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Ranking by Eigenvector Versus Other Methods in the Analytic Hierarchy Process

T. L. SAATY
University of Pittsburgh, 322 Mervis Hall
Pittsburgh, PA 15260, U.S.A.
Saaty@vms.cis.pitt.edu

G. HU
University of Pittsburgh, 329 Mervis Hall
Pittsburgh, PA 15260, U.S.A.

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Abstract—Counter-examples are given to show that in decision making, different methods of deriving priority vectors may be close for every single pairwise comparison matrix, yet they can lead to different overall rankings. When the judgments are inconsistent, their transitivity affects the final outcome, and must be taken into consideration in the derived vector. It is known that the principal eigenvector captures transitivity uniquely and is the only way to obtain the correct ranking on a ratio scale of the alternatives of a decision. Because of this and of the counter-examples given below, one should only use the eigenvector for ranking in making a decision. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Pairwise-comparison, Eigenvector, Logarithmic least squares, Transitivity, Decision making.

1. INTRODUCTION

It is essential that a credible decision theory yields unique answers for the alternatives of a decision, perhaps not only in terms of ranks, but also in terms of priorities. A mathematical approach to decision making may have different algorithms suggested for deducing scales or priority vectors from elicited judgments. The Analytic Hierarchy Process (AHP) is a mathematical theory for deriving ratio scale priority vectors from positive reciprocal matrices with entries established by paired comparisons. The AHP uses a principal Eigenvalue Method (EM) to derive priority vectors (see [1,2]). Several other methods have also been proposed. Among them are the geometric mean or Logarithmic Least Squares Method (LLSM) (see [3,4]), the Least Squares Method (LSM), and others. It has been proved that when the positive reciprocal matrix is consistent ($a_{ij}a_{jk} = a_{ik}$), all these methods lead to the same outcome, and hence, by synthesizing the overall ranking according to a well-prescribed procedure, they lead to the same decision. But in real life, judgments are frequently inconsistent, and these different methods give rise to different priority vectors (see [1,5,6]). In multicriteria processes, different methods may each produce a ranking of the alternatives of a decision that is different than another method. Such variability in ranking violates the uniqueness requirement mentioned above and is, therefore, unacceptable.

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Some people have argued that if the inconsistency is "small", and hence, the results of all the methods are small perturbations of their values at consistency where they coincide, it may not matter much which method one uses. Their thought may be that since some other methods, for example LLSM, are computationally simpler than EM, it is both convenient and efficient to use methods other than EM. But it does matter because of the potential variability in the rankings. Surprisingly, in multicriteria processes even when different methods yield priority vectors that are close, both on criteria and alternatives, after synthesis (weighting and adding) the rank order of the alternatives may differ, one may end up choosing a less desirable alternative over a more desirable one. We show this below with examples.

We need to highlight to the reader the importance of transitivity in this context. There are two kinds of transitivity: one is ordinal and the other is cardinal. The first is that if A is preferred to B and B to C, then A must be preferred to C. The second is that if A is preferred to B three times and B to C twice, then A must be preferred to C six times, a much stronger requirement which implies consistency. A consistent matrix is cardinally transitive and, therefore, ordinally transitive. An inconsistent matrix need not be either. How to capture the necessary numerical transitivities in an inconsistent matrix to produce a vector of priorities is a crucial concern. It has been shown that EM is the only method which "directly deals with the question of inconsistency and captures the rank order inherent in the inconsistent data" (see [1,5]).

2. EXAMPLES

Our examples begin by perturbing several consistent matrices of judgments occurring in a decision to get inconsistent matrices. We then derive priorities for these matrices using different methods. The different methods yield close results for each matrix. However, when these priority vectors are synthesized according to the AHP, the final vectors for the alternatives yield different rankings. It suffices to show that two such methods can give different final rankings. Because EM is known to uniquely capture transitivity in inconsistent matrices, we will compare the outcome of perturbations from consistency of EM with one other method (LLSM) because of the simple way in which we can represent and compute with the latter.

