



A Clebsch–Gordan formula for $SL_3(\mathbb{C})$ and applications to rationality[☆]

Christian Böhning, Hans-Christian Graf v. Bothmer^{*}

*University of Göttingen, Courant Center for Higher Order Structures in Mathematics,
Bunsenstrasse 3, Göttingen, Germany*

Received 12 January 2009; accepted 6 November 2009

Available online 21 December 2009

Communicated by Ludmil Katzarkov

Abstract

If R, S, T are irreducible $SL_3(\mathbb{C})$ -representations, we give an easy and explicit description of a basis of the space of equivariant maps $R \otimes S \rightarrow T$ (Theorem 3.1). We apply this method to the rationality problem for invariant function fields. In particular, we prove the rationality of the moduli space of plane curves of degree 34. This uses a criterion which ensures the stable rationality of some quotients of Grassmannians by an SL -action (Proposition 5.4).

© 2009 Elsevier Inc. All rights reserved.

Keywords: Representation theory; Rationality; Moduli spaces; Plane curves; Group quotients

1. Introduction

For the group $SL_2(\mathbb{C})$ the irreducible representations are the $V(d) := \text{Sym}^d(\mathbb{C}^2)$. If d_1, d_2, n are non-negative integers such that $0 \leq n \leq \min(d_1, d_2)$, and if for $f \in V(d_1)$ and $g \in V(d_2)$ one puts

$$\psi_n(f, g) := \frac{(d_1 - n)! (d_2 - n)!}{d_1! d_2!} \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{\partial^n f}{\partial z_1^{n-i} \partial z_2^i} \frac{\partial^n g}{\partial z_1^i \partial z_2^{n-i}}$$

[☆] Supported by the German Research Foundation (Deutsche Forschungsgemeinschaft (DFG)) through the Institutional Strategy of the University of Göttingen.

^{*} Corresponding author.

E-mail address: bothmer@math.uni-hannover.de (H.-C. Graf v. Bothmer).

then the map $(f, g) \mapsto \psi_n(f, g)$ is a bilinear and $\mathrm{SL}_2(\mathbb{C})$ -equivariant map from $V(d_1) \times V(d_2)$ onto $V(d_1 + d_2 - 2n)$. The map

$$V(d_1) \otimes V(d_2) \rightarrow \bigoplus_{n=0}^{\min(d_1, d_2)} V(d_1 + d_2 - 2n)$$

$$(f, g) \mapsto \sum_{n=0}^{\min(d_1, d_2)} \psi_n(f, g)$$

is an isomorphism of $\mathrm{SL}_2(\mathbb{C})$ -modules (“Clebsch–Gordan decomposition”), cf. [3], p. 122. The maps ψ_n are called transvectants (*Überschiebungen* in German). Their importance derives from the fact that they make the preceding isomorphism *explicit*.

Now let $G := \mathrm{SL}_3(\mathbb{C})$, and let $V(a, b)$ be the irreducible G -module whose highest weight has numerical labels a, b where a, b are non-negative integers. A representation $V(a, b) \otimes V(c, d)$ decomposes similarly into irreducible summands, and the Cartan–Killing theory of highest weights allows us to compute the multiplicity with which $V(e, f)$ occurs (an entirely similar statement holds of course for $\mathrm{SL}_n(\mathbb{C})$ or any semi-simple linear algebraic group); in other words, the theory of highest weights asserts the existence of an isomorphism

$$E \otimes E' \simeq \bigoplus_{i \in I} E_i$$

of irreducible representations of a semi-simple algebraic group, but does not give us the isomorphism, at least it is not easy to unravel from this theory. On the other hand, it is often important to know the isomorphism, e.g.:

- In *the problem of rationality for fields of invariants*, see [4] for a survey. Here one almost always has to check certain non-degeneracy statements for maps of the form $E \otimes E' \rightarrow E''$ (or similar maps constructed by representation theory), and for this one has to know the maps explicitly. Often one is dependent on computer aid when studying these maps, one needs fast methods for computing them.
- In *the geometry of syzygies* (see [11]). Here one wants to understand differentials of certain chain complexes constructed by representation theoretic means, as for example by Kempf’s geometric technique based on taking direct images of Koszul complexes; here computational efficiency is again one of the desiderata.

In the first sections of this article we give a simple method, contained in Theorem 3.1, to obtain a basis for the space

$$\mathrm{Hom}_G(V(a, b) \otimes V(c, d), V(e, f)).$$

In particular, it enables one to immediately write down matrix representatives for the occurring maps. Moreover, Theorem 3.1 gives a factorization of all such maps into certain elementary building blocks and explicit formulas for them. These building blocks are well known (see e.g. [6] and [7]) but have to our knowledge not yet been used to construct a basis of the above space in the case where multiplicities occur.

During the proof, which occupies Sections 2 through 4, we also set up a natural bijection between the basis maps and the expansions of Young diagrams which occur in the combinatorics of the Littlewood–Richardson rule.

One is tempted to think that something of this sort should have been discovered before, but we could not find it in the classical or modern literature.

In any event, for us the main reason for introducing this computational scheme is that it is the one we use and found most convenient for applications to the problem of rationality for invariant function fields; a sample of such applications is contained in Section 5.

First of all, Theorem 3.1 allows one to prove rationality for many spaces $\mathbb{P}(V(a, b))/G$ via the double bundle method [1] where $V(a, b)$ is a space of mixed tensors. We prove rationality of $\mathbb{P}(V(4, 4))/G$ as an example.

