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Note on the 3-graph counting lemma

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Dedicated to Professor Miklós Simonovits on the occasion of his 60th birthday

Abstract

Szemerédi's *regularity lemma* proved to be a powerful tool in extremal graph theory. Many of its applications are based on the so-called *counting lemma*: if G is a k-partite graph with k-partition $V_1 \cup \cdots \cup V_k$, $|V_1| = \cdots = |V_k| = n$, where all induced bipartite graphs $G[V_i, V_j]$ are (d, ε)-regular, then the number of k-cliques K_k in G is $d^{\binom{k}{2}} n^k (1 \pm o(1))$. Frankl and Rödl extended Szemerédi's regularity lemma to 3-graphs and Nagle and Rödl established an accompanying 3-*graph counting lemma* analogous to the graph counting lemma above. In this paper, we provide a new proof of the 3-graph counting lemma.

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1. Introduction

Szemerédi's regularity lemma [20] is a powerful tool in combinatorics with many applications in *extremal graph theory*, *combinatorial number theory*, and *theoretical computer science* (see, e.g., the excellent surveys [8,9] for some of these applications). The lemma asserts that all large graphs can be decomposed into constantly many edge-disjoint, bipartite subgraphs, almost all of which behave "random-like" (see Theorem 1 below).

The broad applicability of Szemerédi's lemma to graph problems suggests that a regularity lemma for hypergraphs might render many applications. Frankl and Rödl [1] established such an extension, hereafter called the *FR-Lemma*, of the regularity lemma to 3-graphs or 3-uniform hypergraphs. (A 3-uniform hypergraph $\mathscr{H}$ on the vertex set V is a family of 3-element subsets of V, i.e., $\mathscr{H} \subseteq \binom{V}{3}$. Note that we identify hypergraphs with their edge set and we write $V(\mathscr{H})$ for the vertex set.) The FR-lemma guarantees that any large 3-graph admits a decomposition into constantly

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many edge-disjoint, tripartite subsystems, almost all of which behave "random-like." Applications of the FR-lemma to 3-graphs can be found in [1,4–6,10,11,15,16,18,19].

Most of the applications of the 3-graph regularity lemma are based on a structural counterpart, the so-called 3-*graph counting lemma*, which was first obtained by the first two authors [12]. As a cogent example, the counting lemma, working within the framework of the FR-lemma, gives a new proof of Szemerédi's theorem for arithmetic progressions of length four (see [1]) and its multidimensional version restricted to four points (see [19]). In this note we give an alternative proof of the 3-graph counting lemma, Theorem 5. This result was originally obtained by the first two authors [12] and follows also from the work of Peng, Skokan and the second author [14]. (In this latter reference, the authors show that hypergraph "regularity", defined precisely in Definition 3, is suitably preserved on complete underlying subgraphs, which then implies the counting lemma.) The proof presented here is substantially different. It is based on Szemerédi's regularity lemma and is somewhat simpler than the earlier proofs. The statement of Theorem 5 requires some notation and we begin by stating Szemerédi's regularity lemma precisely.

1.1. Szemerédi's regularity lemma

In this paper we write $x = y \pm \xi$ for reals x and y and some positive $\xi > 0$ for the inequalities $y - \xi \leqslant x \leqslant y + \xi$. Szemerédi's lemma pivots on the concept of an ε-*regular pair*. Let bipartite graph B be given with bipartition $X \cup Y$. We say the pair (X, Y) is (d, ε)-*regular* if for all $X' \subseteq X$ and $Y' \subseteq Y$ where $|X'| > \varepsilon|X|$ and $|Y'| > \varepsilon|Y|$, we have $d_B(X', Y') = d \pm \varepsilon$ where $d_B(X', Y') = |E(B[X', Y'])||X'|^{-1}|Y'|^{-1}$ is the *density* of the bipartite subgraph $B[X', Y']$ of B induced on $X' \cup Y'$. We say the pair (X, Y) is ε-*regular* if it is (d, ε)-regular for some d. In this paper, we use a well-known variant of Szemerédi's regularity lemma for k-partite graphs G, and therefore present Szemerédi's lemma in this context. Let k-partite graph G be given with k-partition $V = V(G) = V_1 \cup \cdots \cup V_k$.We say a refining partition $W_1^i \cup \cdots \cup W_t^i = V_i$, $1 \leqslant i \leqslant k$, is t-*equitable* if $|W_1^i| \leqslant \cdots \leqslant |W_t^i| \leqslant |W_1^i| + 1$. We say a t-equitable partition $W_1^i \cup \cdots \cup W_t^i = V_i$, $1 \leqslant i \leqslant k$, is ε-*regular* if for all $1 \leqslant i < j \leqslant k$, all but εt^2 pairs (W_a^i, W_b^j), $1 \leqslant a, b \leqslant t$, are ε-regular. Szemerédi's regularity lemma (for k-partite graphs) can then be stated[4] as follows.

Theorem 1 (*Szemerédi's regularity lemma*). *Let integer $k \geqslant 1$ and $\varepsilon > 0$ be given. There exist positive integers $N_0 = N_0(k, \varepsilon)$ and $T_0 = T_0(k, \varepsilon)$ such that any k-partite graph G on the vertex set $V = V_1 \cup \cdots \cup V_k$ with $|V_1|, \ldots, |V_k| \geqslant N_0$, admits an ε-regular and t-equitable partition $W_1^i \cup \cdots \cup W_t^i = V_i$ for $1 \leqslant i \leqslant k$, where $t \leqslant T_0$.*

Central to many applications of Szemerédi's regularity lemma is the assertion that any subgraph F of constant size may be embedded into an appropriately given collection of "dense and regular" pairs from an ε-regular and t-equitable partition. This observation is due to the counting lemma for graphs. For a graph G, we denote by $\mathscr{K}_s^{(2)}(G)$ the set of all s-tuples from $V(G)$ spanning cliques $K_s^{(2)}$ in G.

Fact 2 (*Counting lemma*). *For every integer $s \geqslant 2$ and constants $d > 0$ and $\gamma > 0$ there exists $\varepsilon > 0$ so that whenever G is an s-partite graph with vertex partition $V_1 \cup \cdots \cup V_s$ satisfying that all induced bipartite graphs $G[V_i, V_j]$, $1 \leqslant i < j \leqslant s$, are (d, ε)-regular and $|V_1| = \cdots = |V_s| = n$ for sufficiently large n, then $|\mathscr{K}_s^{(2)}(G)| = d^{\binom{s}{2}} n^s (1 \pm \gamma)$.*

1.2. The counting lemma for 3-graphs

In this section we introduce the notion of regular 3-graphs and state the 3-graph counting lemma. We omit a formulation of the FR-Lemma since it is somewhat technical and, in fact, is not needed to state the corresponding counting lemma. The following definition generalizes the notion of regular graphs to regular 3-graphs.

Definition 3 ((δ, r)-*regularity*). Let a positive integer $r \geqslant 1$ and constants $d \geqslant 0$ and $\delta > 0$ be given along with a 3-graph $\mathscr{H}$ and a 3-partite graph $P = P^{12} \cup P^{13} \cup P^{23}$. We say that $\mathscr{H}$ is (d, δ, r)-*regular with respect to* P if for any family

[4] There are other k-partite formulations of Szemerédi's regularity lemma. A possibly more common formulation would define t-equitable partitions as $W_0^i \cup W_1^i \cup \cdots \cup W_t^i = V_i$, $1 \leqslant i \leqslant t$, where $|W_0^i| < t$ and $|W_1^i| = \cdots = |W_t^i|$ (W_0^i, $1 \leqslant i \leqslant t$, is often referred to as a "garbage" class). Then ε-regular, t-equitable partitions would be defined otherwise the same as we did for Theorem 1; for each $1 \leqslant i < j \leqslant k$, all but εt^2 pairs (W_a^i, W_b^j), $1 \leqslant a, b \leqslant t$, are ε-regular. These two notions of t-equitable ε-regular partitions are the equivalent, however, up to a slight change in ε.

$\boldsymbol{Q} = \{Q_1, \ldots, Q_r\}$ of r subgraphs of P with

$$\left|\bigcup_{i=1}^{r} \mathcal{K}_3^{(2)}(Q_i)\right| > \delta|\mathcal{K}_3^{(2)}(P)| \quad \text{we have } |d_{\mathcal{H}}(\boldsymbol{Q}) - d| < \delta,$$

where

$$d_{\mathcal{H}}(\boldsymbol{Q}) = \begin{cases} \dfrac{|\mathcal{H} \cap \bigcup_{i=1}^{r} \mathcal{K}_3^{(2)}(Q_i)|}{|\bigcup_{i=1}^{r} \mathcal{K}_3^{(2)}(Q_i)|} & \text{if } |\bigcup_{i=1}^{r} \mathcal{K}_3^{(2)}(Q_i)| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

is the *density* of $\mathcal{H}$ on $\boldsymbol{Q}$. We say $\mathcal{H}$ is (δ, r)-*regular with respect to P* if it is (d, δ, r)-regular with respect to P for some $d \geqslant 0$.

In most contexts where $\mathcal{H}$ is (d, δ, r)-regular w.r.t. P, we actually have $\mathcal{H} \subseteq \mathcal{K}_3^{(2)}(P)$. This assumption, however, is not needed to state Definition 3. Moreover, we note that Definition 3 allows some members Q_i of $\boldsymbol{Q}$ to be empty.

While Szemerédi's regularity lemma decomposes the vertex set of a graph, the 3-graph regularity lemma partitions not only the vertex set, but also partitions the set of all pairs between any two such vertex classes into edge-disjoint bipartite graphs. In that environment, the concept corresponding to an ε-regular pair is that of Definition 3, where the three bipartite graphs P^{12}, P^{13}, and P^{23} are also regular (in the sense of Szemerédi). Consequently, a corresponding generalization of Fact 2 takes place in the following environment.

