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On a class of pseudocompact spaces derived from ring epimorphisms

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Abstract

A Tychonoff space X is RG if the embedding of $C(X) \to C(X_\delta)$ is an epimorphism of rings. Compact RG-spaces are known and easily described. We study the pseudocompact RG-spaces. These must be scattered of finite Cantor Bendixon degree but need not be locally compact. However, under strong hypotheses, (countable compactness, or small cardinality) these spaces must, indeed, be compact. The main theorems shows, how to construct a suitable maximal almost disjoint family, and apply it to obtain examples of RG-spaces that are almost compact, locally compact, non-compact, almost-P, and of Cantor Bendixon degree 2. More complicated examples of pseudocompact non-compact RG-spaces ensue.

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1. Introduction

Let X be a Tychonoff space and let $C(X)$ be the ring of continuous real-valued functions on X. As shown in [13, Section 5] and [6, Section 1] the study of epimorphisms in the category of commutative rings yields an algebra of real-valued functions on X, denoted $G(X)$, with some properties of interest. The ring $G(X)$ is Von Neumann regular, it is a subring of $C(X_\delta)$, and it is an epimorphic extension of $C(X)$ in the category of rings [13, 2.7]. (As usual, X_δ denotes the underlying set of X with the topology generated by the G_δ-sets of the space X). The functions in $G(X)$ are finite linear combinations of products of functions in $C(X)$ and their quasi-inverses, taken in the regular ring $C(X_\delta)$. This explicit representation provides a useful notion of "degree". The definition of the quasi-inverse, and the presentation of a function f in $G(X)$ as well as its regularity degree (denoted $rg(f)$) are found in [6, pp. 1, 2] as is the regularity degree of a space X denoted $rg(X)$. We will abuse notation and also speak of 'rank' interchangeably with 'regularity degree'.

The class of RG-spaces was defined and studied in [6]. A space X is RG (for regularly good) if $G(X) = C(X_\delta)$. It is equivalent to demand that the embedding $C(X) \to C(X_\delta)$ be an epimorphism of rings. (One direction is clear since the embedding into $G(X)$ is an epimorphism. The converse follows from the algebraic fact that regular rings like $G(X)$ are "dominant" cf. [16]). Although RG-spaces are generally difficult to determine, they are characterized nicely in the compact case as follows [6, 3.4]: a compact space is RG iff it is scattered and of finite dispersal degree, i.e., of finite Cantor–Bendixon degree, cf. [6].

In this note we are interested in studying pseudocompact spaces that are RG. Being scattered of finite *CB* index is not sufficient because Isbell's space Ψ of [4, 5I] is never RG even though there are versions of Ψ that are almost compact).

We show that pseudocompact RG-spaces must be scattered strongly zero-dimensional of finite *CB* index and of finite regularity degree (Proposition 1). Section 2 presents examples which show that there is quite a variety of pseudocompact RG-spaces (obtained via the Generating Machine). These spaces need not be almost compact, in fact they need not even be locally compact.

Pseudocompact RG-spaces of cardinality ω_1 and countably compact RG-spaces of any cardinality, must be compact (Proposition 2, Theorem 3).

We also show (Theorem 4) that a locally compact pseudocompact RG-space of cardinality $< \mathfrak{p}$ must be compact ($\mathfrak{p}$ is defined in [17]).

Later in Theorem 6 we show that if a cardinal κ admits a certain kind of family of maximal almost disjoint subsets, then an RG-space of *CB* index 2 can be constructed that is pseudocompact, locally compact, almost compact, almost-P, but not compact. The construction mimics that of Isbell's space Ψ. It can always be done for regular cardinals κ satisfying $\kappa = \kappa^\omega = \kappa^{\omega_1}$, in particular for spaces of cardinality $(2^{\omega_1})^+$. It turns out that it is non-decidable in ZFC whether there is a locally compact non-compact pseudocompact RG-space of cardinality ω_2 (Corollary 1).

All spaces discussed will be Tychonoff. When X is locally compact and $x \in X$, then $x \in O \subset K$ will mean that O is an open neighbourhood of x and that K is compact. A space X is called almost compact [4, 6J] if $|\beta X - X| \leqslant 1$. It is called almost-$P$ [9] if every non-empty zero-set has non-empty interior cf. [9]. Our terminology and notation will be that of Gillman and Jerison [4] and Porter and Woods [11]. The paper [12] gives an algebraic context for the study of $G(X)$ and of RG-spaces and [14] has other results on the topic.

2. Elementary results and some examples

Recall that a subspace Y of X is called G-embedded [6, 2.1] if the natural restriction homomorphism $G(X) \to G(Y)$ is onto.

The following replies to [6, 4.5] and is based on an observation by M. Sanchis.

Lemma 1. *Let Y be a G_δ-dense, G-embedded subspace of X. Then X is RG if Y is RG. In particular, if Y is RG and $Y \subset X \subset \upsilon Y$ then X is RG.*

Proof. Take $f \in C((X)_\delta)$. Restricting to Y_δ gives $f|Y = \sum a_i b_i^*$ for $a_i, b_i \in C(Y)$ because Y is RG. By the G-embedding of Y in X, this lifts to $\sum C_j D_j^*$ for some $C_j, D_j \in C(X)$, and now the functions f and $\sum C_j D_j^*$ agree on the dense (in the delta topology) subset Y and therefore are equal. Thus $f \in G(X)$.

For the second claim note that X is C-embedded (and hence G-embedded) in υY and it is G_δ-dense in υY by [4, 8.8(b)] or [11, 5.11(f)].

In fact it is easy to see that $\upsilon(Y_\delta) = (\upsilon Y)_\delta$ as follows. Since Y is an RG-space, $C(Y_\delta) = G(Y)$, so the G-embedding of Y in υY implies that Y_δ is C-embedded in $(\upsilon Y)_\delta$, which is realcompact because υY is. Also Y_δ is dense in $(\upsilon Y)_\delta$ (as Y is G_δ-dense in υY), it follows that $(\upsilon Y)_\delta = \upsilon(Y_\delta)$. □

Remark 1. We should point our here that the claim in [11, 5F(7)]—i.e., that $\upsilon(Y_\delta) = (\upsilon Y)_\delta$ in general—is false. Isbell's space Ψ provides a counterexample.

