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Bounds on the number of lifts of a Brauer character in a p-solvable group

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Abstract

The Fong–Swan theorem shows that for a p-solvable group G and Brauer character $\varphi \in \mathrm{IBr}_p(G)$, there is an ordinary character $\chi \in \mathrm{Irr}(G)$ such that $\chi^0 = \varphi$, where ^{0} denotes restriction to the p-regular elements of G. This still holds in the generality of π-separable groups, where $\mathrm{IBr}_p(G)$ is replaced by $\mathrm{I}_\pi(G)$. For $\varphi \in \mathrm{I}_\pi(G)$, let $L_\varphi = \{\chi \in \mathrm{Irr}(G) \mid \chi^0 = \varphi\}$. In this paper we give a lower bound for the size of L_φ in terms of the structure of the normal nucleus of φ and, if G is assumed to be odd and $\pi = \{p'\}$, we give an upper bound for L_φ in terms of the vertex subgroup for φ.

Keywords: Brauer character; Finite groups; Representations; Solvable groups

1. Introduction

The Fong–Swan theorem (see [9]) asserts that if G is a finite p-solvable group, and φ is an irreducible Brauer character of G, then there must exist an ordinary character $\chi \in \mathrm{Irr}(G)$ such that $\chi^0 = \varphi$, where ^{0} denotes restriction to the set of p-regular elements of G. Such a character is called a lift of φ. Moreover, the set $\mathrm{IBr}_p(G)$ of the irreducible Brauer characters of G forms a basis of the space of class functions on the p-regular elements of G and G has the property that if $\chi \in \mathrm{Irr}(G)$, then χ^0 is an $\mathbb{N}$-linear combination of elements of $\mathrm{IBr}_p(G)$.

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Let π be any set of primes. Isaacs in [3] and [4] has generalized the above to the case where G is π-separable, and the set $\mathrm{IBr}_p(G)$ is replaced by the set $\mathrm{I}_\pi(G)$, the irreducible π-partial characters of G. Again, if $\chi \in \mathrm{Irr}(G)$, it is shown that χ^0 is an $\mathbb{N}$-linear combination of elements of $\mathrm{I}_\pi(G)$, where here ^{0} denotes restriction to the set of π-elements. Isaacs then defines a set $\mathrm{B}_\pi(G) \subseteq \mathrm{Irr}(G)$ of lifts of $\mathrm{I}_\pi(G)$ such that for each $\varphi \in \mathrm{I}_\pi(G)$, there is a unique character $\chi \in \mathrm{B}_\pi(G)$ such that $\chi^0 = \varphi$. There are other ways, however, to construct sets of lifts of the $\mathrm{I}_\pi(G)$ characters, as in [13], where a canonical set of lifts which we will call $\mathrm{N}_\pi(G)$ is constructed. Little is known about the set of all lifts of a particular character $\varphi \in \mathrm{I}_\pi(G)$. For instance, using character triples, Laradji in [6] has given lower bounds for the number of such lifts in the case that $p = 2$.

Let G be a π-separable group. For a fixed character $\varphi \in \mathrm{I}_\pi(G)$, we define L_φ to be $\{\chi \in \mathrm{Irr}(G) \mid \chi^0 = \varphi\}$. This set will be the focus of study in this paper. The first main result of this paper, which follows immediately from the work of Navarro [11], is a lower bound on the size of L_φ in terms of a certain subgroup W_φ determined by φ.

Theorem 1.1. *Let G be a π-separable group, and let $\varphi \in \mathrm{I}_\pi(G)$. Then there is a subgroup W_φ and an irreducible π-partial character α of W_φ such that $|L_\varphi| \geqslant |W_\varphi : W_\varphi'|_{\pi'}$, $\alpha^G = \varphi$, and $\alpha(1)$ is a π-number.*

We will give an example where this inequality is strict. It is not known if there is a better lower bound. The main result of this paper is the following, which gives an upper bound in the case when G has odd order and $\pi = p'$. We define the vertex Q of φ to be simply a Sylow p-subgroup of W_φ.

Theorem 1.2. *Suppose G has odd order, and suppose $\varphi \in \mathrm{IBr}_p(G)$ for some prime p. If Q is a vertex subgroup for φ, then $|L_\varphi| \leqslant |Q : Q'|$.*

Theorem 1.2 is already known in some special cases. If G is p-solvable, and if the Brauer character φ of G has height zero and is in the block B, then every lift of φ must have height zero. Recall in this case that the vertex group of φ is equal to the defect group of B. Therefore, by Olsson's conjecture (see [15]), which is known to be true for p-solvable groups after the $k(GV)$-conjecture is known to be true, we have that the number of lifts of φ is bounded above by $|Q : Q'|$.