Setup 4. *Let positive integers k, r and n and positive constants d_3, δ_3, d_2 and δ_2 be given. Suppose*

(1) $V = V_1 \cup \cdots \cup V_k$, $|V_1| = \cdots = |V_k| = n$, *is a partition of vertex set V.*
(2) $P = \bigcup_{1 \leqslant i<j \leqslant k} P^{ij}$ *is a k-partite graph, with vertex set V and k-partition above, where all* $P^{ij} = P[V_i, V_j]$, $1 \leqslant i < j \leqslant k$, *are* (d_2, δ_2)-*regular.*
(3) $\mathcal{H} = \bigcup_{1 \leqslant h<i<j \leqslant k} \mathcal{H}^{hij} \subseteq \mathcal{K}_3(P)$ *is a k-partite 3-graph, with vertex set V and k-partition above, where all* $\mathcal{H}^{hij} = \mathcal{H}[V_h, V_i, V_j]$, $1 \leqslant h < i < j \leqslant k$, *are* (d_3, δ_3, r)-*regular with respect to* $P^{hi} \cup P^{ij} \cup P^{hj}$.

The counting lemma estimates the number of hypercliques, i.e., complete 3-graphs, $K_k^{(3)}$ in $\mathcal{H}$. We denote by $\mathcal{K}_k^{(3)}(\mathcal{H})$ the set of all k-tuples from $V(\mathcal{H})$ spanning hypercliques $K_k^{(3)}$ in $\mathcal{H}$.

Theorem 5 (*Counting lemma, Nagle and Rödl [12]*). *Let $k \geqslant 3$ be an integer. For every $\gamma > 0$ and $d_3 > 0$ there exists $\delta_3 > 0$ so that for all $d_2 > 0$ there exist integer r and $\delta_2 > 0$ and n sufficiently large so that with these constants, if $\mathcal{H}$ and P are as in Setup 4, then*

$$|\mathcal{K}_k^{(3)}(\mathcal{H})| = d_3^{\binom{k}{3}} d_2^{\binom{k}{2}} n^k (1 \pm \gamma).$$

Proving Theorem 5 is the content of this paper. The first proof of Theorem 5 appeared in [12] and another proof by Peng, Skokan, and one of the authors was given in [14]. The proof we present here is shorter than the previous ones and we believe it is also simpler. We present our proof in Section 2 and conclude this introduction with the following remarks.

The main problem of proving Theorem 5 is working with the given quantification of constants: $\forall \gamma, d_3, \exists \delta_3$: $\forall d_2, \exists \delta_2, \exists r$. This quantification, consistent with the output of the 3-graph regularity lemma, allows for the graph P to be relatively "sparse" compared to δ_3, the measure of regularity of the 3-graph $\mathcal{H}$. If the quantification of constants were allowed as $\forall \gamma, d_3, d_2, \exists \delta_3 = \delta_2$, then such a "dense" version of Theorem 5 is simpler to prove and was proved in [7]. In the present paper, we use Szemerédi's regularity lemma, Theorem 1, to overcome those difficulties arising from the quantification of constants in Theorem 5. Recently Gowers [2,3] developed a regularity lemma and a corresponding counting lemma for ℓ-graphs for general $\ell \geqslant 3$. The approach in [2,3] is different and, for e.g., $\ell = 3$ the notion of 3-graph regularity there differs from that in Definition 3. A regularity lemma for ℓ-graphs ($\ell \geqslant 3$) extending the notion of (δ, r)-regularity was proved by Rödl and Skokan [17] and the current authors [13] proved an accompanying

ℓ-graph counting lemma for that regularity lemma. The proof of the general counting lemma in [13] was inspired by the main idea presented here, i.e., it uses the regularity lemma for ℓ-graphs to overcome difficulties, which are similar to those indicated in the previous remark.

2. Proof of the 3-graph counting lemma

It was shown in [12] that the full statement of Theorem 5 can be deduced from just the lower bound. Hence it suffices to prove the lower bound of Theorem 5 only.

Our proof of Theorem 5 proceeds by induction on $k \geqslant 3$. The base case $k = 3$ is trivial. Indeed, by Definition 3, $\mathscr{H} = \mathscr{H}^{123}$ has (relative) density $d_3 \pm \delta_3$ with respect to $P = P^{12} \cup P^{23} \cup P^{13}$. Fact 2 implies that (with $\delta_2 \ll \gamma$) $|\mathscr{K}_3^{(2)}(P)| = d_2^3 n^3 (1 \pm \gamma/2)$ and the lower bound of Theorem 5 for $k = 3$ then follows from $\delta_3 \ll \gamma$.

To proceed to the induction step, we assume that Theorem 5 holds for $k - 1$. Recalling the quantification of Theorem 5, which is $\forall \gamma, d_3, \exists d_3 : \forall d_2, \exists \delta_2, \exists r$, we may assume that

$$\frac{1}{k}, \frac{\gamma}{2}, d_3 \gg \delta_3 \geqslant \min\{\delta_3, d_2\} \gg \delta_2, \frac{1}{r} \gg \frac{1}{n} \tag{1}$$

holds. Then for a given graph P and a 3-graph $\mathscr{H}$ as in Setup 4, we show $|\mathscr{K}_k^{(3)}(\mathscr{H})| \geqslant d_3^{\binom{k}{3}} d_2^{\binom{k}{2}} n^k (1 - \gamma)$.

We now refine the hierarchy in (1) and introduce some further auxiliary constants. Let $\varepsilon_0 > 0$ and integer $r' > 0$ be chosen so that both ε_0, $1/r' \ll \min\{d_2, \delta_3\}$. Let $T_0 = T_0(k - 1, \varepsilon_0)$ be the constant guaranteed by Szemerédi's regularity lemma, Theorem 1. We choose $\delta_2 > 0$ so small and integers r and n so large (which complies with the quantification of Theorem 5) that the hierarchy (1) extends to

$$\frac{1}{k}, \gamma, d_3 \gg \delta_3 \geqslant \min\{\delta_3, d_2\} \gg \varepsilon_0, \frac{1}{r'}, \frac{1}{T_0} \gg \delta_2, \frac{1}{r} \gg \frac{1}{n}. \tag{2}$$

Before going into the precise details of the induction step, we first give an informal description of the proof.

2.1. Outline of the induction step

The so-called link graphs of $\mathscr{H}$ play a central rôle in our proof of the induction step. In the context of Setup 4, consider a vertex $v \in V_1$ and fix $2 \leqslant i < j \leqslant k$. The (i, j)-*link graph* L_v^{ij} is defined[5] as $L_v^{ij} = \{\{v_i, v_j\} \in P^{ij} : \{v, v_i, v_j\} \in \mathscr{H}\}$ and the *link graph* L_v of v is then set as $L_v = \bigcup_{2 \leqslant i < j \leqslant k} L_v^{ij}$. (Note that L_v is a $(k - 1)$-partite graph.)

A natural place to consider applying the induction hypothesis on the counting lemma is to enumerate cliques $K_{k-1}^{(3)}$ in the $(k - 1)$-partite hypergraph $\mathscr{H} \cap \mathscr{K}_3^{(2)}(L_v)$ (with the $(k - 1)$-partite graph L_v), where $v \in V_1$ is a "typical" vertex. (Indeed, a clique $K_{k-1}^{(3)}$ in $\mathscr{H} \cap \mathscr{K}_3^{(2)}(L_v)$ corresponds to a clique $K_k^{(3)}$ in $\mathscr{H}$ containing the vertex v). For this, one would need to check that the hypothesis of the counting lemma is met (for $(k - 1)$) by $\mathscr{H} \cap \mathscr{K}_3^{(2)}(L_v)$ and L_v (replacing $\mathscr{H}$ and P, as in Setup 4). Unfortunately, this would not often be the case. Indeed, one may show that while the density of the bipartite graphs L_v^{ij} (for most $v \in V_1$), $1 \leqslant i < j \leqslant k$, is about $d_2 d_3$, the regularity of L_v^{ij} depends on δ_3. As we see in (2), $\delta_3 \gg d_3 d_2$, and to apply the induction hypothesis, we would need it the other way around.

The main idea of our proof is to apply the Szemerédi regularity lemma, Theorem 1, to the link graphs L_v, i.e., we 'regularize' the links. With $\varepsilon_0 \ll d_2 d_3$ (cf. (2)), we will regularize each L_v to obtain ε_0-regular partition $\boldsymbol{P}_v$ given by $V_i = W_1^{v,i} \cup \cdots \cup W_{t_v}^{v,i}$, $2 \leqslant i \leqslant k$, where $t_v \leqslant T_0$ for the constant T_0 appearing in (2). We will then show that for each $2 \leqslant i < j \leqslant k$, for most $v \in V_1$, most of the pairs $W_a^{v,i}$, $W_b^{v,j}$, $1 \leqslant a, b \leqslant t_v$, will have density in L_v^{ij} close to $d_2 d_3$ (see part (i) of Claim 7). (Of course, most of these pairs $W_a^{v,i}$, $W_b^{v,j}$ are ε_0-regular where $\varepsilon_0 \ll d_2 d_3$). Showing this will involve using the (d_3, δ_3, r)-regularity of $\mathscr{H}^{1ij}$ w.r.t. $P^{1i} \cup P^{1j} \cup P^{ij}$ and the choice $r \gg T_0$. We will then show that for all $2 \leqslant h < i < j \leqslant k$, for most $v \in V_1$, most triples $W_a^{v,h}$, $W_b^{v,i}$, $W_c^{v,j}$, $1 \leqslant a, b, c \leqslant t_v$, will satisfy that $\mathscr{H}^{hij} \cap \mathscr{K}_3^{(2)}(L_v)$

[5] Note that L_v^{ij} has vertex set $N_{P^{1i}}(v) \cup N_{P^{1j}}(v)$ where, for example, $N_{P^{1i}}(v)$ is the P^{1i}-neighborhood of the vertex v. Note that L_v^{ij} is a subgraph of P_v^{ij}, where $P_v^{ij} = P^{ij}[N_{P^{1i}}(v), N_{P^{1j}}(v)]$ is the subgraph of P^{ij} induced on the neighborhoods $N_{P^{1i}}(v)$ and $N_{P^{1j}}(v)$.

is $(d_3, \delta_3^{1/20}, r')$-regular w.r.t. $L_v[W_a^{v,h}, W_b^{v,i}, W_c^{v,j}]$ (see part (ii) of Claim 7). Showing this will involve using the (d_3, δ_3, r)-regularity of $\mathscr{H}^{hij}$ w.r.t. $P^{hi} \cup P^{hj} \cup P^{ij}$ and the choice $r \gg \max\{r', T_0\}$.