Recall that the relationship between *CB* index and regularity degree in RG-spaces is quite fluid. Clearly P-spaces are always RG; they need not have any isolated points, (in which case the notion of *CB* index is irrelevant), and if scattered, they can have finite *CB* index, or have infinite *CB* index. In the compact case the connection is very tight. It is precisely when the space is scattered and of finite index that the space is RG. No such link holds for almost compact spaces. But the following result underscores the proximity of the pseudocompact case to the compact case.

Proposition 1. *Suppose that X is a pseudocompact RG-space. Then X is scattered, it is of finite CB index, functionally countable, and strongly zero-dimensional. Furthermore $rg(X)$, the regularity degree of the ring $C(X)$, is finite.*

Proof. Since X is pseudocompact $\beta X = \upsilon X$ so by Lemma 1, βX is RG. Thus it is scattered and of finite *CB* index [6, 3.4]. Therefore X is scattered of finite *CB* index. The space X is functionally countable because it is C-embedded in βX and each function on βX has countable range.

Since compact scattered spaces are zero-dimensional [6, 3.3], βX is zero-dimensional and therefore X is strongly zero-dimensional by [3, 6.2.12].

It remains to show that the regularity degree is finite. Let $f \in C(X_\delta)$. As in the proof of Lemma 1, f lifts to a function $F \in C[\upsilon(X_\delta)] = C[(\upsilon X)_\delta]$. But by Lemma 1, $\upsilon X = \beta X$ is an RG-space of finite regularity degree say n. Therefore $F \in G(\upsilon X)$ and $rg(F) \leqslant n$. It follows (by restriction) that $rg(f) \leqslant n$. □

Lemma 2. *A space Y is RG if it contains a cozero set U such that U is RG and $Y - U$ is RG and G-embedded. The space Y is of finite rank if U and $Y - U$ are of finite rank.*

Proof. Since cozero sets induce epimorphisms in the category of commutative rings, [1, 2.1(ii)] they are automatically G-embedded. Now the result is straightforward. Suppose that $U = \operatorname{coz}(m)$. Take $f \in C(Y_\delta)$. Then $f|U \in G(U)$ and $f|Y - U \in G(Y - U)$. As U and $Y - U$ are G-embedded in Y there are functions $A_i, B_i, C_j, D_j \in C(Y)$ so that f and $\sum A_i B_i^*$ agree on U and f and $\sum C_j D_j^*$ agree on $Y - U$. Thus $f = (mm^*)(\sum A_i B_i^*) + (1 - mm^*)(\sum C_j D_j^*) \in G(Y)$. □

For the next theorem we will need to cite the following result:

Theorem 1 (Starbird [15]). *If K is a compact subspace of W, then $X \times K$ is C^*-embedded in $X \times W$.*

Theorem 2. *Let αN be a compactification of N that is RG. Let X be an RG-space of finite rank. Then $Y = X \times \alpha N$ is RG of finite rank. If $CB(X) = n$, then $CB(X \times N^*) = n + 1$.*

Proof. The space αN is of finite CB index because it is RG.

We will induct on $CB(\alpha N)$. Let U be the union of the clopen sets $X \times \{n\}$. The cozero set U is RG because each $X \times \{n\}$ is of the same (finite) regularity degree namely that of X [6, 2.8].

When αN has index 2 (the least possible), $\alpha N - N$ is finite and $Y - U$ is the free union of a finite number of copies of X and thus RG. It is G-embedded because it is C^*-embedded by Starbird's theorem. By Lemma 2, $X \times \alpha N$ is an RG-space of finite regularity degree.

Now assume the result for CB index n and consider the case when $CB(X) = n + 1$. The space $\alpha N - N$ is compact of CB index n so $Y - U = (\alpha N - N) \times X$ is RG and of finite regularity degree by inductive assumption. Also $Y - U$ is G-embedded because it is C^*-embedded again by Starbird's theorem which is applicable since $\alpha N - N$ is compact. The space Y is RG and of finite rank by Lemma 2.

The last claim concerns the raising of the CB index under taking the product with N^* and it is the result of a straightforward consideration of the isolated points in the product. □

2.1. Generating machine

We now present a method for obtaining new examples from existing ones. Throughout M will denote a pseudocompact non-compact RG-space. An example that is almost compact and almost-P is constructed in Theorem 6 below.

By Theorem 2, $Y = M \times N^*$ is an RG-space. It is pseudocompact because it is the product of a pseudocompact space with a compact space. Hence by Glicksberg's Theorem [5], $\beta(M \times N^*) = \beta M \times \beta N^* = \beta M \times N^*$. Therefore $\upsilon Y = \beta M \times N^*$.

Now suppose that T is any space that lies between Y and βY. By Lemma 1, T is RG and it is pseudocompact since it contains Y as a dense subspace. There are many ways of choosing T. Since N^* is not almost-P, neither is Y so it follows by a theorem of Levy [9, 2.2] that βY is not almost-P and hence by the same theorem that none of the space T we generate will be almost-P.

2.2. Examples

1. A pseudocompact RG-space that is not locally compact.
   Let $p \in \beta M - M$, and let $T = Y \cup \{(p, \omega)\}$ where ω is the point at infinity of N^*. Then the point (p, ω) has no compact neighbourhood in T.
2. A pseudocompact locally compact RG-space that is not almost compact.
   Take an instance of M which is almost compact and consequently locally compact. Let $T = Y$. Then $\beta T - T = (\beta M - M) \times N^*$ which is infinite. This means that T is certainly not a finite free union of almost compact spaces.
3. Other possibilities.
   It is clear that for any space T which the procedure produces, we can repeat the procedure beginning with T in the place of M. In particular we can manufacture pseudocompact RG-spaces whose outgrowths are scattered of any finite *CB* index. Thus the structure of the outgrowths of pseudocompact RG-spaces can be complicated.

3. Compactness when the space is of cardinality ω_1 or is countably compact

The following result is obvious for spaces of cardinality ω, since countable pseudocompact spaces are compact.

Proposition 2. *If X is pseudocompact RG of cardinality ω_1, then X is compact.*

Proof. Let X be pseudocompact of cardinality ω_1. By Proposition 1, X is functionally countable. Since X is RG, and given the nature of the functions in $G(X)$, X_δ is also functionally countable. Thus X_δ cannot be written as the free union of an uncountable collection of disjoint clopen subsets.

Suppose X is not compact. As it is pseudocompact, it follows that it is not Lindelöf [4, 5.9, 8.2]. Let $\mathcal{C}$ be an open cover of X with no countable subcover. Let $X = \{x_\alpha\colon \alpha < \omega_1\}$.