For the double bundle method one uses linear fibrations over projective spaces; one may also consider linear fibrations over more general Grassmannians; see Proposition 5.2. This was already mentioned in [10], but has not yet found any application to our knowledge. One problem is that one needs to know the stable rationality for quotients of Grassmannians $\text{Grass}(k, V)/\Gamma$ where V is a linear representation of a reductive group Γ . In Proposition 5.4 we give a criterion for stable rationality that applies in some cases if Γ is a group of type SL. Using this and Theorem 3.1, we prove the rationality of the moduli space of plane curves of degree 34, i.e. $\mathbb{P}(V(0, 34))/G$, in Theorem 5.5. This case cannot be handled by the double bundle method, cf. Remark 5.6, nor has it been treated by any other method so far.

2. The Littlewood–Richardson rule for $\text{SL}_3(\mathbb{C})$

It is well known that isomorphism classes of irreducible $\text{GL}_n(\mathbb{C})$ -modules correspond bijectively to n -tuples of integers $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ via associating to such a representation its highest weight $\lambda_1\epsilon_1 + \dots + \lambda_n\epsilon_n$ where ϵ_i is the i -th coordinate function of the standard diagonal torus in $\text{GL}_n(\mathbb{C})$. The space of the corresponding irreducible representation will be denoted $\Sigma^\lambda(\mathbb{C}^n)$. Here Σ^λ is called the *Schur functor* (cf. [5]). If all λ_j are non-negative, one associates to λ the corresponding *Young diagram* whose number of boxes in its i -th row is λ_i ; λ will often be identified with this Young diagram. For example,

$$\Sigma^{1,1,1}(\mathbb{C}^3) \longleftrightarrow \Lambda^3(\mathbb{C}^3) \longleftrightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$

We list some properties of the Schur functors for future use:

- One has $\Sigma^\lambda(\mathbb{C}^n) \simeq \Sigma^\mu(\mathbb{C}^n)$ as $\text{SL}_n(\mathbb{C})$ -representations if and only if $\lambda_i - \mu_i =: h$ is constant for all i . In fact, in this case

$$\Sigma^\lambda(\mathbb{C}^n) \simeq \Sigma^\mu(\mathbb{C}^n) \otimes (\Lambda^n(\mathbb{C}^n))^{\otimes h}.$$

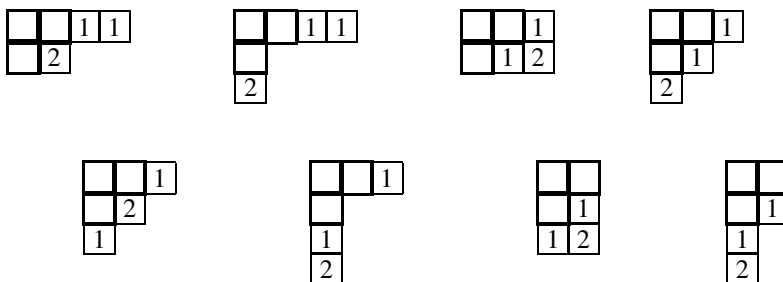
- $\Sigma^{(\lambda_1, \lambda_2, \dots, \lambda_n)}(\mathbb{C}^n)^\vee \simeq \Sigma^{(-\lambda_n, -\lambda_{n-1}, \dots, -\lambda_1)}(\mathbb{C}^n)$.
- The representation $V(a, b)$ of $G = \text{SL}_3(\mathbb{C})$ is isomorphic to $\Sigma^{(a+b, b, 0)}(\mathbb{C}^3)$.
- For a Young diagram λ with more than n rows one has $\Sigma^\lambda(\mathbb{C}^n) = 0$ by definition.

The Littlewood–Richardson rule to decompose $\Sigma^\lambda \otimes \Sigma^\mu$ into irreducible factors where λ, μ are Young diagrams (cf. [5], §A.1) says the following (in this notation we suppress the space which the Schur functors are applied to, since it plays no role): label each box of μ with the number of the row it belongs to. Then expand the Young diagram λ by adding the boxes of μ to the rows of λ subject to the following rules:

- (a) The boxes with labels $\leq i$ of μ together with the boxes of λ form again a Young diagram.
- (b) No column contains boxes of μ with equal labels.
- (c) When the integers in the boxes added are listed from right to left and from top down, then, for any $0 \leq s \leq$ (number of boxes of μ), the first s entries of the list satisfy: each label l ($1 \leq l \leq$ (number of rows of μ) $- 1$) occurs at least as many times as the label $l + 1$.

We will call this configuration of boxes (together with the labels) a μ -expansion of λ . Then the multiplicity of Σ^ν in $\Sigma^\lambda \otimes \Sigma^\mu$ is the number of times the Young diagram ν can be obtained by expanding λ by μ according to the above rules, forgetting the labels.

Example 2.1. For $\Sigma^{(2,1,0)} \otimes \Sigma^{(2,1,0)}$ the following expansions are possible:



Hence we have the following decomposition

$$V(1, 1) \otimes V(1, 1) = V(2, 2) \oplus V(3, 0) \oplus V(0, 3) \oplus 2V(1, 1) \oplus V(0, 0).$$

For $G = \text{SL}_3(\mathbb{C})$ the combinatorics of the Littlewood–Richardson rule can be handled explicitly. For this let $V(e, f)$ be a summand of $V(a, b) \otimes V(c, d)$. In the following we set $\lambda = (a + b, b, 0)$, $\mu = (c + d, d, 0)$ and let $\nu = (e + f + s, f + s, s)$ be the unique Young diagram corresponding to $V(e, f)$ in the decomposition of $\Sigma^\lambda(\mathbb{C}^3) \otimes \Sigma^\mu(\mathbb{C}^3)$.