From the two observations above, we then infer that for most $v \in V_1$, most $(k-1)$-partite graphs $L_v[W_{a_2}^{v,2}, \ldots, W_{a_k}^{v,k}]$, $1 \leqslant a_2, \ldots, a_k \leqslant t_v$, and corresponding 3-graphs $\mathscr{H} \cap \mathscr{K}_3^{(2)}(L_v[W_{a_2}^{v,2}, \ldots, W_{a_k}^{v,k}])$ satisfy the hypothesis (for $(k-1)$) of the counting lemma. (That is, after the adjustment of regularizing the links, we are in a position of using the induction hypothesis (within the pieces)). We then use the induction hypothesis to count the cliques $K_{k-1}^{(3)}$ in $\mathscr{H} \cap \mathscr{K}_3^{(2)}(L_v[W_{a_2}^{v,2}, \ldots, W_{a_k}^{v,k}])$. We then add over all suitable choices of indices $1 \leqslant a_2, \ldots, a_k \leqslant t_v$ and then add over all suitable choices of vertices $v \in V_1$.

We now formalize the details sketched above.

2.2. Transversals and their properties

Let the constants be fixed as in (2) and a k-partite graph P and a 3-graph $\mathscr{H}$ be given as in Setup 4. We first regularize the link graphs. For every vertex $v \in V_1$, we apply Szemerédi's regularity lemma, Theorem 1 with ε_0, to the $(k-1)$-partite link graph L_v to obtain an ε_0-regular and t_v-equitable partition $\boldsymbol{P}_v$ of $V(L_v)$, where $t_v \leqslant T_0$ (see (2)). In other words, $\boldsymbol{P}_v$ refines the partition $N_{P^{12}}(v) \cup \cdots \cup N_{P^{1k}}(v) = V(L_v)$, i.e., we obtain $W_1^{v,i} \cup \cdots \cup W_{t_v}^{v,i} = N_{P^{1i}}(v)$ for $i = 2, \ldots, k$, where for all pairs $2 \leqslant i < j \leqslant k$ all but at most $\varepsilon_0 t_v^2$ pairs $(a, b) \in [t_v] \times [t_v]$ satisfy that $L_v^{ij}[W_a^{v,i}, W_b^{v,j}]$ is ε_0-regular.

For a fixed $v \in V_1$ and a fixed $(k-1)$-vector $\boldsymbol{a}_v = (a_2, \ldots, a_k) \in [t_v] \times \cdots \times [t_v] = [t_v]^{k-1}$ we denote by $L_v(\boldsymbol{a}_v)$ the subgraph of L_v induced on the sets $W_{a_2}^{v,2}, \ldots, W_{a_k}^{v,k}$, i.e.,

$$L_v(\boldsymbol{a}_v) = \bigcup_{2 \leqslant i<j \leqslant k} L_v^{ij}[W_{a_i}^{v,i}, W_{a_j}^{v,j}] = L_v[W_{a_2}^{v,2}, \ldots, W_{a_k}^{v,k}]. \tag{3}$$

Similarly, we define for all $2 \leqslant h < i < j \leqslant k$ and $(a_h, a_i, a_j) \in [t_v]^3$

$$L_v^{hij}[a_h, a_i, a_j] = L_v^{hi}[W_{a_h}^{v,h}, W_{a_i}^{v,i}] \cup L_v^{ij}[W_{a_i}^{v,i}, W_{a_j}^{v,j}] \cup L_v^{hj}[W_{a_h}^{v,h}, W_{a_j}^{v,j}]. \tag{4}$$

Moreover, we set $\mathscr{H}(\boldsymbol{a}_v)$ to be equal to the 3-graph $\mathscr{H}$ induced on the triangles of $L_v(\boldsymbol{a}_v)$, i.e.,

$$\mathscr{H}(\boldsymbol{a}_v) = \mathscr{H} \cap \mathscr{K}_3^{(2)}(L_v(\boldsymbol{a}_v)) = \bigcup_{2 \leqslant h<i<j \leqslant k} \mathscr{H}^{hij}(\boldsymbol{a}_v), \tag{5}$$

where $\mathscr{H}^{hij}(\boldsymbol{a}_v) = \mathscr{H}^{hij} \cap \mathscr{K}_3^{(2)}(L_v^{hij}[a_h, a_i, a_j])$.
We refer to the objects $\mathscr{H}(\boldsymbol{a}_v)$ and $L_v(\boldsymbol{a}_v)$ as *transversals* of the partition $\boldsymbol{P}_v$ (see Fig. 1).

Note that as L_v was regularized, we infer that all but $\varepsilon_0 k^2 t_v^{k-1}$ vectors $\boldsymbol{a}_v = (a_2, \ldots, a_k) \in [t_v]^{k-1}$ satisfy that all $\binom{k-1}{2}$ bipartite graphs $L_v^{ij}[W_{a_i}^{v,i}, W_{a_j}^{v,j}]$, $2 \leqslant i < j \leqslant k$, are ε_0-regular.

It follows directly from the definitions in (3) and (5) that

$$|\mathscr{K}_k^{(3)}(\mathscr{H})| = \sum_{v \in V_1} \sum_{\boldsymbol{a}_v \in [t_v]^{k-1}} |\mathscr{K}_{k-1}^{(3)}(\mathscr{H}(\boldsymbol{a}_v))|. \tag{6}$$

In our proof of the induction step we will use the following well-known fact about the size of typical neighborhoods in δ_2-regular graphs (see, e.g., [8, Fact 1]).

Fact 6. *For all but $2k\delta_2 n$ vertices $v \in V_1$, we have $|N_{P^{1i}}(v)| = (d_2 \pm \delta_2)n$, for all $2 \leqslant i \leqslant k$.*

For future reference, we set

$$V_1' = \{v \in V_1 : |N_{P^{1i}}(v)| = (d_2 \pm \delta_2)n, \text{ for all } 2 \leqslant i \leqslant k\}, \tag{7}$$

so that Fact 6 implies $|V_1'| \geqslant (1 - 2k\delta_2)n$. The following claim is the key observation for the proof of Theorem 5. While technical looking, part (i) of Claim 7 follows from standard arguments, which we present in Section 4. The proof of part (ii) is given in Section 5.

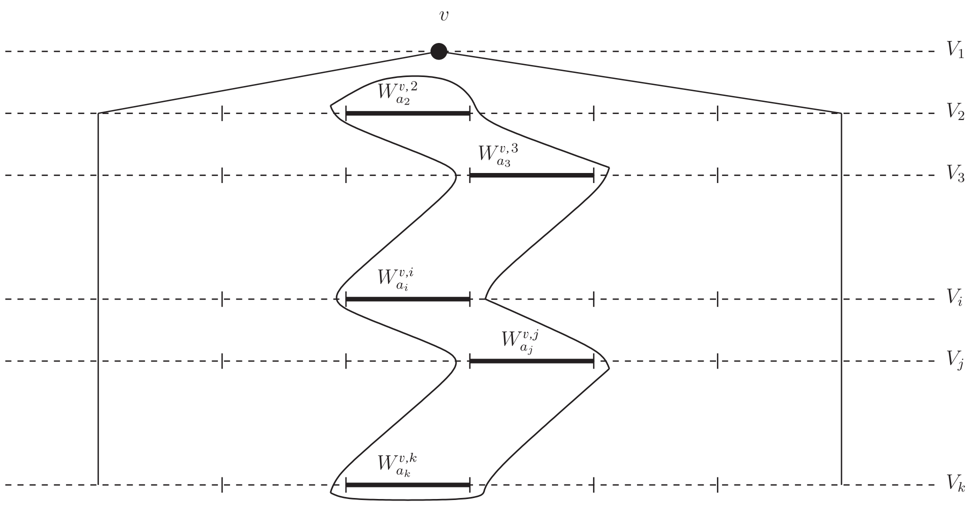

Fig. 1. A transversal of the partition $\boldsymbol{P}_v$.

Claim 7. *For all but $\delta_3^{1/4}n$ vertices $v \in V_1'$ (see (7)), all but $\delta_3^{1/20}k^3t_v^{k-1}$ vectors $\boldsymbol{a}_v = (a_2, \ldots, a_k) \in [t_v]^{k-1}$ yield transversals $L_v(\boldsymbol{a}_v)$ and $\mathscr{H}(\boldsymbol{a}_v)$ satisfying that*

(i) *for all $2 \leqslant i < j \leqslant k$ the bipartite graphs $L_v^{ij}[W_{a_i}^{v,i}, W_{a_j}^{v,j}]$ have density $d_2d_3(1 \pm \delta_3^{1/4})$ and (due to regularization) are ε_0-regular,*

(ii) *for all $2 \leqslant h < i < j \leqslant k$ the 3-partite 3-graph $\mathscr{H}^{hij}(\boldsymbol{a}_v)$ is $(d_3, \delta_3^{1/20}, r')$-regular with respect to the 3-partite graphs $L_v^{hij}[a_h, a_i, a_j]$, where r' is given in (2) (recall the notation in (4)).*

Let V_1^{typ} denote the set of "typical" vertices $v \in V_1$ to which Fact 6 and Claim 7 refer. For each $v \in V_1^{\mathrm{typ}}$, let $[t_v]_{\mathrm{typ}}^{k-1}$ denote the set of "typical" vectors $\boldsymbol{a}_v \in [t_v]^{k-1}$ to which Claim 7 refers.