Let $\delta < \omega_1$ and inductively assume that for each $\alpha < \delta$ we have chosen a cozero-set V_α of X such that:

(i) $x_\alpha \in V_\alpha$,
(ii) if $\alpha_1 < \alpha_2 < \delta$ then V_{α_1} is a proper subset of V_{α_2},
(iii) $V_\alpha \subset C_\alpha$ for some $C_\alpha \in \mathcal{C}$.

By (iii) and our choice of $\mathcal{C}$, $X - \bigcup_{\alpha<\delta} V_\alpha \neq \emptyset$.

Let $\lambda(\delta) = \min\{\gamma < \omega_1\colon\ x_\gamma \notin \bigcup_{\alpha<\delta} V_\alpha\}$. By (i) $\lambda(\delta) \geqslant \delta$. Then $x_{\lambda(\delta)} \in C_\delta$ for some $C_\delta \in \mathcal{C}$. There is a cozero-set W_δ of X such that $x_{\lambda(\delta)} \in W_\delta \subset C_\alpha$. Let $V_\delta = (\bigcup_{\alpha<\delta} V_\alpha) \cup W_\delta$. Then (i)–(iii) hold for $\alpha \leqslant \delta$.

Thus $(V_\alpha)_{\alpha<\omega_1}$ is strictly increasing ω_1-sequence of cozero sets of X. Clearly

$$\left\{V_\delta - \bigcup_{\alpha<\delta} V_\alpha\colon\ \delta < \omega_1\right\}$$

is an uncountable disjoint covering of X by non-empty clopen sets of X_δ, contradicting what we asserted earlier. The proposition follows. $\square$

We will now prove that a countably compact RG-space must be compact. First we introduce some notation and state a simple structural lemma.

Let Y be a scattered space with $CB(Y) = n$. Let $L_1(Y) = I(Y)$ denote the set of isolated points of Y. If $1 < k \leqslant n$, define $L_k(Y)$ to be $I(Y - \bigcup_{i=1}^{k-1} L_i(Y))$. The following is immediate [6, 3.2].

Lemma 3. *Let X be a scattered space with $CB(X) = n$. Then*:

(i) *$\{L_i(X)\colon\ 1 \leqslant i \leqslant n\}$ partitions X.*
(ii) *If $k \in \{1, \ldots, n\}$ then $L_k(X)$ is a closed discrete subspace of $\bigcup_{i=1}^{k} L_i(X)$ and a dense open subspace of $\bigcup_{i=k}^{n} L_i(X)$.*
(iii) *If $1 < k \leqslant n$ and $p \in L_k(X)$ then p has an X-neighbourhood V such that*

$$V - \{p\} \subset \bigcup_{i=1}^{k-1} L_i(X).$$

The subset $L_k(X)$ is sometimes called the “k-th level” of X.

Theorem 3. *A countably compact RG-space is compact.*

Proof. Let X be a countably compact RG-space. Since X is pseudocompact, by Proposition 1, βX is scattered and of finite CB index say n. Thus $\beta X = \bigcup_{i=1}^{n} L_i(\beta X)$.

We will prove that $L_i(\beta X) \subset X$ for each $i \in \{1, \ldots, n\}$. This will show that $\beta X = X$ so X is compact. Clearly $L_1(\beta X) \subset X$ since $L_1(\beta X) = I(\beta X)$ and X is dense in βX.

Suppose if possible that $L_i(\beta X) - X \neq \emptyset$ for some i and let k be the smallest such i. Clearly $k \geqslant 2$. Let $p \in L_k(\beta X) - X$. By (iii) in the preceding lemma, there is a compact βX-neighbourhood A of p such that $A - \{p\} \subset \bigcup_{i=1}^{k-1} L_i(\beta X)$. As $L_{k-1}(\beta X)$ is dense and open in $\bigcup_{i=k-1}^{n} L_i(\beta X)$ (see (ii) above), it is evident that $p \in \mathrm{cl}_{\beta X}(A \cap L_{k-1}(\beta X))$. Thus $A \cap L_{k-1}(\beta X)$ is an infinite discrete space by (ii) above. But as $L_{k-1}(\beta X)$ is closed in $\bigcup_{i=1}^{k-1} L_i(\beta X)$ and also discrete, it is clear that the only βX-limit point of $A \cap L_{k-1}(\beta X)$ is p. Since $p \notin X$, the set $A \cap L_{k-1}(\beta X)$ is an infinite closed discrete subspace of X contradicting the hypothesis that X is countably compact. The theorem follows. □

4. The case of locally compact spaces of cardinality less than $\mathfrak{p}$

As noted earlier pseudocompact RG-spaces need not be locally compact (although it is easy to check that separable ones of *CB* index $\leqslant 3$ are). The results in this section will assume local compactness.

We recall [17, p. 115] that the set A is a pseudo-intersection of a family $\mathcal{F}$ if for each $F \in \mathcal{F}$ the set $A - F$ is finite. The family $\mathcal{F}$ has the sfip (strong finite intersection property) if every nonempty finite subfamily has an infinite intersection. The cardinal $\mathfrak{p}$ is $\min|\mathcal{F}|$ as $\mathcal{F}$ ranges over countably infinite subfamilies of ω with the sfip and with no infinite pseudo-intersection.

The purpose of this section is to establish the following result.

Theorem 4. *If $\kappa < \mathfrak{p}$, X has cardinality κ, and X is locally compact pseudocompact RG, then X is compact.*

The theorem does not imply Proposition 2 because there are models of set theory in which $\mathfrak{p} = \omega_1$—cf. [17, 3.1(a)].

In order to prove the theorem we require a series of lemmas.

Lemma 4. *Let X be a locally compact space that contains a countably infinite set A of isolated points with the property that every compact subset K of X has finite intersection with A. Then X is not pseudocompact.*

Proof. It suffices to show that A is closed in X. If so, it is an infinite discrete clopen subset of X, and it admits an unbounded function that has a continuous extension to all of X. If possible, let q be in the closure of A but not in A. By local compactness $q \in O \subset K$ compact. By assumption $K \cap A$ is a finite set of isolated points so $O \setminus (K \cap A)$ is an open neighbourhood of q disjoint from A, a contradiction. □

As usual, $w(X)$ will denote the weight of the space X (see [11]).