We denote the priority vector derived from the two methods by $w = (w_1, w_2, \ldots, w_n)^\top$ and $x = (x_1, x_2, \ldots, x_n)^\top$, respectively. The judgment matrix is denoted as $A = (a_{ij})$, $a_{ji} = 1/a_{ij}$, $a_{ij} > 0$, $i, j = 1, 2, \ldots, n$. EM solves for the principal eigenvector in $Aw = \lambda_{\max} w$, where $\lambda_{\max}$ is the principal eigenvalue of A. The EM solution is given by

$$w = \lim_{k \to \infty} \left(\frac{A^k e^\top}{e A^k e^\top} \right),$$

where $e = (1, 1, \ldots, 1)$. LLSM minimizes $\sum_{i,j=1}^{n} (\log a_{ij} - \log x_i/x_j)^2$ with respect to x. The LLSM solution is given by the normalized products of the elements in each row:

$$x_i = \frac{\left(\prod_{j=1}^{n} a_{ij} \right)^{1/n}}{\sum_{i=1}^{n} \left(\prod_{j=1}^{n} a_{ij} \right)^{1/n}}, \qquad i = 1, \ldots, n.$$

The LLSM formulation requires the extra assumption of the minimization of differences. The EM formulation requires no such assumption. It is known that EM and LLSM coincide in their results when $n \leq 3$. Thus, for the simplest example to demonstrate variability in ranks, we must use at least four criteria and four alternatives. That is what we do here. It should be even easier to construct examples with a larger number of criteria and of alternatives.

Our first example is a simple AHP model. Under the overall goal, there are four criteria: C_1, C_2, C_3, and C_4. Under each criterion, there are four alternatives: A_1, A_2, A_3, and A_4, which are the same for all the four criteria. The judgment matrices and the corresponding priority vectors derived by using EM and LLSM, respectively, are listed below:

with respect to the GOAL:

$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} 1 & 2 & 2 & 4 \\ 1/2 & 1 & 3 & 3 \\ 1/2 & 1/3 & 1 & 4 \\ 1/4 & 1/3 & 1/4 & 1 \end{pmatrix} \end{array} \begin{array}{c} \text{EM} \\ \begin{pmatrix} 0.412 \\ 0.316 \\ 0.194 \\ 0.079 \end{pmatrix} \end{array} \begin{array}{c} \text{LLSM} \\ \begin{pmatrix} 0.422 \\ 0.307 \\ 0.191 \\ 0.080 \end{pmatrix} \end{array},$$

with respect to criterion C_1:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} 1 & 2 & 2 & 4 \\ 1/2 & 1 & 3 & 3 \\ 1/2 & 1/3 & 1 & 4 \\ 1/4 & 1/3 & 1/4 & 1 \end{pmatrix} \end{array} \begin{array}{c} \text{EM} \\ \begin{pmatrix} 0.412 \\ 0.316 \\ 0.194 \\ 0.079 \end{pmatrix} \end{array} \begin{array}{c} \text{LLSM} \\ \begin{pmatrix} 0.422 \\ 0.307 \\ 0.191 \\ 0.080 \end{pmatrix} \end{array},$$

with respect to criterion C_2:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} 1 & 1/4 & 1/3 & 1/4 \\ 4 & 1 & 4 & 1/2 \\ 3 & 1/4 & 1 & 1/2 \\ 4 & 2 & 2 & 1 \end{pmatrix} \end{array} \begin{array}{c} \text{EM} \\ \begin{pmatrix} 0.075 \\ 0.354 \\ 0.160 \\ 0.410 \end{pmatrix} \end{array} \begin{array}{c} \text{LLSM} \\ \begin{pmatrix} 0.078 \\ 0.347 \\ 0.162 \\ 0.413 \end{pmatrix} \end{array},$$

with respect to criterion C_3:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} 1 & 1/4 & 1/3 & 1/4 \\ 4 & 1 & 4 & 1/2 \\ 3 & 1/4 & 1 & 1/2 \\ 4 & 2 & 2 & 1 \end{pmatrix} \end{array} \begin{array}{c} \text{EM} \\ \begin{pmatrix} 0.075 \\ 0.354 \\ 0.160 \\ 0.410 \end{pmatrix} \end{array} \begin{array}{c} \text{LLSM} \\ \begin{pmatrix} 0.078 \\ 0.347 \\ 0.162 \\ 0.413 \end{pmatrix} \end{array},$$

with respect to criterion C_4:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} 1 & 2 & 2 & 4 \\ 1/2 & 1 & 3 & 3 \\ 1/2 & 1/3 & 1 & 4 \\ 1/4 & 1/3 & 1/4 & 1 \end{pmatrix} \end{array} \begin{array}{c} \text{EM} \\ \begin{pmatrix} 0.412 \\ 0.316 \\ 0.194 \\ 0.079 \end{pmatrix} \end{array} \begin{array}{c} \text{LLSM} \\ \begin{pmatrix} 0.422 \\ 0.307 \\ 0.191 \\ 0.080 \end{pmatrix} \end{array}.$$