Lemma 2.2. *Expand the Young diagram of λ by adding p_i boxes with label 1 to row i and afterwards q_i boxes with label 2 to row i (see Fig. 1). This is a μ -expansion of λ if and only if the following inequalities hold:*

- (1) (a) $p_i \geq 0$,
 (b) $q_i \geq 0$;
- (2) (a) $p_2 \leq a$,
 (b) $p_3 \leq b$,

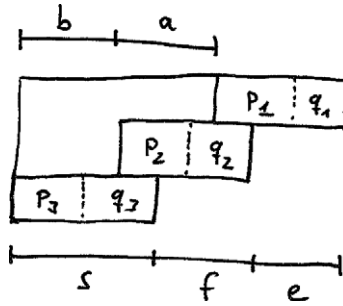


Fig. 1.

- (c) $p_2 + q_2 - a \leq p_1$,
- (d) $p_3 + q_3 - b \leq p_2$;
- (3) (a) $q_1 = 0$,
- (b) $q_2 \leq p_1$,
- (c) $q_2 + q_3 \leq p_1 + p_2$;
- (4) (a) $p_1 + p_2 + p_3 = c + d$,
- (b) $q_1 + q_2 + q_3 = d$.

Proof. The inequalities (1a) and (1b) are obvious positivity conditions. (2a) and (2b) ensure that the boxes of λ together with the boxes of μ with label 1 form again a Young diagram and there is at most one label 1 in every column. (2c) and (2d) guarantee that the boxes of λ together with all boxes of μ form again a Young diagram and there is at most one label 2 in every column. (3a), (3b) and (3c) encode that the string of labels read from right to left and from top down always contains more 1's than 2's. The last two equations reflect that the total number of 1's and 2's is given by the Young diagram describing $V(c, d)$. \square

For given a, b, c, d, e, f the equations above leave only one unknown:

Lemma 2.3. *Let $s = p_3 + q_3$ be the number of labeled boxes in the third row of the μ -expansion. Let furthermore $j = p_3$ be the number of 1's in the third row and $t = p_2 - q_3$ the difference between the number of 1's in the second row and the number of 2's in the third row. With this we obtain*

- (1) $p_3 = j$,
- (2) $q_3 = s - j$,
- (3) $p_2 = s + t - j$,
- (4) $q_2 = d - s + j$,
- (5) $p_1 = c + d - (s + t)$,
- (6) $q_1 = 0$,
- (7) $s = \frac{(a+c-e)+2(b+d-f)}{3}$,
- (8) $t = \frac{(a+c-e)-(b+d-f)}{3}$.

Proof. (1), (2) and (3) follow from the definition of j, s and t . Since the total number of 2's is d we obtain (4). Similarly $p_1 + p_2 + p_3 = c + d$ implies (5). Eq. (6) is true for all μ -expansions.

Since we know that the number of labeled boxes is $c + 2d$, the number of empty boxes is $a + 2b$ and the total number of boxes is $3s + 2f + e$, we obtain

$$s = \frac{(a + c - e) + 2(b + d - f)}{3}.$$

Finally the total length of the first row is $a + b + p_1$, on the one hand, and $s + e + f$ on the other. This gives (8). \square

Proposition 2.4. *For given a, b, c, d, e and f there exists a μ -expansion of λ of shape ν with $p_3 = j$ if and only if j satisfies the following inequalities:*

- (1) (a) $0 \leq j \leq s + t \leq c + d,$
 (b) $0 \leq s - j \leq d;$
- (2) (a) $s + t - j \leq a,$
 (b) $j \leq b,$
 (d) $j \leq b + t;$
- (3) (b) $j \leq c - t,$
 (c) $j \leq c.$

Proof. Substitute the expressions of Lemma 2.3 into the inequalities of Lemma 2.2. The inequality (2c) gives $s + 2t \leq a + c$ which is always true since $s + 2t = a + c - e$. Furthermore (3a), (4a) and (4b) simplify to $0 = 0$. \square

Remark 2.5. The numbering in the list above is taken from the corresponding inequalities in Lemma 2.2.

3. A basis for $\text{Hom}_G(V(a, b) \otimes V(c, d), V(e, f))$

We put

$$S^a := \text{Sym}^a(\mathbb{C}^3), \quad D^b := \text{Sym}^b(\mathbb{C}^3)^\vee$$

and denote by e_1, e_2, e_3 and x_1, x_2, x_3 dual bases in \mathbb{C}^3 resp. $(\mathbb{C}^3)^\vee$ so that $V(a, b)$ can be realized concretely as the kernel of the map

$$\Delta := \sum_{i=1}^3 \frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial x_i} : S^a \otimes D^b \rightarrow S^{a-1} \otimes D^{b-1}. \tag{1}$$

We will always view $V(a, b)$ in this way in the following. By $\pi_{e,f}$ we denote the equivariant projection from $S^e \otimes D^f$ onto $V(e, f)$.