2.3. The induction step

We conclude from Fact 6 and Claim 7 above that for any vertex $v \in V_1^{\mathrm{typ}}$ and any $\boldsymbol{a}_v \in [t_v]_{\mathrm{typ}}^{k-1}$, the transversals $\mathscr{H}(\boldsymbol{a}_v)$ and $L_v(\boldsymbol{a}_v)$, satisfy the hypothesis of Setup 4 with the constants $k-1$, d_3, $\delta_3^{1/20}$, d_2d_3, ε_0, r' and d_2n/t_v. Indeed, as in Setup 4, observe that $\mathscr{H}(\boldsymbol{a}_v)$ replaces $\mathscr{H}$, $L_v(\boldsymbol{a}_v)$ replaces P, $k-1$ replaces k, d_3 remains d_3, $\delta_3^{1/20}$ replaces δ_3, d_2d_3 replaces d_2, ε_0 replaces δ_2 and d_2n/t_v replaces n. (We will take $\gamma/2$ to replace γ)

Due to the hierarchy of the constants in (2), we may assume that

$$\frac{1}{k-1}, \frac{\gamma}{2}, d_3 \gg \delta_3^{1/20} \geqslant \min\{\delta_3^{1/20}, d_2d_3\} \gg \varepsilon_0, \frac{1}{r'} \gg \frac{t_v}{d_2n}. \tag{8}$$

As such, for fixed $v \in V_1^{\mathrm{typ}}$ and $\boldsymbol{a}_v = (a_2, \ldots, a_k) \in [t_v]_{\mathrm{typ}}^{k-1}$, we may apply the induction hypothesis to the transversals $\mathscr{H}(\boldsymbol{a}_v)$ and $L_v(\boldsymbol{a}_v)$ and infer

$$\begin{aligned} |\mathscr{K}_{k-1}^{(3)}(\mathscr{H}(\boldsymbol{a}_v))| &\geqslant d_3^{\binom{k-1}{3}}(d_2d_3)^{\binom{k-1}{2}}\left(\frac{d_2n}{t_v}\right)^{k-1}\left(1-\frac{\gamma}{2}\right) \\ &= d_3^{\binom{k}{3}} d_2^{\binom{k}{2}} \frac{n^{k-1}}{t_v^{k-1}}\left(1-\frac{\gamma}{2}\right). \end{aligned} \tag{9}$$

Consequently, by (6) we have

$$|\mathscr{K}_k^{(3)}(\mathscr{H})| = \sum_{v\in V_1} \sum_{\boldsymbol{a}_v\in[t_v]^{k-1}} |\mathscr{K}_{k-1}^{(3)}(\mathscr{H}(\boldsymbol{a}_v))|$$

$$\overset{(9)}{\geqslant} d_3^{\binom{k}{3}} d_2^{\binom{k}{2}} n^{k-1}\left(1-\frac{\gamma}{2}\right) \sum_{v\in V_1^{\mathrm{typ}}} \frac{|[t_v]_{\mathrm{typ}}^{k-1}|}{t_v^{k-1}}.$$

By Fact 6 and Claim 7, $|V_1^{\mathrm{typ}}| \geqslant (1-\delta_3^{1/4}-2k\delta_2)n > (1-2\delta_3^{1/4})n$ and $|[t_v]_{\mathrm{typ}}^{k-1}| \geqslant (1-k^3\delta_3^{1/20})t_v^{k-1}$ for every $v \in V_1^{\mathrm{typ}}$. Hence we conclude (due to the hierarchy in (8)) that

$$|\mathscr{K}_k^{(3)}\mathscr{H})| \geqslant d_3^{\binom{k}{3}} d_2^{\binom{k}{2}} n^k \left(1-\frac{\gamma}{2}\right)(1-2\delta_3^{1/4})(1-k^3\delta_3^{1/20}) \geqslant d_3^{\binom{k}{3}} d_2^{\binom{k}{2}} n^k (1-\gamma).$$

This concludes our proof of Theorem 5.

3. Proof of Claim 7

In this section, we outline our strategies for proving parts (i) and (ii) of Claim 7. To begin, we find the following notation helpful to discuss Claim 7 and use it in the remainder of this paper.

Definition 8. Fix $v \in V_1$. For fixed $2 \leqslant i < j \leqslant k$ and $2 \leqslant h < i$, set

$$\mathrm{L}_{\mathrm{good}}^{ij}(v) = \{(a,b) \in [t_v]^2 \colon L_v^{ij}[W_a^{v,i}, W_b^{v,j}] \text{ is } (d,\varepsilon_0)\text{-regular for } d = d_2 d_3(1 \pm \delta_3^{1/4})\},$$

$$\mathrm{L}_{\mathrm{good}}^{hij}(v) = \{(a,b,c) \in [t_v]^3 \colon (a,b) \in \mathrm{L}_{\mathrm{good}}^{hi}(v),\ (b,c) \in \mathrm{L}_{\mathrm{good}}^{ij}(v),\ (a,c) \in \mathrm{L}_{\mathrm{good}}^{hj}(v)\},$$

$$\mathrm{H}_{\mathrm{good}}^{hij}(v) = \{(a,b,c) \in [t_v]^3 \colon \mathscr{H}^{hij} \text{ is } (d_3, \delta_3^{1/20}, r')\text{-regular w.r.t } L_v^{hij}[a,b,c]\},$$

where $L_v^{hij}[a,b,c]$ was defined in (4). Finally, set

$$\mathrm{L}_{\mathrm{good}}(v) = \{\boldsymbol{a}_v \in [t_v]^{k-1} \colon (a_i, a_j) \in \mathrm{L}_{\mathrm{good}}^{ij}(v) \text{ for all } 2 \leqslant i < j \leqslant k\},$$

$$\mathrm{H}_{\mathrm{good}}(v) = \{\boldsymbol{a}_v \in [t_v]^{k-1} \colon (a_h, a_i, a_j) \in \mathrm{H}_{\mathrm{good}}^{hij}(v) \text{ for all } 2 \leqslant h < i < j \leqslant k\}.$$

We also define corresponding "bad" sets and fix

$$\mathrm{L}_{\mathrm{bad}}^{ij}(v) = [t_v]^2 \backslash \mathrm{L}_{\mathrm{good}}^{ij}(v),\ \ \mathrm{L}_{\mathrm{bad}}^{hij}(v) = [t_v]^3 \backslash \mathrm{L}_{\mathrm{good}}^{hij}(v),\ \ \mathrm{H}_{\mathrm{bad}}^{hij}(v) = [t_v]^3 \backslash \mathrm{H}_{\mathrm{good}}^{hij}(v),$$

$$\mathrm{L}_{\mathrm{bad}}(v) = [t_v]^{k-1} \backslash \mathrm{L}_{\mathrm{good}}(v) \quad \text{and} \quad \mathrm{H}_{\mathrm{bad}}(v) = [t_v]^{k-1} \backslash \mathrm{H}_{\mathrm{good}}(v).$$

In the notation above, Claim 7 asserts that all but $\delta_3^{1/4} n$ vertices $v \in V_1'$ (see (7)) satisfy

$$|\mathrm{L}_{\mathrm{bad}}(v)| + |\mathrm{H}_{\mathrm{bad}}(v)| \leqslant \delta_3^{1/20} t_v^{k-1}.$$

We consider the sum on the left hand side of the inequality above. Observe that

$$|\mathrm{L}_{\mathrm{bad}}(v)| + |\mathrm{H}_{\mathrm{bad}}(v)| = |\mathrm{L}_{\mathrm{bad}}(v)| + |\mathrm{H}_{\mathrm{bad}}(v) \cap \mathrm{L}_{\mathrm{good}}(v)| + |\mathrm{H}_{\mathrm{bad}}(v) \cap \mathrm{L}_{\mathrm{bad}}(v)|$$

$$\leqslant 2|\mathrm{L}_{\mathrm{bad}}(v)| + |\mathrm{H}_{\mathrm{bad}}(v) \cap \mathrm{L}_{\mathrm{good}}(v)|.$$

Moreover, observe that

$$|\mathrm{L}_{\mathrm{bad}}(v)| \leqslant t_v^{k-3} \sum_{2\leqslant i<j\leqslant k} |\mathrm{L}_{\mathrm{bad}}^{ij}(v)|$$

and

$$|\mathrm{H}_{\mathrm{bad}}(v)\cap \mathrm{L}_{\mathrm{good}}(v)|\leqslant t_v^{k-4}\sum_{2\leqslant h<i<j\leqslant k}|\mathrm{H}^{hij}_{\mathrm{bad}}(v)\cap \mathrm{L}^{hij}_{\mathrm{good}}(v)|$$

hold for all $v\in V_1'$. We may therefore give reformulations of parts (i) and (ii) from Claim 7 in the following form.

Proposition 9 (*Claim 7 part (i)*). *Let P and $\mathscr{H}$ satisfy Setup* 4 *with constants as in* (2). *Then all but $2k^2\delta_3^{1/2}n$ vertices $v\in V_1'$ (see* (7)) *satisfy that $|\mathrm{L}^{ij}_{\mathrm{bad}}(v)|\leqslant 3\delta_3^{1/4}t_v^2$ for all $2\leqslant i<j\leqslant k$.*

Proposition 10 (*Claim 7 part (ii)*). *Let P and $\mathscr{H}$ satisfy Setup* 4 *with constants as in* (2). *Then all but $k^3\delta_2^{1/4}n$ vertices $v\in V_1'$ (see* (7)) *satisfy that $|\mathrm{L}^{hij}_{\mathrm{good}}(v)\cap \mathrm{H}^{hij}_{\mathrm{bad}}(v)|<2\delta_3^{1/20}t_v^3$ for all $2\leqslant h<i<j\leqslant k$.*

Propositions 9 and 10 together imply that all but $2k^2\delta_3^{1/2}n+k^3\delta_2^{1/4}n\leqslant\delta_3^{1/4}n$ vertices $v\in V_1'$ satisfy

$$\begin{aligned}
&2|\mathrm{L}_{\mathrm{bad}}(v)|+|\mathrm{H}_{\mathrm{bad}}(v)\cap \mathrm{L}_{\mathrm{good}}(v)|\\
&\quad\leqslant 2t_v^{k-3}\sum_{2\leqslant i<j\leqslant k}|\mathrm{L}^{ij}_{\mathrm{bad}}(v)|+t_v^{k-4}\sum_{2\leqslant h<i<j\leqslant k}|\mathrm{H}^{hij}_{\mathrm{bad}}(v)\cap \mathrm{L}^{hij}_{\mathrm{good}}(v)|\\
&\quad\leqslant 6\delta_3^{1/4}\binom{k}{2}t_v^{k-1}+2\delta_3^{1/20}\binom{k}{3}t_v^{k-1}\leqslant\delta_3^{1/20}k^3t_v^{k-1},
\end{aligned}$$

as promised by Claim 7.

We give the proofs of Propositions 9 and 10 in Sections 4 and 5, respectively.