Lemma 5. *Let X be locally compact, non-compact, of finite CB index and of cardinality κ. Then $w(X) \leqslant \kappa$.*

Proof. We induct on $CB(X)$. If $CB(X) = 1$ then X is discrete and $\{\{x\}\colon x \in X\}$ is an open base of cardinality κ so $w(X) \leqslant \kappa$.

Now suppose the result holds for all locally compact spaces of *CB*-index $\leqslant n$. Assume that $CB(X) = n + 1$ and that $|X| \leqslant \kappa$. Let $T = L_n(X)$ (see the beginning of Section 3).

Then T is a closed discrete subspace of X, $CB(X - T) = n$ and $X - T$ is locally compact with $|X - T| \leqslant \kappa$. Then $w(X - T) \leqslant \kappa$. Let $\mathcal{C}$ be an open base for $X - T$ of cardinality $\leqslant \kappa$. The members of $\mathcal{C}$ are open in X since $X - T$ is open in X.

Let $p \in T$. As X is locally compact and T is closed discrete, there exists a compact subset A of X such that $p \in \mathrm{int}_X A \subset A$ and $A \cap T = \{p\}$. Therefore $CB(A - p) \leqslant n$ and $A - p$ is locally compact and of cardinality $\leqslant \kappa$ so $w(A - p) \leqslant \kappa$.

It is routine to show that if Y is locally compact and non-compact then $w(Y^*) = w(Y)$. Therefore $w(A) \leqslant \kappa$. But $p \in \mathrm{int}_X A$ so for all $p \in T$, there exists an open neighbourhood base $\mathcal{B}_p$ at p in X such that $|\mathcal{B}_p| \leqslant \kappa$.

But $|T| \leqslant \kappa$. Let $\mathcal{A} = \mathcal{C} \cup \{\mathcal{B}_p\colon\ p \in T\}$. Then $\mathcal{A}$ is an open base for X of cardinality $\leqslant \kappa$ since it is the union of at most κ families of at most κ members. This completes the induction step. □

Lemma 6. *Let X be locally compact, non-compact, scattered, and of finite CB index. Then there is a countable set of isolated points in X whose closure is not compact.*

Proof. We induct on $CB(X)$. The result is clear when $CB(X) = 1$. If X is as hypothesized, then $CB(X - I(X)) < CB(X)$ so by assumption there is a countable subset S of $L_2(X)$ for which $\mathrm{cl}_X S$ is not compact. If $s \in L_2(X)$ then s has a compact neighbourhood $K(s)$ such that $K(s) - \{s\} \subset I(X)$. Let $A(s)$ be a countably infinite subset of $K(s) - \{s\}$, and put $A = \bigcup\{A(s)\colon\ s \in S\}$. Clearly A is a countable subset of $I(X)$ that is dense in $A \cup S$; thus $\mathrm{cl}_X S \subset \mathrm{cl}_X A$ so as $\mathrm{cl}_X S$ is not compact, neither is $\mathrm{cl}_X A$. □

Lemma 7. *Suppose that $\kappa < \mathfrak{p}$. Let X be locally compact, non-compact, scattered, of finite CB index, and suppose that $|X| \leqslant \kappa$. Then X is not pseudocompact.*

Proof. By Lemma 6, X has a countable set of isolated points D whose closure is not compact. The conclusion will hold if we show that D contains an infinite subset A that satisfies the conditions of Lemma 4. The existence of A will follow from the fact that $\kappa < \mathfrak{p}$. By the definition of p, [17, p. 115], a family $\mathcal{F}$ of countably infinite sets will have an infinite pseudo-intersection if the family has the strongly finite intersection property, and if its cardinality is less than p. We choose as members of $\mathcal{F}$ those countably infinite subsets F of D with the property that $D \setminus F = D \cap C$ for some compact open subset C of X. The family is non-empty since $D \in \mathcal{F}$. We check that it has the strong finite intersection property. Let $F_1, \ldots, F_n \in \mathcal{F}$ with associated compact open sets $C_1, \ldots, C_n$. Suppose $\bigcap F_i$ is finite say $\{d_1, \ldots, d_k\} \subset D$. Then D lies in the compact set $\bigcup C_i \cup \{d_i\}$, so the closure of D is compact, which is false. Now we want to show that $\mathcal{F}$ has size at most κ. By Lemma 5, X has an open base of cardinality $\leqslant \kappa$. Let $\mathcal{C}$ be the collection of compact open subsets of X. Then $|\mathcal{F}| \leqslant |\mathcal{C}|$, and as each member of $\mathcal{C}$ is a union of finitely many elements of $\mathcal{B}$, $|\mathcal{C}| \leqslant \kappa$. Thus $|\mathcal{F}| \leqslant \kappa < p$. Now let A be the infinite pseudo-intersection of the family $\mathcal{F}$. □

Lastly we have:

Proof of Theorem 4. The space X is scattered, and of finite *CB* index by Corollary 1. It has weight at most κ by Lemma 5. If it is not compact, then it is not pseudocompact by Lemma 7. $\square$

5. Non-compact examples via Ψ-like spaces

In this section we will construct spaces that are RG, almost compact, locally compact, almost-P, non compact, and of finite *CB* index.

The notion of a maximal almost disjoint family is found in [17, p. 115].

Definition 1 (*RG-MAD: RG-maximal almost disjoint family on* κ). Let κ be a cardinal, $\kappa \geqslant \omega_2$. An *RG-MAD* family, $\mathcal{A}$, is a collection of subsets of κ of cardinality at least ω_2 obeying:

(i) if $A, B \in \mathcal{A}$, $A \neq B$, then $|A \cap B|$ is finite,
(ii) for all $X \in [\kappa]^{\omega_0}$, there is an $A \in \mathcal{A}$ such that $A \cap X$ has cardinality ω_0,
(iii) for every subset $\mathcal{B}$ of $\mathcal{A}$ of cardinality ω_1, and for all sets D in $[\kappa]^{\omega_1}$, there is an $A \in \mathcal{A}$ such that for all $B \in \mathcal{B}$, $(B \setminus D) \cap (A) \neq \emptyset$.

Note that (i) and (ii) imply that the family $\mathcal{A}$ is maximal: if there were a subset S of cardinality at least ω_2 that could be added while retaining the almost disjoint property then a countable subset of S could also be added, and it could play the role of X in (ii).