Our purpose is to determine an explicit basis of the G -equivariant maps

$$\text{Hom}_G(V(a, b) \otimes V(c, d), V(e, f))$$

if $V(e, f)$ is a subrepresentation of $V(a, b) \otimes V(c, d)$. To this end we define the following elementary maps:

$$\alpha : (S^a \otimes D^b) \otimes (S^c \otimes D^d) \rightarrow (S^{a-1} \otimes D^b) \otimes (S^c \otimes D^{d-1}),$$

$$\alpha := \sum_{i=1}^3 \frac{\partial}{\partial e_i} \otimes \text{id} \otimes \text{id} \otimes \frac{\partial}{\partial x_i},$$

$$\beta : (S^a \otimes D^b) \otimes (S^c \otimes D^d) \rightarrow (S^a \otimes D^{b-1}) \otimes (S^{c-1} \otimes D^d),$$

$$\beta := \sum_{i=1}^3 \text{id} \otimes \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial e_i} \otimes \text{id},$$

$$\vartheta : (S^a \otimes S^c) \otimes (D^{b+d}) \rightarrow (S^{a-1} \otimes S^{c-1}) \otimes D^{b+d+1},$$

$$\vartheta := \sum_{\sigma \in \mathfrak{S}_3} (-1)^{\text{sgn}(\sigma)} \frac{\partial}{\partial e_{\sigma(1)}} \otimes \frac{\partial}{\partial e_{\sigma(2)}} \otimes x_{\sigma(3)},$$

$$\omega : S^{a+c} \otimes (D^b \otimes D^d) \rightarrow S^{a+c+1} \otimes (D^{b-1} \otimes D^{d-1}),$$

$$\omega := \sum_{\sigma \in \mathfrak{S}_3} (-1)^{\text{sgn}(\sigma)} e_{\sigma(1)} \otimes \frac{\partial}{\partial x_{\sigma(2)}} \otimes \frac{\partial}{\partial x_{\sigma(3)}}.$$

Note that an easier way of defining ϑ and ω is by saying that ϑ is multiplication by the determinant $x_1 \wedge x_2 \wedge x_3$ and ω multiplication by its inverse $e_1 \wedge e_2 \wedge e_3$.

Theorem 3.1. *Suppose that $V(e, f)$ occurs in the decomposition of $V(a, b) \otimes V(c, d)$ and let s and t be defined as above. Let J be the set of all integers j satisfying the inequalities:*

- (1) (a) $0 \leq j \leq s + t \leq c + d,$
 (b) $0 \leq s - j \leq d;$
- (2) (a) $s + t - j \leq a,$
 (b) $j \leq b,$
 (d) $j \leq b + t;$
- (3) (b) $j \leq c - t,$
 (c) $j \leq c.$

Then a basis of $\text{Hom}_G(V(a, b) \otimes V(c, d), V(e, f))$ is given by the restriction to $V(a, b) \otimes V(c, d)$ of the maps

$$\pi_{e,f} \circ \vartheta^t \circ \beta^j \circ \alpha^{s-j}, \quad j \in J$$

if $t \geq 0$ and

$$\pi_{e,f} \circ \omega^{-t} \circ \beta^j \circ \alpha^{s+t-j}, \quad j \in J$$

if $t \leq 0$.

A few explanatory remarks are in order.

Remark 3.2. When writing a composition like $\pi_{e,f} \circ \vartheta^t \circ \beta^j \circ \alpha^{s-j}$, we suppress the obvious multiplication maps from the notation. For example if $t \geq 0$ the map

$$\beta^j \circ \alpha^{s-j} : (S^a \otimes D^b) \otimes (S^c \otimes D^d) \rightarrow (S^{a-s+j} \otimes D^{b-j}) \otimes (S^{c-j} \otimes D^{d-s+j})$$

is composed with the multiplication map

$$(S^{a-s+j} \otimes D^{b-j}) \otimes (S^{c-j} \otimes D^{d-s+j}) \rightarrow S^{a-s+j} \otimes S^{c-j} \otimes D^{b+d-s}$$

before applying ϑ^t to land in $S^{a-s+j-t} \otimes S^{c-j-t} \otimes D^{b+d-s+t}$. Before applying the equivariant projection $\pi_{e,f}$ we multiply again to map to

$$S^{a+c-s-2t} \otimes D^{b+d-s+t}$$

which one, looking back at the definition of t and s , identifies as $S^e \otimes D^f$. This simplification of notation should cause no confusion.

Proof. Note that the element $m := (e_1^a \otimes x_3^b) \otimes (e_3^c \otimes x_1^d)$ is in the subspace $V(a, b) \otimes V(c, d) \subset (S^a \otimes D^b) \otimes (S^c \otimes D^d)$ by the definition of Δ in formula (1). Note also that the image of the map

$$\delta : S^{e-1} \otimes D^{f-1} \rightarrow S^e \otimes D^f, \quad \delta = \sum_{i=1}^3 e_i \otimes x_i \tag{2}$$

is a complement to the subspace $V(e, f)$ in $S^e \otimes D^f$. If $t \geq 0$ we compute

$$(\vartheta^t \circ \beta^j \circ \alpha^{s-j})(m) = (\text{non-zero constant}) \cdot e_1^{a-s+j-t} e_3^{c-j-t} \otimes x_3^{b-j} x_2^t x_1^{d-s+j}.$$

The inequalities above imply that this is a non-zero monomial in $S^e \otimes D^f$ for all $j \in J$. If $t \leq 0$ then

$$(\omega^{-t} \circ \beta^j \circ \alpha^{s+t-j})(m) = (\text{non-zero constant}) \cdot e_1^{a-s-t+j} e_2^{-t} e_3^{c-j} \otimes x_3^{b+t-j} x_1^{d-s+j}$$

is also a non-zero monomial in $S^e \otimes D^f$. Each non-zero bihomogeneous polynomial in the subspace

$$\text{im}(\delta) = (e_1 \otimes x_1 + e_2 \otimes x_2 + e_3 \otimes x_3) \cdot (S^{e-1} \otimes D^{f-1}) \subset S^e \otimes D^f$$

contains monomials (with non-zero coefficient) divisible by $e_2 \otimes x_2$. Since the preceding monomials in cases $t \geq 0$ resp. $t \leq 0$ are not divisible by $e_2 \otimes x_2$, a linear combination of them can be zero modulo $\text{im}(\delta)$ only if this linear combination is already zero as a polynomial in $S^e \otimes D^f$. But in both cases $t \geq 0$ and $t \leq 0$, the degrees of the above monomials with respect to the variable e_1 are pairwise distinct, so they cannot combine to zero non-trivially in $S^e \otimes D^f$. \square