4. Proof of Proposition 9

Let P and $\mathscr{H}$ be given as in Setup 4 where the constants satisfy (2). Moreover, let $\{\boldsymbol{P}_v\}_{v\in V_1}$ be the family of ε_0-regular, t_v-equitable partitions obtained in Section 2.2. We prove that all but $2k^2\delta_3^{1/2}n$ vertices $v\in V_1'$ (see (7)) satisfy $|\mathrm{L}^{ij}_{\mathrm{bad}}(v)|\leqslant 3\delta_3^{1/4}t_v^2$ for all $2\leqslant i<j\leqslant k$. Let us clarify this goal. Fix $2\leqslant i<j\leqslant k$. Since $\boldsymbol{P}_v$ is ε_0-regular for every $v\in V_1$, at most $\varepsilon_0t_v^2\leqslant\delta_3^{1/4}t_v^2$ pairs $(W_a^{v,i},W_b^{v,j})$, $1\leqslant a,b\leqslant t_v$, can be irregular. Hence we only have to verify the density assertion of Proposition 9, namely, for all but $2\delta_3^{1/2}n$ vertices $v\in V_1'$,

$$d_{L_v^{ij}}(W_a^{v,i},W_b^{v,j})=d_2d_3(1\pm\delta_3^{1/4}),$$

holds for all but $3\delta_3^{1/4}t_v^2$ pairs $(W_a^{v,i},W_b^{v,j})$. We begin with the following definition.

Definition 11. Let $L\subseteq P$ be bipartite graphs with bipartition $U_1\cup U_2$ and let $d,\delta>0$ and integer r be given. We say L is (d,δ,r)-regular with respect to P if every family $\boldsymbol{B}=\{B_1,\dots,B_r\}$ of r *induced* subgraphs $B_i\subseteq P$ satisfying $|\bigcup_{s=1}^r B_s|>\delta|P|$ also satisfies $|L\cap\bigcup_{s=1}^r B_s|=(d\pm\delta)|\bigcup_{s=1}^r B_s|$.

The following fact appeared (in slightly different language) in [1, Claim A] (see also [12]). It asserts that for $\mathscr{H}$ and P as in Setup 4, most vertices $v\in V_1$ satisfy that their links L_v^{ij}, $2\leqslant i\leqslant j\leqslant k$, are regular in the sense of Definition 11.

Fact 12 (*most links are $(d_3,2\delta_3^{1/2},r)$-regular*). *Let k, d_3, δ_3, d_2 and r be given as in* (2). *Then for $\mathscr{H}$ and P are as in Setup* 4, *all but $2k^2\delta_3^{1/2}n$ vertices $v\in V_1'$ (see* (7)) *satisfy that for all $2\leqslant i<j\leqslant k$, L_v^{ij} is $(d_3,2\delta_3^{1/2},r)$-regular with respect to $P_v^{ij}=P[N_{P^{1i}}(v),N_{P^{1j}}(v)]$.*

Fact 12 is essentially the same as Claim A from [1]. For completeness, we sketch a proof of Fact 12 at the end of this section.

As in Fact 12, we say that a vertex $v \in V_1'$ is a *good vertex* if for all $2 \leqslant i < j \leqslant k$, L_v^{ij} is $(d_3, 2\delta_3^{1/2}, r)$-regular with respect to P_v^{ij}. Let $V_1^{\text{good}} = V_1^{\text{good}}(k, d_3, \delta_3, d_2, \delta_2, r)$ be the set of all good vertices $v \in V_1'$.

Proof of Proposition 9. Fact 12 ensures us that almost every vertex $v \in V_1'$ is a good vertex. Now, fix $2 \leqslant i < j \leqslant k$. The key observation is that every *good vertex* $v \in V_1^{\text{good}}$ satisfies that all but $2\delta_3^{1/4} t_v^2$ pairs $W_a^{v,i}$, $W_b^{v,j}$, $1 \leqslant a, b \leqslant t_v$, have density $d_2 d_3(1 \pm \delta_3^{1/4})$.

Indeed, let $v \in V_1^{\text{good}}$ but suppose $\{(W_a^{v,i}, W_b^{v,j})\}_{(a,b)\in I}$ is a collection of pairs, each with density, say, smaller than $d_2 d_3(1 - \delta_3^{1/4})$, such that $|I| \geqslant \delta_3^{1/4} t_v^2$. We claim the family $\boldsymbol{B} = \{P_v^{ij}[W_a^{v,i}, W_b^{v,j}]: (a,b) \in I\}$ contradicts the $(d_3, 2\delta_3^{1/2}, r)$-regularity of L_v^{ij} with respect to P_v^{ij}.

Note that (2) gives that $r \geqslant T_0^2 \geqslant t_v^2 \geqslant |I| = |\boldsymbol{B}|$. The set $\boldsymbol{B}$ is therefore a family of r *induced* subgraphs of $P_v^{ij} = P[N_{P^{1i}}(v), N_{P^{1j}}(v)]$. We claim $\boldsymbol{B}$ is a family of r induced subgraphs of P_v^{ij} satisfying

$$\left| \bigcup_{(a,b)\in I} P_v^{ij}[W_a^{v,i}, W_b^{v,j}] \right| > 2\delta_3^{1/2} |P_v^{ij}| \tag{10}$$

and

$$\left| L_v^{ij} \cap \bigcup_{(a,b)\in I} P_v^{ij}[W_a^{v,i}, W_b^{v,j}] \right| < (d_3 - 2\delta_3^{1/2}) \left| \bigcup_{(a,b)\in I} P_v^{ij}[W_a^{v,i}, W_b^{v,j}] \right|. \tag{11}$$

Once (10) and (11) are established, we see that $\boldsymbol{B}$ contradicts the $(d_3, 2\delta_3^{1/2}, r)$-regularity of L_v^{ij} with respect to P_v^{ij}. This will prove Proposition 9.

We first verify (10). Observe that

$$\left| \bigcup_{(a,b)\in I} P_v^{ij}[W_a^{v,i}, W_b^{v,j}] \right| = \sum_{(a,b)\in I} |P_v^{ij}[W_a^{v,i}, W_b^{v,j}]|. \tag{12}$$

Fix $(a,b) \in I$. Recall that $\delta_2 \leqslant 1/T_0 \leqslant 1/t_v$ in (2) and since $v \in V_1'$

$$|W_a^{v,i}| = \frac{|N_{P^{1i}}(v)|}{t_v} \pm 1 = (d_2 \pm 2\delta_2)\frac{n}{t_v},$$

(recall (7)). Consequently, the (d_2, δ_2)-regularity of P^{ij} implies that

$$\begin{aligned} |P_v^{ij}[W_a^{v,i}, W_b^{v,j}]| &= (d_2 \pm \delta_2)|W_a^{v,i}||W_b^{v,j}| \\ &= (d_2 \pm \delta_2)\left((d_2 \pm 2\delta_2)\frac{n}{t_v}\right)^2 = (d_2 \pm 2\delta_2)^3 \frac{n^2}{t_v^2}. \end{aligned} \tag{13}$$

The (d_2, δ_2)-regularity of P^{ij} also implies (recalling $v \in V_1'$ (cf. (7))

$$|P_v^{ij}| = (d_2 \pm \delta_2)((d_2 \pm \delta_2)n)^2 = (d_2 \pm \delta_2)^3 n^2. \tag{14}$$

Consequently, with $|I| \geqslant \delta_3^{1/4} t_v^2$, (12), (13) and (14) establish (10).

Observe that (11) is equivalent to

$$\sum_{(a,b)\in I} |L_v^{ij}[W_a^{v,i}, W_b^{v,j}]| < (d_3 - 2\delta_3^{1/2}) \sum_{(a,b)\in I} |P_v^{ij}[W_a^{v,i}, W_b^{v,j}]|. \tag{15}$$

Fix $(a,b) \in I$. Our assumption is that

$$|L_v^{ij}[W_a^{v,i}, W_b^{v,j}]| < d_2 d_3(1 - \delta_3^{1/4})|W_a^{v,i}||W_b^{v,j}|$$

which, with (13), implies

$$|L_v^{ij}[W_a^{v,i}, W_b^{v,j}]| < d_3 \frac{(1-\delta_3^{1/4})}{(1-\delta_2 d_2^{-1})} |P_v^{ij}[W_a^{v,i}, W_b^{v,j}]|$$

$$< (d_3 - 2\delta_3^{1/2}) |P_v^{ij}[W_a^{v,i}, W_b^{v,j}]| \tag{16}$$

where the last inequality follows from $\delta_2 \ll d_2, \delta_3$ in (2). As (16) holds for each $(a,b) \in I$, (15) follows. □

Proof of Fact 12. It suffices to consider just the case $k=3$, for which we prove all but $2\delta_3^{1/2} n$ vertices $v \in V_1'$ (see (7)) satisfy that L_v^{23} is $(d_3, 2\delta_3^{1/2}, r)$-regular w.r.t. P_v^{23}. We note that while the constants $d_3, \delta_3, d_2, \delta_2$ and r satisfy the hierarchy in (2) (due to the quantification of the counting lemma), all that is required to enable the present sketch is that $0 < \delta_2 = \delta_2(d_2) \ll d_2$ is sufficiently small.

For each fixed vertex $v \in V_1'$ (see (7)) for which L_v^{23} is not $(d_3, 2\delta_3^{1/2}, r)$-regular w.r.t. P_v^{23}, fix a family $\boldsymbol{B}_v = \{B_{v1}, \dots, B_{vr}\}$ of r induced subgraphs $B_{vs} \subseteq P_v^{23}$, $1 \leqslant s \leqslant r$, for which

$$\left| \bigcup_{s=1}^{r} B_{vs} \right| > 2\delta_3^{1/2} |P_v^{23}| \tag{17}$$

but for which either

$$\left| L_v^{23} \cap \bigcup_{s=1}^{r} B_{vs} \right| < (d_3 - 2\delta_3^{1/2}) \left| \bigcup_{s=1}^{r} B_{vs} \right| \quad \text{or} \quad \left| L_v^{23} \cap \bigcup_{s=1}^{r} B_{vs} \right| > (d_3 + 2\delta_3^{1/2}) \left| \bigcup_{s=1}^{r} B_{vs} \right|.$$

Let V_1^- be the set of such vertices $v \in V_1'$ for which the first condition holds and let V_1^+ be the set of such vertices $v \in V_1'$ for which the second condition holds. We claim $|V_1^-| < \delta_3^{1/2} n$ and $|V_1^+| < \delta_3^{1/2} n$. The proofs of these two inequalities are symmetric, so w.l.o.g., we prove only the first.