Theorem 5. *Let $\kappa = \kappa^{\omega_0} = \kappa^{\omega_1}$ be a regular cardinal. Then there is an RG-MAD family A_α of cardinality κ on the set κ.*

Proof. Enumerate $[\kappa]^{\omega_0}$ as $\{X_\alpha\colon \alpha < \kappa\}$ and $[\kappa]^{\omega_1}$ as $\{D_\alpha\colon \alpha < \kappa\}$ making sure that each D_α occurs with κ repetitions.

We will recursively construct a family that satisfies the conditions of Definition 1. (In fact, something stronger than the third condition will hold.)

The sets A_α will be of cardinality κ and will be non-stationary, i.e., their individual complements will contain a closed unbounded subset of κ cf. [7, p. 78], [8, p. 57].

As well, the family A_α will satisfy the following three conditions:

(1) if $\beta < \alpha$, $A_\alpha \cap A_\beta$ will be finite,
(2) if $|X_\alpha \cap A_\beta| < \omega$ for all $\beta < \alpha$, then $X_\alpha \subset A_\alpha$ (either X_α has an infinite intersection with a preceding A_β, or else its intersection with A_α is infinite),
(3) $\forall \beta < \alpha$, $A_\alpha \cap (A_\beta \setminus D_\alpha) \neq \emptyset$.

First we note that once this is done, the three conditions of Definition 1 will be satisfied by $\{A_\alpha\colon \alpha < \kappa\}$. Condition (i) is identitical and condition (ii) is immediate, so it suffices to check condition (iii). Let $\mathcal{B}$ be a family of ω_1 sets from $\mathcal{A}$, and let D be a subset of κ of cardinality ω_1. Recall that the set D occurs at least κ times in the set $\{D_\alpha\}$. Since the cofinality of κ is greater than ω_1, there is a γ greater than each α occurring as a subscript in $\mathcal{B}$. The corresponding A_γ is the set A that one wants for condition (iii).

To begin the construction, choose a set A_0 from $[\kappa]^\kappa$ that is non-stationary and contains X_0. This is possible as follows: X_0 has cardinality ω_0 so it is non-stationary. Its complement contains a closed unbounded set S, which is of cardinality κ by [7, 6.12]. Now choose C a non-stationary subset of S of cardinality κ (say its set of non-limit points) and let $A_0 = X_0 \cup C$. Thus A_0 is non-stationary and of the right cardinality.

Conditions (1) and (3) are automatic and condition (2) holds by the choice of A_0.

Now assume that all $A_\beta, \beta < \alpha$ have been defined. Since κ is a regular cardinal, $\bigcup\{A_\beta\colon \beta < \alpha\}$ is non-stationary [7, p. 78]. We now do a construction that "disjointifies" the A_β by ignoring small intersections. Each D_α is of cardinality ω_1 so it is non-stationary. For each β, let $B_\beta = A_\beta \setminus (\bigcup A_\gamma\colon \gamma < \alpha,\ \gamma \neq \beta) \cup D_\alpha$. Now B_β is not empty because A_β has cardinality κ and one is deleting fewer than κ finite sets from it. The B_β are disjoint by construction. The set $\bigcup B_\beta \subset \bigcup A_\beta$ so it is non-stationary.

Choose C_α, non-stationary from $[\kappa]^\kappa$, so that C_α is disjoint from $\bigcup A_\beta,\ \beta < \alpha$. (We need C_α to make sure that the A_α that we construct is of size κ.) Again, this is possible because the complement of the non-stationary set $\bigcup A_\beta,\ \beta < \alpha$, contains a closed unbounded set which, in turn, contains, a non-stationary set of cardinality κ.

Now for each $\beta < \alpha$ choose $b_\beta \in B_\beta$ and define A_α as follows:

Case 1. If there is a $\beta < \alpha$ for which $X_\alpha \cap A_\beta$ is infinite, let $A_\alpha = C_\alpha \cup \{b_\beta\colon \beta < \alpha\}$.

Case 2. If for all $\beta < \alpha,\ X_\alpha \cap A_\beta$ is finite, let $A_\alpha = C_\alpha \cup \{b_\beta\colon \beta < \alpha\} \cup X_\alpha$.

We need to check that the three conditions hold:

(1) In case 1, $A_\alpha \cap A_\beta = \{b_\beta\}$. In case 2, $A_\alpha \cap A_\beta = \{b_\beta\} \cup (X_\alpha \cap A_\beta)$ which is finite in this case.
(2) This holds by the construction of A_α, because we are in case 2.
(3) By construction, $b_\beta \in A_\alpha \cap (A_\beta \setminus D_\alpha)$, so the intersection is not empty. □

Recall from Proposition 1 that a pseudocompact RG-space must be of finite CB-index.

Theorem 6. *Let $\mathcal{A}$ be an RG-MAD family on $\kappa \geqslant \omega_2$. Then there is a pseudocompact, locally compact, almost compact, almost-P, noncompact RG-space of CB index* 2 *of size* $|\mathcal{A}| + \kappa$.

Furthermore there is a pseudocompact, locally compact, almost compact, RG-space of each finite CB-index.

Proof. Let L be a set of cardinality $|\mathcal{A}|$, and let $A \to p(A)$ be a bijection from $\mathcal{A}$ onto L. Let $X = \kappa \cup \{p(A)\colon A \in \mathcal{A}\} = \kappa \cup L$. Define a topology τ on X as follows:

$$\tau = \bigl\{V \subset X\colon\ p(A) \in V \Rightarrow A - V < \omega\bigr\}.$$

It is straightforward to verify that (X, τ) is a locally compact Hausdorff (hence Tychonoff) space. It is reminiscent of "Ψ-like" spaces used frequently as examples; see [4, 5I] for a discussion of Ψ. Observe that $\{p(A)\} \cup A = K(A)$ is the one point compactification of the discrete open subspace A of X, and is a compact open X-neighbourhood of $p(A)$. Also note that $I(X) = \kappa$.

We will prove that X is an almost compact RG-space. It is clear that L is an infinite closed discrete subspace of X so X is not countably compact, while X is scattered and $CB(X) = 2$.