Combining the above results, we see that if G has odd order and $\varphi \in \mathrm{I}_{p'}(G) = \mathrm{IBr}_p(G)$, then $|W_\varphi : W_\varphi'|_p \leqslant |L_\varphi| \leqslant |Q : Q'|$. Thus if the two above indices are equal (for instance if W_φ is nilpotent), then we get an exact count on the size of L_φ. In fact, it is enough to assume that W_φ has a normal p-complement to get the above equality. These are the nilpotent Brauer characters discussed in [12].

2. Properties of π-special characters

In this section we summarize the properties of π-special and π-factorable characters of a finite π-separable group G. All of the results in this section can be found in [2] and [7]. For the remainder of this paper, assume that G is π-separable, unless otherwise indicated.

Recall that the order $o(\chi)$ of an irreducible character χ is the order of the linear character $\lambda = \det(\chi)$, i.e. the order $o(\chi)$ is the smallest positive integer n such that $(\det(\chi(x)))^n = 1$ for every element x of G. If G is a π-separable group, a character $\alpha \in \mathrm{Irr}(G)$ is called π-special if (a) $\alpha(1)$ is a π-number and (b) for every subnormal subgroup S of G, and every irreducible constituent γ of α_S, we have that the order of γ is a π-number. Recall that π' denotes the complement of the set of primes π. If α and β are π-special and π'-special, respectively, then one can show that in fact $\chi = \alpha\beta$ is irreducible. If an irreducible character χ can be written in the form $\chi = \alpha\beta$ where α is π-special and β is π'-special, we say that χ is π-factorable. If $\chi = \alpha_1\beta_1$ is another such factorization, then $\alpha_1 = \alpha$ and $\beta_1 = \beta$, i.e. the factorization is unique. We will frequently write that if $\gamma \in \mathrm{Irr}(G)$ is π-factorable, then $\gamma = \gamma_\pi\gamma_{\pi'}$, where γ_π and $\gamma_{\pi'}$ are the π-special and π' special factors of γ, respectively. Note that if $\gamma \in \mathrm{Irr}(G)$ is π-factorable, and if $S \lhd\lhd G$, then all of the constituents of γ_S must be π-factorable.

The π-special characters of G (denoted by $\mathcal{X}_\pi(G)$) behave well with respect to normal subgroups. It is clear from the definition that if $\alpha \in \mathcal{X}_\pi(G)$ and $S \lhd\lhd G$ then all the irreducible constituents of α_S are in $\mathcal{X}_\pi(S)$. If $N \lhd G$ and G/N is a π'-group and if $\alpha \in \mathcal{X}_\pi(G)$, then α restricts irreducibly to $\alpha_N \in \mathcal{X}_\pi(N)$. Moreover, if $\gamma \in \mathcal{X}_\pi(N)$ and γ is invariant in G, then there is a unique π-special extension of γ to G. If $M \lhd G$ and G/M is a π-group and if $\delta \in \mathcal{X}_\pi(M)$, then every irreducible constituent of γ^G is in $\mathcal{X}_\pi(G)$.

Finally, we note that $\mathcal{X}_\pi(G)$ is related to $\mathrm{I}_\pi(G)$. If $\varphi \in \mathrm{I}_\pi(G)$ and $\varphi(1)$ is a π-number, then there necessarily exists a unique character $\alpha \in \mathcal{X}_\pi(G)$ such that $\alpha^0 = \varphi$. Also, if $\gamma \in \mathcal{X}_\pi(G)$, then $\gamma^0 \in \mathrm{I}_\pi(G)$. In fact, if $\gamma \in \mathcal{X}_\pi(G)$ and if H is a Hall π-subgroup of G, then γ restricts irreducibly to H, and γ is the unique π-special extension of γ_H.

3. The normal nucleus and vertices

In this section we summarize Navarro's construction of the normal nucleus of $\chi \in \mathrm{Irr}(G)$, and we define a set $\mathrm{N}_\pi(G)$ of canonical lifts of $\mathrm{I}_\pi(G)$. We then define the vertex pair (Q, δ) for $\chi \in \mathrm{Irr}(G)$. Proofs of these results can all be found in [13].

In [13], Navarro defines, for a p-solvable group G and a character $\chi \in \mathrm{Irr}(G)$, the normal nucleus (W, γ) of χ. In [13], this normal nucleus is defined only for p-solvable groups, but the same definitions and results go through if G is assumed only to be π-separable for some set of primes π. We here briefly review this construction and highlight some of the properties we need.