4. Equivariant projections

To complete the picture, we will give in this section a method to compute the equivariant projection

$$\pi_{a,b} : S^a \otimes D^b \rightarrow V(a, b) = \ker(\Delta) \subset S^a \otimes D^b.$$

Lemma 4.1. *One has*

$$\pi_{a,b} = \sum_{j=0}^N \mu_j \delta^j \Delta^j$$

for some $N \in \mathbb{N}$ and certain $\mu_j \in \mathbb{Q}$ (the map δ is defined in formula (2)).

Proof. Let us denote by $\pi_{a,b,i}$ the equivariant projection

$$\pi_{a,b,i} : S^a \otimes D^b \rightarrow V(a - i, b - i) \subset S^{a-i} \otimes D^{b-i}$$

so that $\pi_{a,b} = \pi_{a,b,0}$. Look at the diagram

$$\begin{array}{ccc} S^a \otimes D^b & \xrightarrow{\Delta^i} & S^{a-i} \otimes D^{b-i} \\ & \swarrow \delta^i & \downarrow \pi_{a-i,b-i} \\ & & V(a - i, b - i) \subset S^{a-i} \otimes D^{b-i} \end{array}$$

By Schur’s lemma,

$$\pi_{a,b,i} = \lambda_i \delta^i \pi_{a-i,b-i} \Delta^i \tag{3}$$

for some non-zero constants λ_i . On the other hand,

$$\pi_{a,b} = \text{id} - \sum_{i=1}^{\min(a,b)} \pi_{a,b,i}.$$

Therefore, since the assertion of the lemma holds trivially if one of a or b is zero, the general case follows by induction on $\min(a, b)$. \square

Note that to compute the μ_j in the expression of $\pi_{a,b}$ in Lemma 4.1, it suffices to calculate the λ_i in formula 3 which can be done by the rule

$$\frac{1}{\lambda_i} (e_1^{a-i} \otimes x_3^{b-i}) = (\Delta^i \circ \delta^i)(e_1^{a-i} \otimes x_3^{b-i})$$

which uses (3) and the injectivity of δ^i .

5. Examples and applications

In the following example we write down explicit matrix representatives for the maps given in 3.1 in one special case.

Example 5.1. In the decomposition of $V(1, 1) \otimes V(1, 1)$, the representation $V(1, 1)$ occurs with multiplicity 2, corresponding to a two-dimensional space

$$V(1, 1) \otimes V(1, 1) \rightarrow V(1, 1)$$

of $SL_3(\mathbb{C})$ -equivariant maps. Here $a = b = c = d = e = f = s = 1$ and $t = 0$. Therefore a basis for this space of equivariant homomorphisms is given by α and β .

To give matrix representatives of α and β we use the vectors

$$\begin{aligned} q_{12} &= e_1x_2, & q_{13} &= e_1x_3, \\ q_{21} &= e_2x_1, & q_{23} &= e_2x_3, \\ q_{31} &= e_3x_1, & q_{32} &= e_3x_2, \\ q_{22} &= e_1x_1 - e_2x_2, & q_{33} &= e_1x_1 - e_3x_3 \end{aligned}$$

(in this order) as a basis of the 8-dimensional space $V(1, 1)$. Using the definition of α and β we obtain:

$$\alpha = \begin{pmatrix} 0 & 0 & -\frac{2}{3}q_{22} + \frac{1}{3}q_{33} & 0 & q_{32} & 0 & q_{12} & q_{12} \\ 0 & 0 & q_{23} & 0 & \frac{1}{3}q_{22} - \frac{2}{3}q_{33} & 0 & q_{13} & q_{13} \\ \frac{1}{3}q_{22} + \frac{1}{3}q_{33} & 0 & 0 & 0 & 0 & q_{31} & -q_{21} & 0 \\ q_{13} & 0 & 0 & 0 & 0 & \frac{1}{3}q_{22} - \frac{2}{3}q_{33} & -q_{23} & 0 \\ 0 & \frac{1}{3}q_{22} + \frac{1}{3}q_{33} & 0 & q_{21} & 0 & 0 & 0 & -q_{31} \\ 0 & q_{12} & 0 & -\frac{2}{3}q_{22} + \frac{1}{3}q_{33} & 0 & 0 & 0 & -q_{32} \\ -q_{12} & 0 & q_{21} & 0 & q_{31} & -q_{32} & -\frac{1}{3}q_{22} + \frac{2}{3}q_{33} & \frac{1}{3}q_{22} + \frac{1}{3}q_{33} \\ 0 & -q_{13} & q_{21} & -q_{23} & q_{31} & 0 & \frac{1}{3}q_{22} + \frac{1}{3}q_{33} & \frac{2}{3}q_{22} - \frac{1}{3}q_{33} \end{pmatrix},$$