Claim 1. *X is pseudocompact.*

For if $f \in C(X) - C^*(X)$ assume without loss of generality that $f \geqslant 0$. For each $n \in N$ inductively choose $x_n \in X$ such that $f(x_{n+1}) > f(x_n) + 1$. As κ is dense in X, for each $n \in \omega$ choose $a_n \in f^{-1}[(f(x_n) - 1/4, f(x_n) + 1/4)] \cap \kappa$ and let $S = \{a_n\colon n \in \omega\}$. By property (ii) of Definition 1 there exists an $A \in \mathcal{A}$ such that $|A \cap S| = \omega$. Then $f|K(A)$ is continuous and unbounded, while $K(A)$ is compact, a contradiction. Hence no such f can exist and X is pseudocompact.

Claim 2. *If $f \in C(X_\delta)$ and $A \in \mathcal{A}$, there is a countable subset C of A for which f is constant on $K(A) - C$.*

To see this, suppose that $f(p(A)) = r$. Then as $f^{-1}(r)$ is open in X_δ and the G_δ-sets of X form an open base for X_δ, there is a countable family $\{W_n\colon n \in N\}$ of open sets of X for which $p(A) \in K(A) \cap (\bigcap_{n \in N} W_n) \subset K(A) \cap f^{-1}(r)$.

As X-open sets of $K(A)$ that contain $p(A)$ are co-finite, $\bigcap_{n\in\omega} W_n$ is co-countable, so $\{x \in K(A)\colon f(x) = r\}$ is co-countable. The claim follows.

Claim 3. *Let $f \in C(X_\delta)$. Then $|f[L]| \leqslant \omega$.*

To see this, suppose there is a subset $\{r_i\}$ of $f[L]$ of cardinality ω_1. For each $i < \omega_1$, there exists $A_i \in \mathcal{A}$ for which $f(p(A_i)) = r_i$. Let $D_i = K(A_i) - f^{-1}(r_i)$. By Claim 2, D_i is countable. Let $D = \bigcup_{i<\omega_1} D_i$; then $|D| = \omega_1$. By (iii) of Definition 1 there exists $A \in \mathcal{A}$ such that for each $i < \omega_1$, $(A_i - D) \cap A \neq \emptyset$. Let $s_i \in (A_i - D) \cap A$; then $\{s_i\colon i < \omega_1\} \subset A$ and $f(s_i) = r_i$. Thus f assumes uncountably many values on A in contradiction to Claim 2. Thus our claim follows.

Claim 4. *Let $f \in C(X_\delta)$. Then there exists $r \in \Re$ such that $L - f^{-1}(r)$ is countable.* (*We call the number r the 'principal value' of f*).

To see this, note that by Claim 3 there is a countable subset $\{r_i\colon i \in N\}$ of $\Re$ such that $L = \bigcup_{i\in N} f^{-1}(r_i)$. As L is uncountable, there is some $a \in N$ such that $f^{-1}(r_a)$ is uncountable. Then $L - f^{-1}(r_a) = \bigcup_{i\in N-\{a\}} f^{-1}(r_i)$. Suppose that $L - f^{-1}(r_a)$ is uncountable. Then the same argument applied to $L - f^{-1}(r_a)$ yields a $b \in N - \{a\}$ such that $f^{-1}(r_b)$ is uncountable. Thus there exist points $\{p(A_i)\colon i < \omega_1\}$ and $\{p(B_i)\colon i < \omega_1\}$ such that $f(p(A_i)) = r_a$ and $f(p(B_i)) = r_b$ for each $i < \omega_1$.

Let $D_1 = (\bigcup_{i<\omega_1} A_i - f^{-1}(r_a)) \cup (\bigcup_{i<\omega_1} B_i - f^{-1}(r_b))$. By Claim 2, using the argument employed in the proof of Claim 3, $|D_1| \leqslant \omega_1$.

Let $D_2 = \{s \in \kappa\colon s \in A_i \cap B_j$ for some $(i, j) \in \omega_1 \times \omega_1$. By (i) of the definition $|D_2| \leqslant \omega_1$. Let $D = D_1 \cup D_2$. By (iii) of the definition there exists an $A \in \mathcal{A}$ such that $(A_i - D) \cap A \neq \emptyset$ and $(B_i - D) \cap A \neq \emptyset$ for all $i < \omega_1$. As $(A_i - D) \cap (B_j - D) = \emptyset$ for each (i, j), both $A \cap (\bigcup_{i<\omega_1} A_i)$ and $A \cap (\bigcup_{i<\omega_1} B_i)$ are uncountable sets, with $f[A \cap (\bigcup_{i<\omega_1} A_i)] = r_a$ and $f[A \cap (\bigcup_{i<\omega_1} B_i)] = r_b$. This contradicts Claim 2. Hence $L - f^{-1}(r_a)$ is countable and Claim 4 follows.

Claim 5. *X is almost compact.*

It suffices to show that if $f, g \in C(X)$ and $Z(f) \cap Z(g) = \emptyset$ then either $Z(f)$ or $Z(g)$ is compact [4]. As $C(X) \subset C(X_\delta)$ it follows from Claim 4 that $L \cap Z(f)$ and $L \cap Z(g)$ are either countable or co-countable. As they are disjoint they cannot both be co-countable, so assume without loss of generality that $L \cap Z(f)$ is countable. Using Claim 4 again, we see that there exists an $r \in \Re - \{0\}$ such that $f^{-1}(r)$ is co-countable. (In fact, we are showing that if $s \in \Re - 0$, then $L \cap f^{-1}(s)$ is finite.)

We next show that $L \cap Z(f)$ is finite. If not, there exists a countable infinite subset $\{p(A_i)\colon i \in N\} \subset L \cap Z(f)$. Choose $\{p(B_j)\colon j \leqslant \omega_1\} \subset L \cap f^{-1}(r)$.

Let

$$D = \left(\bigcup_{j<\omega_1} \bigl(B_j - f^{-1}(r)\bigr)\right) \cup \left(\bigcup_{i\in N} A_i - Z(f)\right) \cup \left[\bigcup \bigl\{S \cap T\colon S, T \in \{B_j\colon j < \omega_1\} \cup \{A_i\colon i \in N\},\ S \neq T\bigr\}\right].$$

By arguments similar to those used in Claims 3 and 4, $|D| \leqslant \omega_1$. By (iii) of Definition 1 there exists $A \in \mathcal{A}$ such that $(B_j - D) \cap A \neq \emptyset$ and $(A_i - D) \cap A \neq \emptyset$ for $j < \omega_1$ and $i \in N$. Then by our choice of D, there exists an uncountable subset $\{s_j\colon j < \omega_1\}$ of $A \cap f^{-1}(r)$ and a countably infinite subset $\{t_i\colon i \in N\}$ of $A \cap Z(f)$. The existence of these sets contradicts the continuity of f at $p(A)$, and we conclude that $L \cap Z(f)$ is finite as claimed—say $L \cap Z(f) = \{p(A_i)\colon i = 1, \dots, n\}$.

