stationary processes from the unstable process $S_t$ and prove the law of large numbers for frequencies of some statistics with application to technical analysis, which can help us to find many interesting things, unfortunately, we do not find a method such that the derived constant $\tilde{C}$ in Theorem 5.1 is concrete. So it is hoped that future research will propose a method(s) to give sharper rate of convergence, and in addition, the related questions both in financial investment tactics and theoretical research should be worthy for further study.

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