Assume, on the contrary, that $|V_1^-| \geqslant \delta_3^{1/2} n$. We show V_1^- leads to a contradiction with the (d_3, δ_3, r)-regularity of $\mathscr{H}^{123}$ w.r.t. $P^{12} \cup P^{13} \cup P^{23}$. In particular, we show the set V_1^- implies the existence of a family $\boldsymbol{Q} = \boldsymbol{Q}_{V_1^-} = \{Q_1, \dots, Q_r\}$ satisfying

$$\left| \bigcup_{s=1}^{r} \mathscr{K}_3^{(2)}(Q_s) \right| > \delta_3 \left| \mathscr{K}_3^{(2)}(P^{12} \cup P^{13} \cup P^{23}) \right| \quad \text{and} \quad d_{\mathscr{H}^{123}}(\boldsymbol{Q}) < d_3 - \delta_3. \tag{18}$$

Indeed, fix $v \in V_1^-$ and fix $1 \leqslant s \leqslant r$. Define $Q_{vs}^{12} \subseteq P^{12}$ (respectively $Q_{vs}^{13} \subseteq P^{13}$) as the set of all edges of P^{12} (resp. P^{13}) containing vertex v and define $Q_{vs}^{23} = B_{vs}$. Set $Q_{vs} = Q_{vs}^{12} \cup Q_{vs}^{13} \cup Q_{vs}^{23}$ and $Q_s = \bigcup_{v \in V_1^-} Q_{vs}$. Set $\boldsymbol{Q} = \{Q_1, \dots, Q_r\}$. Note that

$$\left| \bigcup_{s=1}^{r} \mathscr{K}_3^{(2)}(Q_s) \right| = \sum_{v \in V_1^-} \left| \bigcup_{s=1}^{r} B_{vs} \right| \tag{19}$$

$$\text{and} \quad \left| \mathscr{H}^{123} \cap \bigcup_{s=1}^{r} \mathscr{K}_3^{(2)}(Q_s) \right| = \sum_{v \in V_1^-} \left| L_v^{23} \cap \bigcup_{s=1}^{r} B_{vs} \right|. \tag{20}$$

Note that the second inequality of (18) is trivial. Indeed, using both equalities in (19) and (20) and the definition of V_1^-, we have

$$\sum_{v \in V_1^-} \left| L_v^{23} \cap \bigcup_{s=1}^{r} B_{vs} \right| < (d_3 - 2\delta_3^{1/2}) \sum_{v \in V_1^-} \left| \bigcup_{s=1}^{r} B_{vs} \right| = (d_3 - 2\delta_3^{1/2}) \left| \bigcup_{s=1}^{r} \mathscr{K}_3^{(2)}(Q_s) \right|,$$

so that $d_{\mathscr{H}^{123}}(\boldsymbol{Q}) < d - 2\delta_3^{1/2} < d_3 - \delta_3$ follows.

To see the first inequality of (18), we use (17) to see

$$\sum_{v\in V_1^-}\left|\bigcup_{s=1}^{r} B_{vs}\right| > \sum_{v\in V_1^-} 2\delta_3^{1/2}|P_v^{23}| > 2\delta_3^{1/2}(d_2-\delta_2)((d_2-\delta_2)n)^2|V_1^-|,$$

where the last inequality follows from $v \in V_1'$ (as in (14) cf. (7)). Then our assumption about V_1^- implies

$$2\delta_3^{1/2}(d_2-\delta_2)^3 n^2|V_1^-| > 2\delta_3(d_2-\delta_2)^3 n^3.$$

Since $\delta_2 \ll d_2$, Fact 2 implies $|\mathscr{K}_3^{(2)}(P^{12}\cup P^{13}\cup P^{23})| \leqslant (3/2)d_2^3 n^3$, and so the first inequality of (18) follows from (19) and from $\delta_2 d_2^{-1} \ll d_2$ in (2). □

5. Proof of Proposition 10

We show that all but $k^3\delta_2^{1/4}n$ vertices $v \in V_1'$ (see (7)) satisfy $|\mathrm{L}_{\mathrm{good}}^{hij}(v)\cap \mathrm{H}_{\mathrm{bad}}^{hij}(v)| < 2\delta_3^{1/20}t_v^3$ for all $2\leqslant h<i<j\leqslant k$. In the remainder of this paper, we fix $2\leqslant h<i<j\leqslant k$. It suffices to prove that all but $\delta_2^{1/4}n$ vertices $v\in V_1'$ satisfy $|\mathrm{L}_{\mathrm{good}}^{hij}(v)\cap \mathrm{H}_{\mathrm{bad}}^{hij}(v)| < 2\delta_3^{1/20}t_v^3$ for the fixed indices $2\leqslant h<i<j\leqslant k$.

Remark 13. In the remainder of this paper, the indices $2\leqslant h<i<j\leqslant k$ are *fixed*.

Assume, on the contrary, there exists a set $A^{hij} \subseteq V_1'$ of size

$$|A^{hij}| > \delta_2^{1/4}n \tag{21}$$

consisting of vertices for which

$$|\mathrm{L}_{\mathrm{good}}^{hij}(v)\cap \mathrm{H}_{\mathrm{bad}}^{hij}(v)| \geqslant 2\delta_3^{1/20}t_v^3. \tag{22}$$

We show that (21) leads to a contradiction to our hypothesis of Setup 4 that the triad $\mathscr{H}^{hij}$ is (d_3,δ_3,r)-regular with respect to $P^{hi}\cup P^{ij}\cup P^{hj}$. We outline our approach in the following remark.

Remark 14. Fix $v\in A^{hij}$ and fix $(a,b,c)\in \mathrm{L}_{\mathrm{good}}^{hij}(v)\cap \mathrm{H}_{\mathrm{bad}}^{hij}(v)$. Since $(a,b,c)\in \mathrm{H}_{\mathrm{bad}}^{hij}(v)$, we appeal to Definitions 3 and 8 to infer that there exists a family $\boldsymbol{Q}_{vabc}^{hij}=\{Q_{vabc}^{hij}(p): 1\leqslant p\leqslant r'\}$, $Q_{vabc}^{hij}(p)\subseteq L_v^{hij}[a,b,c]$ (see (4)), satisfying

$$\left|\bigcup_{p=1}^{r'}\mathscr{K}_3^{(2)}(Q_{vabc}^{hij}(p))\right| > \delta_3^{1/20}|\mathscr{K}_3^{(2)}(L_v^{hij}[a,b,c])|, \tag{23}$$

but

$$|d_{\mathscr{H}}(\boldsymbol{Q}_{vabc}^{hij})-d_3| \geqslant \delta_3^{1/20}. \tag{24}$$

In (32), we collect a witness $\boldsymbol{Q}_{vabc}^{hij}$ for each $(a,b,c)\in \mathrm{L}_{\mathrm{good}}^{hij}(v)\cap \mathrm{H}_{\mathrm{bad}}^{hij}(v)$ and $v\in A^{hij}$ to create a "big witness" $\boldsymbol{Q}^{hij}$ that will contradict the (d_3,δ_3,r)-regularity of $\mathscr{H}^{hij}$ with respect to $P^{hi}\cup P^{ij}\cup P^{hj}$.

In the process of collecting the witnesses $\boldsymbol{Q}_{vabc}^{hij}$ over $(a,b,c)\in \mathrm{L}_{\mathrm{good}}^{hij}(v)\cap \mathrm{H}_{\mathrm{bad}}^{hij}(v)$ and $v\in A^{hij}$, we do not need the entire set A^{hij}, and in fact, we need only a small subset thereof. Over two steps, we refine the set A^{hij} into two nested subsets $C^{hij}\subseteq B^{hij}\subseteq A^{hij}$ where the final subset C^{hij} produces the big witness $\boldsymbol{Q}^{hij}$ promised.

5.1. *Refining the set A^{hij}*

We obtain the intermediate subset $B^{hij}\subseteq A^{hij}$ using Fact 15 below. This fact states that from A^{hij} we may find a subset of vertices B^{hij}, every pair from which has the "right" shared P^{1q}-neighborhood, $q\in\{h,i,j\}$.

Fact 15. *Set*

$$f = 128\frac{\delta_3^{2/5}}{d_3^3 d_2^3}. \tag{25}$$

Assuming (21), *there exists a set* $B^{hij} \subseteq A^{hij}$ *of size* $|B^{hij}| = 2f$ *such that for each* $q \in \{h, i, j\}$ *and for every distinct vertices* $u, v \in B^{hij}$,

$$|N_{P^{1q}}(u) \cap N_{P^{1q}}(v)| = (d_2 \pm \delta_2)^2 n. \tag{26}$$

Fact 15 is not difficult to prove and was shown, in a slightly different context, in [1, page 155]. For completeness, we prove Fact 15 in Section 5.5.