The π-analog of Corollary 2.3 of [13] states that if G is π-separable and if $\chi \in \mathrm{Irr}(G)$, then there is a unique normal subgroup N of G maximal with the property that if $\theta \in \mathrm{Irr}(N)$ lies under χ, then θ is π-factorable. (Compare this to Theorem 3.2 of [3].) We call the pair (N, θ) a maximal factorable normal pair of G lying under χ. Let G_θ be the stabilizer of θ and let $\psi \in \mathrm{Irr}(G_\theta)$ be the Clifford correspondent for χ lying above θ. If θ is invariant in G, it is shown that $G = N$, so $\theta = \chi$. In this case, the normal nucleus of χ is defined to be the pair (G, χ), and note that $\chi = \theta$ is π-factorable. If $G_\theta < G$, then the normal nucleus (W, γ) of χ is defined recursively to be the normal nucleus of $\psi \in \mathrm{Irr}(G_\theta|\theta)$. Note, then, that γ is π-factorable, and $\gamma^G = \chi$. Also, the normal nucleus (W, γ) of χ is uniquely defined up to conjugacy.

If G is π-separable and $\chi \in \mathrm{Irr}(G)$ has the normal nucleus (W, γ), we define the vertex pair of χ to be the pair (Q, δ), where Q is a Hall π'-subgroup of W and $\delta = (\gamma_{\pi'})_Q$, which is necessarily in $\mathrm{Irr}(Q)$ since $\gamma_{\pi'}$ is π'-special. Since (W, γ) is unique up to conjugacy, then (Q, δ) is unique up to conjugacy. We define $\mathrm{vtx}(\chi) = (Q, \delta)$. In the case that G is p-solvable, Navarro defines the set $\mathrm{Irr}(G|Q, \delta)$ as those irreducible characters of G with vertex pair (Q, δ). It is then shown that as Q runs over all of the p-subgroups of G, then $\mathrm{Irr}(G|Q, 1_Q)$ is a set of lifts of $\mathrm{IBr}_p(G)$, and each $\varphi \in \mathrm{IBr}_p(G)$ has precisely one lift in that set. In the general case where G is π-separable, we introduce a slightly different notation and define $\mathrm{N}_\pi(G)$ to be the set of characters of G with a π-special normal nucleus character, i.e. those irreducible characters of G with a trivial vertex character. Again, the set $\mathrm{N}_\pi(G)$ forms a canonical set of lifts of $\mathrm{I}_\pi(G)$. Also, the characters of $\mathrm{N}_\pi(G)$ with π-degree are precisely the π-special characters of G.

Classically, a vertex subgroup of a character $\varphi \in \mathrm{IBr}_p(G)$ of a p-solvable group G has been defined as a Sylow p-subgroup of any subgroup U of G with the property that there is an irreducible Brauer character α of U with p'-degree such that $\alpha^G = \varphi$ (see [9]). This same definition can be extended to a π-separable group G if π plays the role of p'. Note that if $\chi \in \mathrm{N}_\pi(G)$ is a lift of $\varphi \in \mathrm{I}_\pi(G)$, then if (W, γ) is a normal nucleus character of χ, then γ is necessarily π-special, and $\gamma^0 = \alpha \in \mathrm{I}_\pi(G)$ is such that $\alpha^G = \varphi$ and α has π-degree. Thus if Q is a Hall π'-subgroup of W, then Q is a vertex in the classical sense for φ. The following theorem, from [5], shows essentially that vertex subgroups of φ are the same no matter how one defines them.

Theorem 3.1. *Suppose G is π-separable, and let $\varphi \in \mathrm{I}_\pi(G)$. Then there exists a subgroup $U \subseteq G$ and a character $\alpha \in \mathrm{I}_\pi(U)$ such that $\alpha^G = \varphi$ and $\alpha(1)$ is a π-number. If we let Q be a Hall π-complement in U, then up to G-conjugacy, Q is uniquely determined by φ, and it is independent of the choice of U.*

The existence part of the above theorem can be established by the construction of $\mathrm{N}_\pi(G)$. The uniqueness, however, requires some work to prove.

The construction of the set $\mathrm{N}_\pi(G)$ is obviously very similar to the construction of the set $\mathrm{B}_\pi(G)$ in [3]. One might ask if these two sets are in fact equal. It is shown in [1] that $\mathrm{B}_\pi(G) = \mathrm{N}_\pi(G)$ if $|G|$ is odd or if $2 \in \pi$, but they need not be equal if $2 \in \pi'$.

4. Theorem 1.1 and related results

In this section we prove some easy preliminary results and then prove Theorem 1.1 of the introduction.

The following theorem, which is used to prove Theorem 4.1, is essentially the same as Navarro's Theorem C in [11], only with the subnormal nucleus being replaced by the normal nucleus. To prove this theorem, then, simply replace the subnormal nucleus in Navarro's proof in [11] with the normal nucleus, and the set $\mathrm{B}_\pi(G)$ with the set $\mathrm{N}_\pi(G)$.