We can see that for each judgment matrix above, the corresponding EM and LLSM priority vectors are close by any measure, and there is no difference in the ranks given by EM and LLSM. Nevertheless, after synthesis, we obtain the following overall ranking:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \text{EM} \\ \begin{pmatrix} 0.240 \\ 0.335 \\ 0.177 \\ 0.248 \end{pmatrix} \end{array} \begin{array}{c} \text{LLSM} \\ \begin{pmatrix} 0.251 \\ 0.327 \\ 0.176 \\ 0.246 \end{pmatrix} \end{array}.$$

We note that the two methods rank the alternatives differently. We have

$$\begin{array}{ll} \text{EM} & A_2 > A_4 > A_1 > A_3, \\ \text{LLSM} & A_2 > A_1 > A_4 > A_3. \end{array}$$

The ranks of A_1 and A_4 by the two methods are different here. The differences may seem to be small, but they are not. Note that the difference of the weights of A_1 and A_4 in EM is $-0.008(0.240\text{–}0.248)$, and in LLSM is $+0.005(0.251\text{–}0.246)$. The sum of these differences caused by using the two methods is 0.013. Dividing 0.013 by the average weight of the alternatives, which is $0.246 = (0.240 + 0.248 + 0.251 + 0.246)/4$, we obtain 5.3%. Under some circumstances of an important and very tight decision, one may consider this difference to be nontrivial and should not be ignored.

Incidentally, the AHP synthesizes the final ranks of the alternatives in two ways. One is the distributive mode which involves weighting and adding just used in the above example. It allows rank reversal when alternatives are added or deleted. The other is the ideal or performance mode which involves dividing the priorities of the alternatives under each criterion by the largest among them, and then weighting by the priorities of the criteria and adding. It allows for rank preservation with respect to irrelevant or dominated alternatives. We also repeated the foregoing calculations with the ideal mode, and obtained the same ranks for the alternatives as above for each method, which means that variability between the two methods occurs with either of these two modes.

In passing, we note that there has been a proposal by some users of LLSM to synthesize the weights of the alternatives by raising the priority of each to the power of its corresponding criterion weight, multiplying the results for each alternative, and finally normalizing with respect to the overall weights of all the alternatives. We obtain for the foregoing example:

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{pmatrix} 0.422^{0.422} * 0.078^{0.307} * 0.078^{0.191} * 0.422^{0.080} \\ 0.307^{0.422} * 0.347^{0.307} * 0.347^{0.191} * 0.307^{0.080} \\ 0.191^{0.422} * 0.162^{0.307} * 0.162^{0.191} * 0.191^{0.080} \\ 0.080^{0.422} * 0.413^{0.307} * 0.413^{0.191} * 0.080^{0.080} \end{pmatrix} = \begin{pmatrix} 0.183 \\ 0.327 \\ 0.176 \\ 0.181 \end{pmatrix} \xrightarrow{\text{normalized}} \begin{pmatrix} 0.211 \\ 0.377 \\ 0.203 \\ 0.209 \end{pmatrix}.$$

This process yields the same ranking as the additive composition of LLSM given above. In other words, it also yields $A_2 > A_1 > A_4 > A_3$, which is again different from the ranking given by EM. When this approach to composition is applied to the EM data above, one obtains the following ranks for the alternatives: $A_2 > A_4 > A_3 > A_1$, which is still at odds with the above result for EM. Note that hierarchic composition in the AHP as illustrated above is a special case of network composition that involves dependence and feedback. It has a strong theoretical foundation and has been used in numerous cases of successful numerical prediction that give credibility to the process of weighting and adding to obtain the overall results.

A better example than the above would be that rank changes on all four alternatives so that $A_4 > A_3 > A_2 > A_1$ becomes $A_1 > A_2 > A_3 > A_4$, which we call strong variability in ranks as opposed to the one in our first example, which we call weak variability in ranks occurring between only two alternatives. Again, in this example, the derived vectors are "close" for each matrix, yet the final ranks (but not the numerical values) are as far apart as possible:

with respect to the GOAL:

$$\begin{array}{c|c} & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} & \begin{pmatrix} 1 & 4 & 2 & 2 \\ 1/4 & 1 & 1/3 & 1/4 \\ 1/2 & 3 & 1 & 3 \\ 1/2 & 4 & 1/3 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \text{EM} \\ \begin{pmatrix} 0.412 \\ 0.079 \\ 0.316 \\ 0.194 \end{pmatrix} \end{array} \quad \begin{array}{c} \text{LLSM} \\ \begin{pmatrix} 0.422 \\ 0.080 \\ 0.307 \\ 0.191 \end{pmatrix} \end{array},$$

with respect to criterion C_1:

$$\begin{array}{c|c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 1 & 4 & 2 & 2 \\ 1/4 & 1 & 1/3 & 1/4 \\ 1/2 & 3 & 1 & 3 \\ 1/2 & 4 & 1/3 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \text{EM} \\ \begin{pmatrix} 0.412 \\ 0.079 \\ 0.316 \\ 0.194 \end{pmatrix} \end{array} \quad \begin{array}{c} \text{LLSM} \\ \begin{pmatrix} 0.422 \\ 0.080 \\ 0.307 \\ 0.191 \end{pmatrix} \end{array},$$

with respect to criterion C_2:

$$\begin{array}{c|c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 1 & 4 & 2 & 2 \\ 1/4 & 1 & 1/3 & 1/4 \\ 1/2 & 3 & 1 & 3 \\ 1/2 & 4 & 1/3 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \text{EM} \\ \begin{pmatrix} 0.412 \\ 0.079 \\ 0.316 \\ 0.194 \end{pmatrix} \end{array} \quad \begin{array}{c} \text{LLSM} \\ \begin{pmatrix} 0.422 \\ 0.080 \\ 0.307 \\ 0.191 \end{pmatrix} \end{array},$$

with respect to criterion C_3:

$$\begin{array}{c|c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 1 & 1/4 & 1/4 & 1/3 \\ 4 & 1 & 4 & 2 \\ 4 & 1/2 & 1 & 1/3 \\ 3 & 1/2 & 3 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \text{EM} \\ \begin{pmatrix} 0.079 \\ 0.412 \\ 0.194 \\ 0.316 \end{pmatrix} \end{array} \quad \begin{array}{c} \text{LLSM} \\ \begin{pmatrix} 0.080 \\ 0.422 \\ 0.191 \\ 0.307 \end{pmatrix} \end{array},$$

with respect to criterion C_4:

$$\begin{array}{c|c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 1 & 1/4 & 1/4 & 1/3 \\ 4 & 1 & 4 & 2 \\ 4 & 1/2 & 1 & 1/3 \\ 3 & 1/2 & 3 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \text{EM} \\ \begin{pmatrix} 0.079 \\ 0.412 \\ 0.194 \\ 0.316 \end{pmatrix} \end{array} \quad \begin{array}{c} \text{LLSM} \\ \begin{pmatrix} 0.080 \\ 0.422 \\ 0.191 \\ 0.307 \end{pmatrix} \end{array}.$$

Here again, for each judgment matrix, the corresponding EM and LLSM priority vectors are close, and there is no difference in the ranks given by the two methods. Nevertheless, after synthesis, we obtain the following overall ranking:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \text{EM} \\ \begin{pmatrix} 0.242 \\ 0.248 \\ 0.253 \\ 0.256 \end{pmatrix} \end{array} \begin{array}{c} \text{LLSM} \\ \begin{pmatrix} 0.2518 \\ 0.2503 \\ 0.2492 \\ 0.2487 \end{pmatrix} \end{array}.$$

We note that this is an example of strong variability in ranks, in which we have

$$\begin{array}{ll} \text{EM} & A_4 > A_3 > A_2 > A_1, \\ \text{LLSM} & A_1 > A_2 > A_3 > A_4. \end{array}$$

We now summarize with the following theorem.

THEOREM. *The eigenvalue method is necessary and sufficient to uniquely capture the ratio scale rank order inherent in inconsistent pairwise comparison judgments.*

3. CONCLUSION

Our example illustrates the difference between metric topology and order topology. In the former, the central concern is closeness; in the latter, it is both closeness and order preservation as in EM.

To recapitulate, EM is the only valid method for deriving the priority vector from a pairwise comparison matrix, particularly when the matrix is inconsistent. In the context of multicriteria decisions, even if variability in ranks does not occur for each individual judgment matrix, it may still occur in the overall ranking of the final alternatives due to the multicriteria process itself. Because all other proposed methods, for example LLSM, can give different rankings, they are unacceptable for deriving priorities in decision making as they do not capture the essential idea of transitivity.

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