$$\beta = \begin{pmatrix} 0 & 0 & \frac{1}{3}q_{22} + \frac{1}{3}q_{33} & q_{13} & 0 & 0 & -q_{12} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3}q_{22} + \frac{1}{3}q_{33} & q_{12} & 0 & -q_{13} \\ -\frac{2}{3}q_{22} + \frac{1}{3}q_{33} & q_{23} & 0 & 0 & 0 & 0 & q_{21} & q_{21} \\ 0 & 0 & 0 & 0 & q_{21} & -\frac{2}{3}q_{22} + \frac{1}{3}q_{33} & 0 & -q_{23} \\ q_{32} & \frac{1}{3}q_{22} - \frac{2}{3}q_{33} & 0 & 0 & 0 & 0 & q_{31} & q_{31} \\ 0 & 0 & q_{31} & \frac{1}{3}q_{22} - \frac{2}{3}q_{33} & 0 & 0 & -q_{32} & 0 \\ q_{12} & q_{13} & -q_{21} & -q_{23} & 0 & 0 & -\frac{1}{3}q_{22} + \frac{2}{3}q_{33} & \frac{1}{3}q_{22} + \frac{1}{3}q_{33} \\ q_{12} & q_{13} & 0 & 0 & -q_{31} & -q_{32} & \frac{1}{3}q_{22} + \frac{1}{3}q_{33} & \frac{2}{3}q_{22} - \frac{1}{3}q_{33} \end{pmatrix}.$$

Notice that $\alpha = \beta^t$.

Theorem 3.1 is of particular importance in applications to the question of rationality of quotient spaces $\mathbb{P}(V(a, b))/G$. In the following, if a linear algebraic group Γ acts on a variety X , the quotient X/Γ is always taken in the sense of Rosenlicht: there is a non-empty Γ -invariant open subset $U \subset X$ for which a geometric quotient U/Γ exists, and X/Γ denotes any birational model for this quotient.

We need the following extension of the double bundle method of [1], see also [10].

Proposition 5.2. *Let V and W be representations of a connected reductive group Γ with $\dim V - \dim W =: k > 0$. Let U be a subrepresentation of $\text{Hom}(V, W)$ such that for generic $u \in U$ the corresponding map in $\text{Hom}(V, W)$ has full rank so that we get a rational map*

$$\begin{aligned} \varphi : \mathbb{P}(U) &\dashrightarrow \mathbb{G} := \text{Grass}(k, V), \\ [u] &\mapsto \ker(u). \end{aligned}$$

Let us assume furthermore that

- φ is dominant; this is equivalent to saying that a fiber $\varphi^{-1}(\varphi([u]))$ has dimension $\dim \mathbb{P}(U) - \dim \mathbb{G}$.
- \mathbb{G}/Γ is stably rational in the sense that $(\mathbb{G}/\Gamma) \times \mathbb{P}^r$ is rational for some $r \leq \dim \mathbb{P}(U) - \dim \mathbb{G}$.
- Let Z be the kernel of the action of Γ on \mathbb{G} : assume Γ/Z acts generically freely on \mathbb{G} , Z acts trivially on $\mathbb{P}(U)$, and there exists a Γ/Z -linearized line bundle \mathcal{L} on the product $\mathbb{P}(U) \times \mathbb{G}$ cutting out $\mathcal{O}(1)$ on the fibers of the projection to \mathbb{G} .

Then $\mathbb{P}(U)/\Gamma$ is rational.

Proof. Let $X :=$ the (closure of) the graph of φ , $p : X \rightarrow \mathbb{G}$ the restriction of the projection which (maybe after shrinking \mathbb{G}) we may assume to be a projective space bundle for which \mathcal{L} is a relatively ample bundle cutting out $\mathcal{O}(1)$ on the fibers. The main technical point is the following result from descent theory [8], §7.1: there are non-empty open subsets $X_0 \subset X$ and $\mathbb{G}_0 \subset \mathbb{G}$ such that we have a fiber product square with the bottom horizontal arrow a Γ/Z -principal bundle:

$$\begin{array}{ccc} X_0 & \longrightarrow & X_0/(\Gamma/Z) \\ p \downarrow & & \bar{p} \downarrow \\ \mathbb{G}_0 & \longrightarrow & \mathbb{G}_0/(\Gamma/Z) \end{array}$$

and by [8], §7.1, \mathcal{L} descends to a line bundle $\bar{\mathcal{L}}$ on $X_0/(\Gamma/Z)$ cutting out $\mathcal{O}(1)$ on the fibers of \bar{p} . Hence \bar{p} is also a Zariski locally trivial projective bundle (of the same rank as p). It then follows that $\mathbb{P}(U)/\Gamma$ is rational. \square

Theorem 3.1 in conjunction with Proposition 5.2 yields rationality results for spaces of mixed tensors of which the following is a sample:

Theorem 5.3. *The space $\mathbb{P}(V(4, 4))/\text{SL}_3(\mathbb{C})$ is rational.*

Proof. In fact

$$\begin{aligned} V(1, 7) &\subset V(4, 4) \otimes V(2, 5), \\ \dim V(4, 4) &= 125, \quad \dim V(2, 5) = 81, \quad \dim V(1, 7) = 80, \end{aligned}$$

and the multiplicity of $V(1, 7)$ in $V(4, 4) \otimes V(2, 5)$ is 2. More precisely here $s = 3$ and $t = 1$. The most restrictive inequality of Proposition 3.1 is

$$0 \leq j \leq c - t = 1$$

in this situation. Therefore $\psi = \vartheta \circ \beta \circ \alpha^2$ and $\phi = \vartheta \circ \alpha^3$ are independent equivariant projections to $V(1, 7)$. We will use ψ in this argument.