Theorem 4.1. *Suppose that G is a π-separable group and suppose $\chi \in \mathrm{N}_\pi(G)$ has normal nucleus (W, α). If $\gamma \in \mathcal{X}_{\pi'}(W)$, then $(\alpha\gamma)^G \in \mathrm{Irr}(G)$. If $\beta \in \mathcal{X}_{\pi'}(W)$, then $(\alpha\gamma)^G = (\alpha\beta)^G$ if and only if $\gamma = \beta$.*

We are now ready to prove the first main theorem of the introduction. The following is a slight restatement of Theorem 1.1. Essentially, we use the linear π'-special characters of the nucleus of the N_π lift of $\varphi \in \mathrm{I}_\pi(G)$ to construct lifts of φ. We also point out that although this argument works equally well with subnormal nuclei and $\mathrm{B}_\pi(G)$, the arguments in the following sections require normality instead of merely subnormality, so in order to relate these results to each other we need to use the normal nucleus and $\mathrm{N}_\pi(G)$.

Theorem 4.2. *Let G be a π-separable group, and let $\varphi \in \mathrm{I}_\pi(G)$. If (W, γ) is the normal nucleus of the unique lift of φ in $\mathrm{N}_\pi(G)$, then for each linear character $\beta \in \mathcal{X}_{\pi'}(W)$, $(\gamma\beta)^G$ is a lift of φ, and if $\delta \in \mathcal{X}_{\pi'}(W)$ is linear, then $(\gamma\beta)^G = (\gamma\delta)^G$ if and only if $\beta = \delta$. Thus $|W : W'|_{\pi'} \leqslant |L_\varphi|$.*

Proof. Let (W, γ) be the normal nucleus for the unique character $\chi \in \mathrm{N}_\pi(G)$ such that $\chi^0 = \varphi$. Let $\lambda \in \mathrm{Irr}(W/W')$ be π'-special. Then by Theorem 4.1, $(\gamma\lambda)^G \in \mathrm{Irr}(G)$. Since λ is a linear π'-special character of W, then $(\gamma\lambda)^0 = \gamma^0$, and therefore $((\gamma\lambda)^G)^0 = (\gamma^0)^G = \varphi \in \mathrm{I}_\pi(G)$.

Moreover, if $\lambda_1 \in \mathrm{Irr}(W/W')$ is another linear π'-special character, then $(\gamma\lambda)^G = (\gamma\lambda_1)^G$ if and only if $\lambda = \lambda_1$. Thus we have constructed an injective map

$$\lambda \to (\gamma\lambda)^G$$

from the set of linear π'-special characters of W into L_φ, and we therefore have $|W : W'|_{\pi'} \leqslant |L_\varphi|$. □

We now give an example to show that this lower bound may be strict.

Example 4.3. There exists a solvable group G and a character $\varphi \in \mathrm{I}_\pi(G)$ such that if W is the normal nucleus of φ, then $1 = |W : W'|_{\pi'} < |L_\varphi|$.

Let Γ be the nonabelian group of order 21. Let Γ act on E, an elementary abelian group of order 5^{21} such that every subgroup of Γ is a stabilizer of some character of E (see [1] for details of this construction). Let G be the semidirect product of Γ acting on E, and let $K \subseteq G$ be such that $E \subseteq K$ and $|G : K| = 3$. Note $K \lhd G$. Let $\pi = \{3, 5\}$. Choose a character $\alpha \in \mathrm{Irr}(E)$ such that $G_\alpha = K$. Now α is invariant in K, thus α must extend to a π-special character $\hat{\alpha} \in \mathrm{Irr}(K)$, and $\beta = (\hat{\alpha})^G$ is necessarily π-special. Note that if $\beta^0 = \varphi \in \mathrm{I}_\pi(G)$, then since β is π-special, $W = G$, and thus $|W : W'|_{\pi'} = 1$. However, for each linear π'-special character $\lambda \in \mathrm{Irr}(K/E)$, we have that $\hat{\alpha}\lambda \in \mathrm{Irr}(K)$ and $((\hat{\alpha}\lambda)^G)^0 = \varphi$. If $(\hat{\alpha}\lambda_1)^G = (\hat{\alpha}\lambda_2)^G$, note that λ_1 and λ_2 must be conjugate in G, but all of the elements of G not in K move $\hat{\alpha}$, thus this forces $\lambda_1 = \lambda_2$. Therefore $|L_\varphi| \geqslant 7$, and in fact it is easily seen that $|L_\varphi| = 7$.

5. Vertices and correspondences

In this section we discuss some correspondences between certain sets of characters that will be necessary to prove the upper bound stated in the introduction. In particular, we will discuss Navarro's star map, which relates certain irreducible characters of an odd group G with certain characters of a subgroup of G. In the next section we will extend Navarro's result to provide a connection between lifts of a Brauer character of an odd group G to certain Brauer characters of subgroups of G.