Then $S = \bigcup\{K(A_i)\colon i = 1, \dots, n\}$ is compact.

Finally, we claim that $H = Z(f) - S$ is a finite set (and hence $Z(f)$ is compact). Clearly $H \subset \kappa$. If H were infinite then by (ii) of Definition 1, there exists $A \in \mathcal{A}$ such that $A \cap H$ is countably infinite. This implies that $f(p(A)) = 0$, so A is one of the A_i which contradicts the fact that $S \cap A = \emptyset$. The claim follows.

Claim 6. *If $f \in C(X_\delta)$ then $f[X]$ is a countable set.*

To see this note that by Claim 5, $\beta X = X \cup \{p\}$, the one-point compactification of X. Clearly βX is scattered and $CB(\beta X) = 3$. By [10, 5.7] it follows that $(\beta X)_\delta$ is Lindelöf. As $\{f^{-1}(f(x))\colon x \in f[\beta X]\}$ is a partition of $(\beta X)_\delta$ into $(\beta X)_\delta$-open sets, it follows that $f[\beta X]$ is countable. As X is pseudocompact (by Claim 1), $\upsilon X = \beta X$. By [4, 5.6 and 5.7] $\upsilon(X_\delta) = (\upsilon X)_\delta = (\beta X)_\delta$. Thus X_δ is dense and C-embedded in $(\beta X)_\delta$, so $f[X]$ is countable.

Claim 7. *If $f \in C(X_\delta)$, there is a countable subset $\{p(A_i)\colon i \in N\}$ of L and a countable subset S of κ such that $|f[L - (S \cup (\bigcup_{i\in N} K(A_i)))]| = 1$. (In other words, f is constant on the complement of a σ-compact cozero-set of X.)*

To see this, note that there is an $r \in \Re$ such that $L - f^{-1}(r) = \{p(A_i)\colon\ i \in N\}$ (using Claim 4). Then as both $f^{-1}(r)$ and $\bigcup_{i\in N} K(A_i)$ are clopen in the P-space X_δ, the set $V = X - [f^{-1}(r) \cup \bigcup_{i\in N} K(A_i)]$ is a clopen subset of X_δ that is contained in κ. If V is uncountable, there is a bijection b from an uncountable subset $T \subset V$ onto a subset of $\Re$. Extend b so that $b[X_\delta - T] = 0$. Then $b \in C(X_\delta)$ and $|b[X_\delta]|$ is uncountable, contradicting Claim 6. Thus V is countable, and $(\bigcup_{i\in N} K(A_i)) \cup V$ is a σ-compact cozero-set of X whose complement is mapped by f to r.

Claim 8. *X is an RG-space and $rg(X) \leqslant 4$.*

Let $f \in C(X_\delta)$ and let $r \in \Re$ such that $L \cap f^{-1}(r)$ is a co-countable set of L. Let $g = f - \mathbf{r}$. Then $L \cap Z(g)$ is a co-countable subset of L and $\mathrm{coz}(g)$ is a σ-compact cozero set W of X (see Claim 7). Let $j \in C(X)$ such that $W = \mathrm{coz}(j)$.

Now $\mathrm{coz}(j)$ is a Lindelöf scattered space of *CB* index 1 or 2 so by [6, 2.11 and 2.12] $\mathrm{coz}(j)$ is an RG-space of *CB*-index no greater than 3. Hence there are $h_i, g_i \in C(\mathrm{coz}(j))$, $i = 1, 2, 3$, such that $g|\mathrm{coz}(j) = \sum_{i=1}^{3} h_i g_i^*$. By [2, 3.1] there are, for each i, s_i, t_i, u_i, and $w_i \in C(X)$ such that $h_i = (s_i t_i^*)|\mathrm{coz}(j)$ and $g_i = (u_i w_i^*)|\mathrm{coz}(j)$. It is a straightforward computation to show that $g = jj^*(\sum_{i=1}^{3}(s_i t_i^*)(u_i w_i^*)$. Thus $f = \mathbf{r} + \sum_{i=1}^{3}(j s_i u_i)(j t_i w_i)^*$ so X is an RG-space and $rg(X) \leqslant 4$.

Claim 9. *That X is almost-P is almost immediate. Suppose that $f \in C(X)$ is a function with non-empty zero-set $Z(f)$. If a point from κ lies in $Z(f)$ then $Z(F)$ has non-empty interior. If no point from κ lies in $Z(F)$ then f vanishes only at points $p(A)$. But if f vanishes at $p(A)$ it also vanishes at points of $K(A)$ by Claim 2.*

Claim 10. *Lastly we must show that we can get our spaces with arbitrary CB-index. This follows by repeatedly taking the product of X with the space N^* and using the arguments of the discussion that follows Theorem 2. The successive new spaces are RG, they are pseudocompact, they are almost compact, and their CB indices increase by 1 at each stage.* □

Remark 2. It is interesting to compare the space X of Theorem 6 with Ψ of [4, 5I]. Both spaces are scattered, locally compact, and pseudocompact of *CB* index 2. X is functionally countable even in the G_δ topology, and Ψ is functionally countable when it is almost compact. Yet X is RG and Ψ never is. It would be interesting to have a precise internal (i.e., without reference to the G_δ-topology) explanation as to why one is RG and the other is not (see open question 5 below).

Remark 3. Although X is scattered of *CB* index 2, the following considerations show that $G(X)$ does not have regularity degree 2 over $C(X)$.

First we need to describe the functions in $CX)$

Let $f \in C(X)$ with principal value r.

(i) If $f(p(A)) = s$ and if f differs from s on $p(A) \cup A$ at infinitely many points then the values (different from s) assumed on A by f must have r (and only r) as a limit point.
(ii) If there are countably infinitely many different points on the upper level where f is different from r then the values different from r must have r (and only r) as a limit point.

Part (i) holds because the relative topology on $p(A) \cup A$ is the one-point compactification of A.