We now consider the induced map

$$\Psi : \mathbb{P}(V(4, 4)) \dashrightarrow \mathbb{P}(V(2, 5)).$$

On $\mathbb{P}(V(4, 4)) \times \mathbb{P}(V(2, 5))$ we can use $\mathcal{L} = \mathcal{O}(1) \boxtimes \mathcal{O}(1)$ as $\mathrm{PGL}_3(\mathbb{C})$ -linearized line bundle. Moreover, $\mathbb{P}(V(2, 5))/\mathrm{PGL}_3(\mathbb{C})$ is stably rational of level 19 since the action of $\mathrm{PGL}_3(\mathbb{C})$ on pairs of 3×3 matrices by simultaneous conjugation is almost free, and the quotient is known to be rational.

Now consider a point $x_0 \in V(4, 4)$. If the map

$$\psi(x_0, \cdot) : V(2, 5) \rightarrow V(1, 7)$$

has maximal rank 80, Ψ is well defined. In this situation let y_0 be a generator of $\ker \psi(x_0, \cdot)$. If the map

$$\psi(\cdot, y_0) : V(4, 4) \rightarrow V(1, 7)$$

has also rank 80 we obtain that the fiber $\Psi^{-1}(\Psi([x_0]))$ has the expected dimension. For a random x_0 it is straightforward to check all of this using a computer algebra program. Notice that this can even be checked over a finite field, since the rank of a matrix is semi-continuous over $\mathrm{Spec} \mathbb{Z}$. See [2] for a Macaulay2-script. We can therefore apply Proposition 5.2 and obtain that $\mathbb{P}(V(4, 4))/\mathrm{SL}_3(\mathbb{C})$ is rational. \square

The following result allows us to make use of Grassmannians other than projective spaces in some cases as well.

Proposition 5.4. *Let V be a (finite dimensional as always) representation of $\Gamma = \mathrm{SL}_p(\mathbb{C})$, p prime. Let $\mathbb{G} := \mathrm{Grass}(k, V)$ be the Grassmannian of k -dimensional subspaces of V . Assume:*

- *The kernel Z of the action of Γ on $\mathbb{P}(V)$ coincides with the center $\mathbb{Z}/p\mathbb{Z}$ of $\mathrm{SL}_p(\mathbb{C})$ and the action of Γ/Z on $\mathbb{P}(V)$ is almost free. Furthermore, the action of Γ on V is almost free and each element of Z not equal to the identity acts homothetically as multiplication by a primitive p th root of unity.*
- $k \leq \dim V - \dim \Gamma - 1$.
- p does not divide k .

Then \mathbb{G}/Γ is stably rational, in fact, $\mathbb{G}/\Gamma \times \mathbb{P}^{\dim \Gamma + 1}$ is rational.

Proof. Let $C \subset \Lambda^k(V)$ be the affine cone over \mathbb{G} consisting of pure (complete decomposable) k -vectors. We will show that under the assumptions of the proposition, the action of Γ on C

is almost free. This will accomplish the proof since C/Γ is generically a torus bundle over \mathbb{G}/Γ hence Zariski-locally trivial since tori are special groups, and the group $\Gamma = \mathrm{SL}_p(\mathbb{C})$ is also special. Recall that a linear algebraic group is called special if every étale locally trivial principal bundle for the group in question is Zariski locally trivial. See [9] for the related theory. So $C/\Gamma \times \Gamma$ is birational to C , hence rational, and Γ is of course rational as a variety.

Let $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ be a general k -vector in $\Lambda^k(V)$. Since $k \leq \dim V - \dim \Gamma - 1$ and, in $\mathbb{P}(V)$, $\dim(\Gamma \cdot [v_1]) = \dim \Gamma$ since Z is finite and Γ/Z acts almost freely on $\mathbb{P}(V)$, the $(k - 1)$ -dimensional projective linear subspace spanned by v_1, \dots, v_k in $\mathbb{P}(V)$ will intersect the $\dim V - 1 - \dim \Gamma$ codimensional orbit $\Gamma \cdot [v_1]$ only in $[v_1]$. Hence, if an element $g \in \Gamma$ stabilizes $v_1 \wedge \cdots \wedge v_k$, it must lie in Z . Thus $g \cdot (v_1 \wedge \cdots \wedge v_k) = \zeta^k (v_1 \wedge \cdots \wedge v_k)$ for a primitive p -th root of unity ζ if $g \neq 1$. But since p does not divide k , the case $g \neq 1$ cannot occur. \square

As an application we prove the following result which has not been obtained by other techniques so far.

Theorem 5.5. *The moduli space $\mathbb{P}(\mathrm{Sym}^{34}(\mathbb{C}^3)^\vee)/\mathrm{SL}_3(\mathbb{C})$ of plane curves of degree 34 is rational.*

Proof. We have

$$V(0, 34) \subset \mathrm{Hom}(V(14, 1), V(0, 21)),$$

with multiplicity one and $\dim V(0, 34) = 630$, $\dim V(14, 1) = 255$, $\dim V(0, 21) = 253$. In this case $s = 14$ and $t = 0$ and the strongest restriction in Theorem 3.1 is

$$14 = s + t - a \leq j \leq b + t = 14.$$

The projection

$$\psi : V(0, 34) \otimes V(14, 1) \rightarrow V(0, 21)$$

is therefore given by $\psi = \beta^{14}$. Using this we get an induced rational map

$$\Psi : \mathbb{P}(V(0, 34)) \dashrightarrow \mathrm{Grass}(2, V(14, 1))$$

with $\dim \mathbb{P}(V(0, 34)) = 629$ and $\dim \mathrm{Grass}(2, V(14, 1)) = 506$. Moreover, Proposition 5.4 shows that $\mathrm{Grass}(2, V(14, 1))/\mathrm{SL}_3(\mathbb{C}) \times \mathbb{P}^9$ is rational, and the action of $\mathrm{PGL}_3(\mathbb{C}) = \mathrm{SL}_3(\mathbb{C})/Z$, where Z is the center of $\mathrm{SL}_3(\mathbb{C})$, is almost free on $\mathrm{Grass}(2, V(14, 1))$. Moreover, let $\mathcal{O}_P(1)$ be the $\mathrm{SL}_3(\mathbb{C})$ -linearized line bundle induced by the Plücker embedding