For a set of primes π, let $\mathbb{Q}_\pi$ denote the field obtained by adjoining all complex nth roots of unity of $\mathbb{Q}$ for all π-numbers n. Recall (see [3, Corollary 12.1]) that if G is a π-separable group and if $2 \in \pi$ or $|G|$ is odd, and if $\chi \in \mathrm{B}_\pi(G)$, then $\chi(g) \in \mathbb{Q}_\pi$ for every element $g \in G$. The following lemma is well known, though the proof does not seem to appear in the literature.

Lemma 5.1. *Let G be a group of odd order, and let $\chi \in \mathrm{Irr}(G)$. If χ is π-factorable and $\chi^0 \in \mathrm{I}_\pi(G)$, then χ has π-degree.*

Proof. Suppose that χ factors as $\alpha\beta$, where $\alpha \in \mathcal{X}_\pi(G)$ and $\beta \in \mathcal{X}_{\pi'}(G)$. Since $\chi^0 \in \mathrm{I}_\pi(G)$, then $\beta^0 \in \mathrm{I}_\pi(G)$ and thus the values of β^0 must be in $\mathbb{Q}_\pi$. However, since β is π'-special, then the values of β must be in $\mathbb{Q}_{\pi'}$, and thus the values of β^0 must be in $\mathbb{Q}$, and therefore $\beta^0 = \bar{\beta}^0$. Let $\eta \in \mathrm{B}_\pi(G)$ be such that $\eta^0 = \beta^0$. Thus $\bar{\eta}$ is the unique lift of $\bar{\beta}^0$, and therefore $\eta = \bar{\eta}$. Since G has odd order, this implies $\eta = 1_G$, and therefore β is linear and $\chi = \alpha\beta$ has π-degree. □

One can show that $GL_2(3)$ is a counterexample to the above lemma if $|G|$ is not assumed to be odd.

Corollary 5.2. *Suppose G is a group of odd order, and suppose $\chi \in \mathrm{Irr}(G)$ is a lift of $\varphi \in \mathrm{I}_\pi(G)$. Let $(W, \alpha\beta)$ be a normal nucleus for χ, where α is π-special and β is π'-special. Then β is linear.*

Proof. Note that since $((\alpha\beta)^G)^0 \in \mathrm{I}_\pi(G)$, then $(\alpha\beta)^0 \in \mathrm{I}_\pi(W)$, and thus β is linear by the above lemma. □

Recall (see [10], for example) that if p is a prime and G is any finite group (we do not need to assume that G is even p-solvable), and if $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$, then we say that $\chi \in \mathrm{Irr}(G|\theta)$ is a relative defect zero character if $(\chi(1)/\theta(1))_p = |G : N|_p$. We denote this by $\chi \in \mathrm{rdz}_p(G|\theta)$, or if p is clear from the context, just $\mathrm{rdz}(G|\theta)$. For instance, it is clear that if $N \subseteq H \subseteq G$, and if $\eta \in \mathrm{Irr}(H|\theta)$ is such that $\eta^G = \chi$, then $\eta \in \mathrm{rdz}(H|\theta)$ if and only if $\chi \in \mathrm{rdz}(G|\theta)$.

The following result (see, for example, Chapter 9 of [9]) does not require any separability assumptions on G.

Theorem 5.3. *Suppose that G is a finite group and Q is a normal p-subgroup of G. Suppose that $\delta \in \mathrm{Irr}(Q)$ is linear and invariant in G. If $\chi \in \mathrm{rdz}(G|\theta)$, then $\chi^0 \in \mathrm{IBr}_p(G)$.*

The following important result, from [10], constructs a bijection between π-special characters and characters of certain subgroups. We define, for a normal subgroup N of G and a Hall π'-subgroup H of G, the set $\mathcal{X}_{\pi,H}(N)$ to be the set of H-invariant π-special characters of N.

Theorem 5.4. *Let G be a group of odd order and let Q be a p-subgroup of G. For every normal subgroup N of G such that $Q \cap N \in \mathrm{Syl}_p(N)$, there is a natural bijection*

$$\widetilde{\ }: \mathcal{X}_{p',Q}(N) \to \mathrm{Irr}\big(\mathbf{N}_N(Q)\big).$$

Moreover, suppose that $M \lhd G$ is such that $M \subseteq N$ and $\theta \in \mathcal{X}_{p',Q}(N)$ and $\eta \in \mathcal{X}_{p',Q}(M)$. Then $\widetilde{\theta}$ lies over $\widetilde{\eta}$ if and only if θ lies over η.