Part (ii) follows from property (iii) of Definition 1 as follows: let $p(A_i)$ be a countably infinite set from the upper level on which f never equals r. Since f is bounded by Claim 1 the values $f(p(A_i))$ have a limit point say $r' \neq r$. Let $p(A'_j)$ be a set of ω_1 points on the upper level where f equals r. Let $\mathcal{B} = \{A_i, A'_j\}$. Let D be the union of the exceptional points in each of the sets A_i, A'_j. The set D has cardinality ω_1 because $\mathcal{B}$ does and because of property (i). By property (iii) there is a set $A \in \mathcal{A}$ that meets each set in $\mathcal{B}$. So A has ω_1 points where f equals r and it also has countably many points whose functional values converge to r'. This is not possible on the one-point compactification $p(A) \cup A$.

Now let us see why we do not have regularity degree 2 for X.

Choose a countably infinite set of points $\{p(A_n), n \in \omega\}$ from the upper level of X. Each one point compactification $\{p(A) \cup A\}$ is compact open in X. It is an easy consequence of the countability of the discrete set $\{p(A_n),\ n \in \omega\}$ (see [4, 3L.2]) and the zero-dimensionality of X, that one can find disjoint compact sets B_n each open in X, so that for each n one has $B_n \subset p(A) \cup A$ and $p(A_n) \in B_n$. Clearly each B_n is cocountable in $p(A_n) \cup A_n$ because the family $\mathcal{A}$ has pairwise intersections finite. Inside each B_n choose a countably infinite subset $C_n = \{c_{n,m}\}$ of points from κ.

Now define a function f as follows: f equals $n+1$ on $B_n \setminus C_n$, and $f(c_{n,m}) = n+m$. Also let $f = 1$ on $X \setminus \bigcup B_n$. It is clear that $f \in C(X_\delta)$, and that f has empty zeroset. Now suppose that there existed $a_1, a_2, b_1, b_2 \in C(X)$ so that $f = a_1(b_1)^* + a_2(b_2)^*$. If both b_1 and b_2 were non-zero at a point A_n then f coincides with $\frac{a_1}{b_1} + \frac{a_2}{b_2}$ on $\mathrm{coz}(b_1) \cap \mathrm{coz}(b_2)$ an open neighbourhood of A_n in the X-topology, i.e., on a cofinite subset (and therefore compact) of $p(A) \cup A_n$. But this is clearly false because f is unbounded on set C_n. Thus for each A_n we know that exactly one of the pair $\{b_1, b_2\}$ must vanish at $p(A_n)$ and there are infinitely many $p(A_n)$ that lie in the zero set of one of b_1, b_2, say with loss of generality b_1. Call them $p(A_{n_k})$. (Notice that the principal value of b_1 has to be zero and that from the principal value of f we get $1 = s_2/t_2$, where s_2, t_2 are respectively the principal values of a_2 and b_2. So for each n_k, $f(p(A_{n_k})) = a_2/b_2(p(A_{n_k})) \neq 0$. Since $t_2 \neq 0$, we have a contradiction because the values of f on the A_{n_k} are supposed to approach s_2/t_2, whereas they go to infinity.

Corollary 1. *It is non-decidable by* ZFC *whether there is a non-compact locally compact pseudocompact RG-space of cardinality ω_2.*

Proof. Under the GCH Lemmas 5 and 6 apply to ω_2 and give a non-compact example. On the other hand there are models of set theory in which $p > \omega_2$ and for them Theorem 4 gives compactness. □

Corollary 2. (ZFC) *There is a pseudocompact non-compact almost-P, RG-space of cardinality* $(2^{\omega_1})^+$.

Proof. The successor cardinal $(2^{\omega_1})^+$ is regular, and it satisfies the two exponential conditions of Theorem 5. □

Corollary 3. *There is a pseudocompact non-compact almost-P, RG-space of cardinality* 2^{ω_1}.

Proof. This is established by a reflection argument. One begins with an *RG-MAD* family $\mathcal{A}$ from $[(2^{\omega_1})^+]^{(2^{\omega_1})^+}$.

Take M an elementary submodel of the universe inside $H(\theta)$ with θ big enough. We ask that M be of cardinality 2^{ω_1} and that M be closed under ω_1-sequences. Let $\mathcal{A}'$ be the family $\{A \cap |M|\colon\ A \in \mathcal{A}\}$.

One checks that $\mathcal{A}'$ and $\bigcup \mathcal{A}'$ both have cardinality 2^{ω_1}.

Now we claim that $\mathcal{A}'$ is an *RG-MAD* family on M.

Property (1) holds trivially because the sets in $\mathcal{A}$ were almost disjoint to begin with.

For Property (2) let X be a countable subset of $M \cap (2^{\omega_1})^+$. Since M is closed under ω_1-sequences there is an A in M that works. For condition (3) let $\mathcal{B}' \subset [\mathcal{A}']^{\omega_1}$. Now note that since there is a 1–1 correspondence between the elements of $\mathcal{A}$ and those of $\mathcal{A}'$ we can let $\mathcal{B} \subset [\mathcal{A}]^{\omega_1}$ be defined by $\mathcal{B}' = \{B \cap |M|\colon\ B \in \mathcal{B}\}$.

Now condition (3) gives the existence of a set $D \in M \cap [(2^{\omega_1})^+]^{\omega_1}$.

Now by assumption $D \in M$ and $\mathcal{B} \in M$ because $\mathcal{A} \in M$. Now by elementarity, the formula (3) holds, i.e., $M \models \exists A \in \mathcal{A} \forall B \in \mathcal{B} \ni B \setminus D \cap A \neq \emptyset$ and since $A \in M$ the intersection with M is non-empty giving $(B \cap M \setminus D) \cap A \cap M \neq \emptyset$. □

6. Open questions

1. If $\kappa \geqslant \max(\omega_2, p)$ does there exist a pseudocompact non-compact RG-space of cardinality κ? If so then an example that is not locally compact exists by the discussion after Theorem 2.
2. Are separable pseudocompact RG-spaces compact? Again, if not, then there will be an example that is not locally compact.
3. Does Theorem 4 hold without assuming local compactness?
4. Let D^* be the one-point compactification of the uncountable discrete space D. Suppose the X is RG. Must it follow that $X \times D^*$ is RG?
5. Suppose that X is a pseudocompact scattered space of finite *CB* index. Give necessary and sufficient conditions for X to be RG. For example, give such conditions in the case where X is almost compact, and the *CB* index is 2.
6. Is there an example of a pseudocompact RG-space that is almost-P but not locally compact?

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