To identify the subset $C^{hij} \subseteq B^{hij}$, we use the following considerations. Fix $v \in B^{hij}$ and set

$$\mathrm{LH}_-(v) = \{(a, b, c) \in \mathrm{L}_{\mathrm{good}}^{hij}(v) \cap \mathrm{H}_{\mathrm{bad}}^{hij}(v) \colon d_{\mathscr{H}}(\boldsymbol{Q}_{vabc}^{hij}) < d_3 - \delta_3^{1/20}\}, \tag{27}$$

$$\mathrm{LH}_+(v) = \{(a, b, c) \in \mathrm{L}_{\mathrm{good}}^{hij}(v) \cap \mathrm{H}_{\mathrm{bad}}^{hij}(v) \colon d_{\mathscr{H}}(\boldsymbol{Q}_{vabc}^{hij}) > d_3 + \delta_3^{1/20}\}. \tag{28}$$

Moreover, we define

$$B_-^{hij} = \{v \in B^{hij} \colon |\mathrm{LH}_-(v)| \geqslant \tfrac{1}{2}|\mathrm{L}_{\mathrm{good}}^{hij}(v) \cap \mathrm{H}_{\mathrm{bad}}^{hij}(v)|\},$$

$$B_+^{hij} = \{v \in B^{hij} \colon |\mathrm{LH}_+(v)| \geqslant \tfrac{1}{2}|\mathrm{L}_{\mathrm{good}}^{hij}(v) \cap \mathrm{H}_{\mathrm{bad}}^{hij}(v)|\}.$$

Clearly, one of $|B_-^{hij}| \geqslant \frac{1}{2}|B^{hij}| = f$ or $|B_+^{hij}| \geqslant \frac{1}{2}|B^{hij}| = f$ holds. In our proof, it does not matter which holds as the cases are symmetric. We assume, without loss of generality, that the former holds and we fix some set

$$C^{hij} \subset B_-^{hij} \subseteq B^{hij} \quad \text{such that } |C^{hij}| = f. \tag{29}$$

We construct the witness $\boldsymbol{Q}^{hij}$ from C^{hij}. Before doing so, however, we state the following fact for future reference.

Fact 16. *Let* $v \in C^{hij}$. *From* (22) *and the definition of* T_0 (*see* (2)), *we infer*

$$\delta_3^{1/20} t_v^3 \leqslant \tfrac{1}{2}|\mathrm{L}_{\mathrm{bad}}^{hij}(v) \cap \mathrm{H}_{\mathrm{bad}}^{hij}(v)| \leqslant |\mathrm{LH}_-(v)| \leqslant t_v^3 \leqslant T_0^3. \tag{30}$$

For each $(a, b, c) \in \mathrm{LH}_-(v)$, *we recall from* (24) *that* $d_{\mathscr{H}}(\boldsymbol{Q}_{vabc}^{hij}) < d_3 - \delta_3^{1/20}$, *and so,*

$$\left|\mathscr{H}^{hij} \cap \bigcup_{p=1}^{r'} \mathscr{K}_3^{(2)}(Q_{vabc}^{hij}(p))\right| < (d_3 - \delta_3^{1/20}) \left|\bigcup_{p=1}^{r'} \mathscr{K}_3^{(2)}(Q_{vabc}^{hij}(p))\right|. \tag{31}$$

5.2. Constructing the witness

With the set C^{hij} above, we proceed to construct the promised witness $\boldsymbol{Q}^{hij}$. Define

$$\boldsymbol{Q}^{hij} = \{Q_{vabc}^{hij}(p) \colon v \in C^{hij},\ (a, b, c) \in \mathrm{LH}_-(v),\ \text{and } p = 1, \ldots, r'\}. \tag{32}$$

We assert $\boldsymbol{Q}^{hij}$ is the promised family witnessing the (d_3, δ_3, r)-irregularity of $\mathscr{H}^{hij}$ with respect to $P^{hi} \cup P^{ij} \cup P^{hj}$.

We first claim that $\boldsymbol{Q}^{hij}$ has at most r members. Indeed, we have

$$|\boldsymbol{Q}^{hij}| \overset{(32)}{=} r' \sum_{v \in C^{hij}} |\mathrm{LH}_-(v)| \overset{(30)}{\leqslant} r' f T_0^3 \overset{(25)}{=} 128 r' T_0^3 \frac{\delta_3^{2/5}}{d_3^3 d_2^3} \overset{(2)}{\leqslant} r,$$

as desired.

Now, as $\boldsymbol{Q}^{hij}$ has at most r members consisting of subgraphs from $P^{hi} \cup P^{ij} \cup P^{hj}$, the following observation, Claim 17 and 18, provide a direct contradiction to the (d_3, δ_3, r)-regularity of $\mathscr{H}^{hij}$ with respect to $P^{hi} \cup P^{ij} \cup P^{hj}$. For that set

$$\mathscr{K}_3^{(2)}(\boldsymbol{Q}^{hij}) = \bigcup\{\mathscr{K}_3^{(2)}(Q_{vabc}^{hij}(p)) \colon v \in C^{hij},\ (a,b,c) \in \mathrm{LH}_-(v),\ p = 1, \ldots, r'\}.$$

Claim 17. $|\mathscr{K}_3^{(2)}(\boldsymbol{Q}^{hij})| > \delta_3 |\mathscr{K}_3^{(2)}(P^{hi} \cup P^{ij} \cup P^{hj})|$.

Claim 18. $|\mathscr{H}^{hij} \cap \mathscr{K}_3^{(2)}(\boldsymbol{Q}^{hij})| < (d_3 - \delta_3)|\mathscr{K}_3^{(2)}(\boldsymbol{Q}^{hij})|$.

Since Claims 17 and 18 provide a contradiction to the (d_3, δ_3, r)-regularity of $\mathscr{H}^{hij}$ with respect to $P^{hi} \cup P^{ij} \cup P^{hj}$, our proof of Proposition 10 will be complete upon proving these two claims.

5.3. Proof of Claim 17

Inclusion–exclusion gives

$$|\mathscr{K}_3^{(2)}(\boldsymbol{Q}^{hij})| \geqslant \sum_{v \in C^{hij}} \left| \bigcup\{\mathscr{K}_3^{(2)}(Q_{vabc}^{hij}(p)) \colon (a,b,c) \in \mathrm{LH}_-(v),\ p = 1, \ldots, r\} \right| - \sum_{v \neq v' \in C^{hij}} \left| \bigcup\{\mathscr{K}_3^{(2)}(Q_{vabc}^{hij}(p)) \cap \mathscr{K}_3^{(2)}(Q_{v'a'b'c'}^{hij}(p'))\} \right|, \tag{33}$$

where the last union runs over all $(a,b,c) \in \mathrm{LH}_-(v)$, $(a',b',c') \in \mathrm{LH}_-(v')$, and $p, p' = 1, \ldots, r'$. We bound the two terms on the right hand side of (33) in the following two facts.[6]

Fact 19. *For every* $v \in C^{hij}$

$$\left| \bigcup\{\mathscr{K}_3^{(2)}(Q_{vabc}^{hij}(p)) \colon (a,b,c) \in \mathrm{LH}_-(v),\ p = 1, \ldots, r\} \right| \geqslant \frac{\delta_3^{1/10}}{128} d_3^3 d_2^6 n^3.$$

Fact 20. *For all distinct vertices* $v, v' \in C^{hij}$

$$\Big| \bigcup\{\mathscr{K}_3^{(2)}(Q_{vabc}^{hij}(p)) \cap \mathscr{K}_3^{(2)}(Q_{v'a'b'c'}^{hij}(p')) \colon (a,b,c) \in \mathrm{LH}_-(v),\ (a',b',c') \in \mathrm{LH}_-(v'),\ \textit{and}\ p, p' = 1, \ldots, r'\} \Big| \leqslant 16 d_2^9 n^3.$$

Facts 19 and 20 conclude the proof of Claim 17.

Proof of Claim 17. Applying Facts 19 and 20 to (33), we obtain the lower bound

$$|\mathscr{K}_3^{(2)}(\boldsymbol{Q}^{hij})| \geqslant f \frac{\delta_3^{1/10}}{128} d_3^3 d_2^6 n^3 - 16 \binom{f}{2} d_2^9 n^3 \geqslant d_2^3 n^3 \left(\frac{f d_3^3 d_2^3 \delta_3^{1/10}}{128} - 8 f^2 d_2^6 \right).$$

Inserting the value $f = 128\delta_3^{2/5}/(d_3^3 d_2^3)$ from (25), we infer the further lower bound

$$|\mathscr{K}_3^{(2)}(\boldsymbol{Q}^{hij})| \geqslant d_2^3 n^3 \left(\delta_3^{1/2} - \frac{2^{17}}{d_3^6} \delta_3^{4/5} \right) = \delta_3^{1/2} d_2^3 n^3 \left(1 - \frac{2^{17}}{d_3^6} \delta_3^{3/10} \right) \geqslant \frac{1}{2} \delta_3^{1/2} d_2^3 n^3, \tag{34}$$

[6] These two facts will also be useful in our proof of Claim 18, as will the inclusion–exclusion of (33).

where the last inequality follows from the fact that $\delta_3 \ll d_3$ from (2). On the other hand, since $\delta_2 \ll d_2$ in (2), we conclude from Fact 2, the counting lemma for graphs, that

$$|\mathscr{K}_3^{(2)}(P^{hi} \cup P^{ij} \cup P^{hj})| \leqslant 2d_2^3 n^3.$$

Comparing this inequality against (34) proves Claim 17. □

Thus, it remains to verify Facts 19 and 20.

Proof of Fact 19. Fix a vertex $v \in C^{hij}$. Observe from (32) that

$$\left|\bigcup\{\mathscr{K}_3^{(2)}(Q_{vabc}^{hij}(p)) \colon (a,b,c) \in \mathrm{LH}_-(v) \text{ and } p = 1, \ldots, r\}\right|$$
$$= \sum_{(a,b,c)\in \mathrm{LH}_-(v)} \left|\bigcup_{p=1}^{r'} \mathscr{K}_3^{(2)}(Q_{vabc}^{hij}(p))\right|$$
$$\overset{(23)}{\geqslant} \sum_{(a,b,c)\in \mathrm{LH}_-(v)} \delta_3^{1/20} |\mathscr{K}_3^{(2)}(L_v^{hij}[a,b,c])|. \tag{35}$$

We further estimate (35) by appealing to Fact 2.