Let $\mathrm{Y}_p(G) \subseteq \mathrm{IBr}_p(G)$ denote the characters in $\mathrm{IBr}_p(G)$ with p'-degree, and assume that the order of G is odd. Suppose χ is a p'-special character of G such that $\chi^0 = \varphi \in \mathrm{IBr}_p(G)$ and let Q be a Sylow p-subgroup of G. Since $\mathbf{N}_G(Q)/Q$ is necessarily a p'-group, then $(\widetilde{\chi})^0$ is trivially in $\mathrm{IBr}_p(\mathbf{N}_G(Q)/Q)$. In this case we abuse notation and say $\varphi \mapsto \widetilde{\varphi}$ is a map from $\mathrm{Y}_p(G)$ to $\mathrm{IBr}_p(\mathbf{N}_G(Q)/Q)$. The following result from [14] essentially extends this correspondence to a map from the Brauer characters of an odd group G to irreducible Brauer characters of certain subgroups of G. We use the notation $\varphi \in \mathrm{IBr}_p(G|Q)$ to signify that the vertex (see Section 3) subgroup of φ is Q.

Theorem 5.5. *Let G be a group of odd order and let $\varphi \in \mathrm{IBr}_p(G)$. Suppose W is a subgroup of G such that there exists a Brauer character $\alpha \in \mathrm{IBr}_p(W)$ of p'-degree such that $\alpha^G = \varphi$, and suppose Q is a Sylow p-subgroup of W. Then $\widetilde{\alpha} \in \mathrm{IBr}_p(\mathbf{N}_W(Q))$ induces irreducibly to $(\widetilde{\alpha})^{\mathbf{N}_G(Q)} \in \mathrm{IBr}_p(\mathbf{N}_G(Q))$. Moreover, the map from $\mathrm{IBr}_p(G|Q)$ to $\mathrm{IBr}_p(\mathbf{N}_G(Q)|Q)$ given by $\varphi \to (\widetilde{\alpha})^{\mathbf{N}_G(Q)}$ is a well defined natural bijection.*

Note that in theory, for a given Brauer character φ of G, there are many subgroups W which might have a Brauer character α that satisfy the hypotheses of the above theorem. However, by

Theorem 3.1, the subgroup Q is unique up to conjugacy, and therefore part of the content of the above theorem is that the particular choice of W and α does not affect the image of φ in $\mathrm{IBr}_p(\mathbf{N}_G(Q))$ under this map.

Now suppose that $\chi \in \mathrm{Irr}(G)$ is a lift of the irreducible Brauer character $\varphi \in \mathrm{IBr}_p(G)$, and suppose that G has odd order. Note that by Corollary 5.2, if $(W, \gamma\delta)$ is a normal nucleus for χ, then $\delta(1) = 1$ and thus $(\gamma\delta)^0 = \gamma^0$ is an irreducible Brauer character of p'-degree, and thus the pair (W, γ^0) satisfies the hypotheses of the above theorem. Moreover, by Theorem 3.1, any Sylow p-subgroup Q of W is conjugate to a vertex of φ. The following theorem essentially extends the correspondence in the above theorem to lifts of φ, and shows that the images of distinct lifts of φ must map to the same Brauer character of $\mathbf{N}_G(Q)$.

Theorem 5.6. *Let G be a group of odd order and suppose $\varphi \in \mathrm{IBr}_p(G)$. Suppose $\chi, \psi \in \mathrm{Irr}(G)$ are lifts of φ, and suppose that χ has normal nucleus (W, γ) and ψ has normal nucleus (V, ϵ), and suppose that a vertex subgroup Q of φ is contained in both W and V. Then $\widetilde{(\gamma^0)}^{\mathbf{N}_G(Q)} \in \mathrm{IBr}_p(\mathbf{N}_G(Q))$ and $\widetilde{(\gamma^0)}^{\mathbf{N}_G(Q)} = \widetilde{(\epsilon^0)}^{\mathbf{N}_G(Q)}$.*

Proof. The first statement follows from Theorem 5.5, since $\gamma \in \mathrm{Irr}(W)$ has π-degree and $\gamma^0 \in \mathrm{IBr}_p(W)$. Since $\chi^0 = \psi^0 = \varphi$, then the injectivity in Theorem 5.5 implies that $\widetilde{(\gamma^0)}^{\mathbf{N}_G(Q)} = \widetilde{(\epsilon^0)}^{\mathbf{N}_G(Q)}$. □

In order to prove our upper bound on the number of lifts of the Brauer character φ, we will need to combine the above result with a result about Navarro's star map, which we now discuss.

Recall (see Section 3) that if G is π-separable with $\chi \in \mathrm{Irr}(G)$ and if χ has normal nucleus $(W, \alpha\beta)$ and if Q is any Hall π'-subgroup of W, then we say the pair (Q, β_Q) is the vertex of χ, and we denote this by $\mathrm{vtx}(\chi) = (Q, \beta_Q)$. Note that β_Q is necessarily irreducible. For a π'-subgroup Q and a character $\delta \in \mathrm{Irr}(Q)$, recall that we use $\mathrm{Irr}(G|Q, \delta)$ to denote all of the irreducible characters of G that have (Q, δ) for the vertex. Finally, we adopt the notation that G_δ is the stabilizer in $\mathbf{N}_G(Q)$ of the character $\delta \in \mathrm{Irr}(Q)$.