$$\mathrm{Grass}(2, V(14, 1)) \subset \mathbb{P}(\Lambda^2(V(14, 1))).$$

If we choose on $\mathbb{P}(V(0, 34)) \times \mathrm{Grass}(2, V(14, 1))$ the bundle $\mathcal{L} := \mathcal{O}(1) \boxtimes \mathcal{O}_P(2)$, all the assumptions of Proposition 5.2 except the dominance of φ have been checked. The latter dominance follows from an explicit computer calculation, as follows:

Choose a random point $x_0 \in V(0, 34)$. If the map

$$\psi(x_0, \cdot) : V(14, 1) \rightarrow V(0, 21)$$

has maximal rank 253, Ψ is well defined. In this case compute a basis y_1, y_2 of $\ker \psi(x_0, \cdot)$. Compute then the two 253×630 -matrices M_1 resp. M_2 representing $\psi(\cdot, y_1)$ resp. $\psi(\cdot, y_2)$. If

$$M := \begin{pmatrix} M_1 \\ M_2 \end{pmatrix},$$

which is a 506×630 matrix, has maximal rank $506 = 2 \cdot 253$, the kernel of M represents the fiber $\Psi^{-1}(\Psi([x_0]))$ and is of expected dimension. Again one can easily do this calculation over a finite field using a computer algebra program. See [2] for a Macaulay2 script. \square

Remark 5.6. As far as we can see, the rationality of $\mathbb{P}(V(0, 34))/\mathrm{SL}_3(\mathbb{C})$ cannot be obtained by direct application of Proposition 5.2 with base of the projection a projective space. In fact, a computer search yields that the inclusion $V(0, 34) \subset \mathrm{Hom}(V(30, 0), V(0, 4) \oplus V(5, 9))$ is the only candidate to be taken into consideration for dimension reasons: $\dim V(30, 0) = \dim(V(0, 4) \oplus V(5, 9)) + 1$ and $\dim \mathbb{P}(V(0, 34)) > \dim \mathbb{P}(V(30, 0))$. However, on $\mathbb{P}(V(0, 34)) \times \mathbb{P}(V(30, 0))$ there does not exist a $\mathrm{PGL}_3(\mathbb{C})$ -linearized line bundle cutting out $\mathcal{O}(1)$ on the fibers of the projection to $\mathbb{P}(V(30, 0))$; for such a line bundle would have to be of the form $\mathcal{O}(1) \boxtimes \mathcal{O}(k)$, $k \in \mathbb{Z}$, and none of these is $\mathrm{PGL}_3(\mathbb{C})$ -linearized: since $\mathcal{O} \boxtimes \mathcal{O}(1)$ is $\mathrm{PGL}_3(\mathbb{C})$ -linearized it would follow that the $\mathrm{SL}_3(\mathbb{C})$ action on $H^0(\mathbb{P}(V(0, 34), \mathcal{O}(1))) \simeq V(34, 0)$ factors through $\mathrm{PGL}_3(\mathbb{C})$ which is not the case.

References

- [1] F. Bogomolov, P. Katsylo, Rationality of some quotient varieties, *Mat. Sb.* 126 (1985) 584–589.
- [2] Christian Böhning, H.-Chr. Graf v. Bothmer, Macaulay2 scripts for A Clebsch–Gordan formula for $\mathrm{SL}_3(\mathbb{C})$ and applications to rationality, available at <http://www.uni-math.gwdg.de/bothmer/ClebschGordan>, 2008.
- [3] M. Brion, G.W. Schwarz, *Théorie des invariants & Géométrie des variétés quotients*, Collection Travaux en Cours, Hermann Éditeurs des Sciences et des Arts, Paris, 2000.
- [4] I. Dolgachev, Rationality of fields of invariants, *Algebraic Geometry*, Bowdoin, Proc. Symp. Pure Math. 46 (1987) 3–16.
- [5] W. Fulton, J. Harris, *Representation Theory. A First Course*, Springer G.T.M., vol. 129, Springer-Verlag, 1991.
- [6] P. Katsylo, On the birational geometry of $(\mathbb{P}^1)^{(m)}/\mathrm{GL}_{n+1}$, Max-Planck-Institut preprint, MPI/94-144, 1994.
- [7] M. Maliakas, P.J. Olver, Explicit generalized Pieri maps, *J. Algebra* 148 (1992) 68–85.
- [8] D. Mumford, J. Fogarty, F. Kirwan, *Geometric Invariant Theory*, third enlarged edition, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 34, Springer-Verlag, 1994.
- [9] J.-P. Serre, *Espaces fibrés algébriques*, *Sém. Claude Chevalley* 3 (1) (1958) 1–37.
- [10] N.I. Shepherd-Barron, Rationality of moduli spaces via invariant theory, in: *Topological Methods in Algebraic Transformation Groups*, New Brunswick, NJ, 1988, in: *Progr. Math.*, vol. 80, Birkhäuser Boston, Boston, MA, 1989, pp. 153–164.
- [11] J. Weyman, *Cohomology of Vector Bundles and Syzygies*, *Cambridge Tracts in Mathematics*, vol. 149, Cambridge University Press, 2003.