Fix $(a,b,c) \in \mathrm{LH}_-(v) \subseteq \mathrm{L}_{\mathrm{good}}^{hij}(v)$ (see (27)). By the definition of $\mathrm{L}_{\mathrm{good}}^{hij}(v)$, each of the three bipartite graphs $L_v^{hi}[W_a^{v,h}, W_b^{v,i}]$, $L_v^{ij}[W_b^{v,i}, W_c^{v,j}]$, and $L_v^{hj}[W_a^{v,h}, W_c^{v,j}]$, is ε_0-regular with density $d_3 d_2(1 \pm \delta_3^{1/4})$, where $\varepsilon_0 \ll d_2 d_3$ from (2). Applying Fact 2 to $L_v^{hij}[a,b,c]$, we therefore conclude

$$|\mathscr{K}_3^{(2)}(L_v^{hij}[a,b,c])| \geqslant \frac{1}{2}(d_3 d_2(1-\delta_3^{1/4}))^3 |W_a^{v,h}||W_b^{v,i}||W_c^{v,j}|$$
$$\geqslant \frac{(d_3 d_2)^3}{16} |W_a^{v,h}||W_b^{v,i}||W_c^{v,j}| \geqslant \frac{d_3^3 d_2^6}{128} \frac{n^3}{t_v^3}, \tag{36}$$

where the last inequality follows from the fact that $v \in V_1'$ (see (7)). Applying (36) to (35), we conclude

$$\left|\bigcup\{\mathscr{K}_3^{(2)}(Q_{vabc}^{hij}(p)) \colon (a,b,c) \in \mathrm{LH}_-(v) \text{ and } p = 1, \ldots, r\}\right|$$
$$\geqslant \frac{\delta_3^{1/20}}{128} \frac{d_3^3 d_2^6}{t_v^3} n^3 \, |\mathrm{LH}_-(v)| \overset{(30)}{\geqslant} \frac{\delta_3^{1/10}}{128} d_3^3 d_2^6 n^3,$$

as claimed. □

Proof of Fact 20. Let two distinct vertices v and $v' \in C^{hij}$ be fixed. We use the notation $P_{vv'}^{hi}$ for the subgraph of P^{hi} induced on $N_{P^{1h}}(v,v') \cup N_{P^{1i}}(v,v')$ where, for example, $N_{P^{1h}}(v,v') = N_{P^{1h}}(v) \cap N_{P^{1h}}(v')$. Define $P_{vv'}^{ij}$ and $P_{vv'}^{hj}$ similarly. Then,

$$\left|\bigcup\{\mathscr{K}_3^{(2)}(Q_{vabc}^{hij}(p)) \cap \mathscr{K}_3^{(2)}(Q_{v'a'b'c'}^{hij}(p')) \colon (a,b,c) \in \mathrm{LH}_-(v),\right.$$
$$\left.(a',b',c') \in \mathrm{LH}_-(v'), \text{ and } p, p' = 1, \ldots, r'\}\right| \leqslant |\mathscr{K}_3^{(2)}(P_{vv'}^{hi} \cup P_{vv'}^{ij} \cup P_{vv'}^{hj})|. \tag{37}$$

To bound the right hand side of (37), we apply Fact 2 to the graph $P_{vv'}^{hi} \cup P_{vv'}^{ij} \cup P_{vv'}^{hj}$, but first check that it is appropriate to do so.

To see that Fact 2 applies to the graph $P_{vv'}^{hi} \cup P_{vv'}^{ij} \cup P_{vv'}^{hj}$, we claim that each of $P_{vv'}^{hi}$, $P_{vv'}^{ij}$ and $P_{vv'}^{hj}$ is $(d_2, \delta_2^{1/2})$-regular, and check this assertion for $P_{vv'}^{hi}$. Recall from (26) that each of $|N_{P^{1h}}(v,v')|, |N_{P^{1i}}(v,v')| = (d_2 \pm \delta_2)^2 n \geqslant \delta_2 n$. Since

P^{hi} is (d_2, δ_2)-regular, and since $P^{hi}_{vv'}$ is the subgraph of P^{hi} induced on $N_{P^{1h}}(v, v') \cup N_{P^{1i}}(v, v')$, we have that $P^{hi}_{vv'}$ inherits[7] $(d_2, \delta_2^{1/2})$-regularity from P^{hi}.

Returning to (37), we apply Fact 2 (with $\delta_2^{1/2} \ll d_2$) to obtain

$$|\mathscr{K}_3^{(2)}(P^{hi}_{vv'} \cup P^{ij}_{vv'} \cup P^{hj}_{vv'})| \leqslant 2d_2^3 |N_{P^{1h}}(v, v')||N_{P^{1i}}(v, v')||N_{P^{1j}}(v, v')|,$$

from which it follows (via (26)) that

$$|\mathscr{K}_3^{(2)}(P^{hi}_{vv'} \cup P^{ij}_{vv'} \cup P^{hj}_{vv'})| \leqslant 16 d_2^9 n^3. \tag{38}$$

Combining (37) and (38) proves Fact 20. □

5.4. Proof of Claim 18

The proof of Claim 18 follows largely from work of the proof of Claim 17. First, observe that

$$\begin{aligned} |\mathscr{H}^{hij} \cap \mathscr{K}_3^{(2)}(\boldsymbol{Q}^{hij})| &\leqslant \sum_{v \in C^{hij}} \sum_{(a,b,c) \in \mathrm{LH}_-(v)} \left| \mathscr{H}^{hij} \cap \bigcup_{p=1}^{r'} \mathscr{K}_3^{(2)}(Q^{hij}_{vabc}(p)) \right| \\ &\overset{(31)}{<} (d_3 - \delta_3^{1/20}) \sum_{v \in C^{hij}} \sum_{(a,b,c) \in \mathrm{LH}_-(v)} \left| \bigcup_{p=1}^{r'} \mathscr{K}_3^{(2)}(Q^{hij}_{vabc}(p)) \right| \\ &= (d_3 - \delta_3^{1/20}) \sum_{v \in C^{hij}} \left| \bigcup_{(a,b,c) \in \mathrm{LH}_-(v)} \bigcup_{p=1}^{r'} \{\mathscr{K}_3^{(2)}(Q^{hij}_{vabc}(p))\} \right|. \end{aligned}$$

Recall that we saw the right-most sum above in our inclusion–exclusion of (33). In particular, we may use (33) and Fact 20 to obtain the further upper bound

$$|\mathscr{H}^{hij} \cap \mathscr{K}_3^{(2)}(\boldsymbol{Q}^{hij})| < (d_3 - \delta_3^{1/20}) \left(|\mathscr{K}_3^{(2)}(\boldsymbol{Q}^{hij})| + 16 \binom{f}{2} d_2^9 n^3 \right).$$

As such, we use Fact 19 and the definition of $\boldsymbol{Q}^{hij}$ in (32) to infer

$$d_{\mathscr{H}}(\boldsymbol{Q}^{hij}) < (d_3 - \delta_3^{1/20}) \left(1 + \frac{16 \binom{f}{2} d_2^9 n^3}{f \delta_3^{1/10} d_3^3 d_2^6 n^3 / 128} \right) \leqslant (d_3 - \delta_3^{1/20}) \left(1 + \frac{2^{10} f d_2^3}{\delta_3^{1/10} d_3^3} \right).$$

Using the value $f = 128 \delta_3^{2/5} / (d_3^3 d_2^3)$ (see (25)), we obtain further upper bound

$$d_{\mathscr{H}}(\boldsymbol{Q}^{hij}) < (d_3 - \delta_3^{1/20}) \left(1 + \frac{2^{17} \delta_3^{3/10}}{d_3^6} \right) < d_3 - \delta_3,$$

where the last inequality follows from $\delta_3 \ll d_3$ in (2). This proves Claim 18.

5.5. Proof of Fact 15

The proof depends only on the hypotheses that the bipartite graphs P^{1h}, P^{1i}, and P^{1j} are each (d_2, δ_2)-regular and that $|A^{hij}| > \delta_2^{1/4} n$ (as we assumed in (21)). In particular, the hypothesis in (22) will play no rôle in what follows.

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[7] As one may show, in fact, $P^{hi}_{vv'}$ inherits $(2\delta_2/d_2^2)$-regularity from P^{hi}.

We shall apply Turán's theorem [21] to the auxiliary graph $\Gamma = (V(\Gamma), E(\Gamma))$ whose vertices are given by $V(\Gamma) = A^{hij} \subseteq V_1'$ and whose edges are given by

$$E(\Gamma) = \left\{ \{v, v'\} \in \binom{A^{hij}}{2} : |N_{P^{1q}}(v, v')| = (d_2 \pm \delta_2)^2 n,\ q \in \{h, i, j\} \right\}$$

(where, for vertices v, v' and index $q \in \{h, i, j\}$, $N_{P^{1q}}(v, v') = N_{P^{1q}}(v) \cap N_{P^{1q}}(v')$). Indeed, with $f = 128\delta_3^{2/5}/(d_3^3 d_2^3)$ given in (25), note that we may take the desired set $B^{hij} \subset A^{hij}$ as the vertex set of any clique K_{2f} in Γ. Suppose, on the contrary, that Γ contains no cliques K_{2f}. Then Turán's theorem ensures

$$|E(\Gamma)| \leqslant \left(1 - \frac{1}{2f-1} + o(1)\right) \binom{|A^{hij}|}{2},$$

where $o(1) \to 0$ as $|A^{hij}| \to \infty$. Since $|A^{hij}| > \delta_2^{1/4} n$, where (2) ensures n may be taken as large as we need, we infer

$$|E(\Gamma)| \leqslant \left(1 - \frac{1}{2(2f-1)}\right) \binom{|A^{hij}|}{2} \leqslant \left(1 - \frac{1}{8f}\right) \binom{|A^{hij}|}{2}. \tag{39}$$

We now show that (39) leads to a contradiction with our choice of constants in (2).

Indeed, for an index $q \in \{h, i, j\}$, the (d_2, δ_2)-regularity of the graph P^{1q} implies that all but $4\delta_2 n^2$ pairs of vertices $\{v, v'\} \in \binom{V_1}{2}$ satisfy $|N_{P^{1q}}(v, v')| = (d_2 \pm \delta_2)^2 n$. As such,

$$\begin{aligned} |E(\Gamma)| &\geqslant \binom{|A^{hij}|}{2} - 12\delta_2 n^2 \geqslant \left(1 - \frac{24\delta_2 n^2}{|A^{hij}|^2}\right) \binom{|A^{hij}|}{2} \\ &\overset{(21)}{\geqslant} (1 - 24\delta_2^{1/2}) \binom{|A^{hij}|}{2}. \end{aligned} \tag{40}$$

Now, comparing (39) and (40) and using $f = 128\delta_3^{2/5}/(d_3^3 d_2^3)$ from (25) yields

$$\frac{d_3^3 d_2^3}{2^{10}\delta_3^{2/5}} = \frac{1}{8f} \leqslant 24\delta_2^{1/2}$$

contradicting (2).

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