The following theorem is, for our purposes, the key result from [10]. In that paper, Navarro constructs, for a group of odd order G, a map $\chi \to \chi^*$ from $\mathrm{Irr}(G|Q, \delta)$ to $\mathrm{rdz}(G_\delta|\delta)$.

Theorem 5.7. *Suppose that G is a group of odd order, and let Q be a p-subgroup of G and $\delta \in \mathrm{Irr}(Q)$. Then the map $\chi \to \chi^*$ is a natural injection from $\mathrm{Irr}(G|Q, \delta)$ to $\mathrm{rdz}(G_\delta|\delta)$.*

Before we discuss some of the specifics of the construction of the star map, we first prove a very useful corollary.

Corollary 5.8. *Suppose that G is a group of odd order, and suppose $\chi \in \mathrm{Irr}(G)$ is such that $\chi^0 \in \mathrm{IBr}_p(G)$. Moreover, suppose $\chi \in \mathrm{Irr}(G|Q, \delta)$. Then $(\chi^*)^0 \in \mathrm{IBr}_p(G_\delta)$.*

Proof. Since χ is a lift of some Brauer character of G, then by Corollary 5.2, the vertex character δ of χ is linear. Thus Theorem 5.3 applied to G_δ implies that $(\chi^*)^0 \in \mathrm{IBr}_p(G_\delta)$. □

For the full details of the construction of the star map, see [10]. We point out here a useful fact about the star map that will be needed. If (W, γ) is a normal nucleus for χ such that $(Q, \delta) \leqslant (W, \gamma)$, then since Q is a Sylow p-subgroup of W then we can apply Theorem 5.4 to obtain a p'-special character $\widetilde{\gamma_{p'}} \in \mathrm{Irr}(\mathbf{N}_W(Q))$. Also, since γ_p is p-special, γ_p restricts irreducibly to $\mathbf{N}_W(Q)$. By the construction of the star map, $\chi^* = (\widetilde{\gamma_{p'}}(\gamma_p)_{\mathbf{N}_W(Q)})^{G_\delta}$.

6. The upper bound

In this section we prove Theorem 1.2 of the introduction, which gives an upper bound on the number of lifts that a Brauer character of a group of odd order may have, in terms of the vertex subgroup Q. Recall that Theorem 5.6 shows that distinct lifts of the same Brauer character φ of G map to the same Brauer character of $\mathbf{N}_G(Q)$, where Q is the vertex subgroup of φ. In this section we will use some results of Navarro to show that distinct lifts of φ with the same vertex character δ map to distinct Brauer characters of G_δ, and we will then use this to show the upper bound on the number of lifts.

Before continuing, we need some definitions and results from [8]. Recall that an irreducible character χ of G has defect zero if $\chi(1)_p = |G|_p$. Now suppose G is any finite group (we make no separability assumptions) and Q is a normal p-subgroup of G, and δ is a G-invariant character of Q. In Theorem 2.1 of [8], Navarro constructs a bijection between $\mathrm{rdz}(G|\delta)$ and $\mathrm{dz}(G/Q)$, the defect zero characters of G/Q. Here we will denote this map by

$$N_\delta : \mathrm{rdz}(G|\delta) \to \mathrm{dz}(G/Q).$$

By construction, if χ and μ are in $\mathrm{rdz}(G|\delta)$ and χ and μ agree on p-regular elements of G, then $N_\delta(\chi) = N_\delta(\mu)$.

Theorem 6.1. *Let G be a group of odd order and let $\pi = p'$. Assume the characters χ and μ are in $\mathrm{Irr}(G|Q, \delta)$, and suppose χ^0 and μ^0 are in $\mathrm{IBr}_p(G)$. If $(\chi^*)^0 = (\mu^*)^0$, then $\chi = \mu$.*

Proof. Suppose $(\chi^*)^0 = (\mu^*)^0$. Note that Q is a normal subgroup of G_δ and clearly δ is invariant in G_δ. Since χ^* and μ^* agree on p-regular elements, by assumption, and both χ^* and μ^* lie over δ, then $N_\delta(\chi^*) = N_\delta(\mu^*)$. Since N_δ is a bijection, then $\chi^* = \mu^*$ Thus by the injectivity of the star map, we have that $\chi = \mu$. □

Before proving Theorem 1.2, we make some definitions and prove one final lemma. If δ is a linear character of a p-subgroup Q of G, then we define $[\delta]$ to be the orbit of δ under the action of $\mathbf{N}_G(Q)$. We also let $\mathcal{O}$ denote a set of representatives of these orbits. Finally, for a fixed character $\varphi \in \mathrm{IBr}_p(G)$, we define $L_\varphi(\delta) = L_\varphi([\delta]) = L_\varphi \cap \mathrm{Irr}(G|Q, \delta)$. Also, recall that by Theorem 3.1, if G has odd order and if the character $\chi \in \mathrm{Irr}(G)$ is a lift of the character $\varphi \in \mathrm{IBr}_p(G)$, then the vertex subgroup is the same (up to conjugacy) as the vertex subgroup of φ.

Of course, the following lemma is true for ordinary irreducible characters as well, though we will only need it for Brauer characters.

Lemma 6.2. *Let H be a subgroup of the finite group G, and suppose that φ is an irreducible character of G. Suppose that there are distinct irreducible Brauer characters $\psi_1, \ldots, \psi_h$ of H such that $\psi_i^G = \varphi$. Then $h \leqslant |G : H|$.*

Proof. Note $\varphi(1) = \psi_i(1)|G : H|$. Since each ψ_i lies under φ, then $h\psi_1(1) \leqslant \varphi(1)$. Therefore we have $h \leqslant \frac{\varphi(1)}{\psi(1)} = |G : H|$. □

Proof of Theorem 1.2. Notice that by Theorem 3.1, if $\chi \in \mathrm{Irr}(G)$ is a lift of $\varphi \in \mathrm{IBr}_p(G)$, then the vertex subgroup of χ is (up to conjugation) Q, the vertex subgroup of φ. Note that $L_\varphi = \bigcup_{[\delta]\in\mathcal{O}} L_\varphi([\delta])$, and this is a disjoint union. Thus we see that $|L_\varphi| = \sum_{[\delta]\in\mathcal{O}} |L_\varphi([\delta])|$.

We claim that $|L_\varphi([\delta])| \leqslant |\mathbf{N}_G(Q) : G_\delta|$. By Corollary 5.8, for each lift $\chi \in L_\varphi(\delta)$, we have $(\chi^*)^0$ is an irreducible Brauer character of G_δ. Let $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(G)$ be distinct members of $L_\varphi(\delta)$, and suppose (W, γ) is a normal nucleus for χ such that $(Q, \delta) \leqslant (W, \gamma)$ and (U, ρ) is a normal nucleus for ψ such that $(Q, \delta) \leqslant (U, \rho)$. Then Theorem 5.6 implies that $(\widetilde{\gamma^0})^{\mathbf{N}_G(Q)} = (\widetilde{\rho^0})^{\mathbf{N}_G(Q)}$. However, since $\chi^* = (\widetilde{\gamma_{p'}}(\gamma_p)_{\mathbf{N}_W(Q)})^{G_\delta}$ and γ_p is linear by Corollary 5.2, then $(\chi^*)^0 = (\widetilde{\gamma^0})^{G_\delta}$ and similarly $(\psi^*)^0 = (\widetilde{\rho^0})^{G_\delta}$. Therefore

$$\big((\chi^*)^0\big)^{\mathbf{N}_G(Q)} = \big((\widetilde{\gamma^0})^{G_\delta}\big)^{\mathbf{N}_G(Q)} = \big((\widetilde{\rho^0})^{G_\delta}\big)^{\mathbf{N}_G(Q)} = \big((\psi^*)^0\big)^{\mathbf{N}_G(Q)}.$$

However, Theorem 6.1 shows that $(\chi^*)^0$ is not equal to $(\psi^*)^0$. Thus the images of the members of the set $L_\varphi(\delta)$ under the map given by $\chi \mapsto (\chi^*)^0$ must be distinct characters in $\mathrm{IBr}_p(G_\delta)$, each of which induces to the same irreducible Brauer character of $\mathbf{N}_G(Q)$. Therefore, by Lemma 6.2, the size of the image of $L_\varphi([\delta])$ under the star map must be bounded above by $|\mathbf{N}_G(Q) : G_\delta|$. Since the star map is an injection, this proves that $|L_\varphi([\delta])| \leqslant |\mathbf{N}_G(Q) : G_\delta|$.

Therefore the sum becomes

$$|L_\varphi| \leqslant \sum_{[\delta]\in\mathcal{O}} \big|\mathbf{N}_G(Q) : G_\delta\big| = |Q : Q'|$$

and we are done. □

The above result leads to some interesting questions. For instance, is the requirement that the order of G is odd really necessary? Certain aspects of the above proof seem to heavily depend on the oddness of $|G|$, but other arguments could be made. One could probably begin by studying the case where $|G|$ is not necessarily odd, but $2 \in \pi'$, which seems to be the case where the lifts are more "under control" (see [1]).

Also, given the definition of the vertex (Q, δ) as above, one could ask which of the properties of a "classical" vertex extend to a vertex pair (Q, δ). This might in turn lead to more precise statements about the number of lifts of a Brauer character.

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