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Note

The existence and uniqueness of strong kings in tournaments[☆]

An-Hang Chen[a], Jou-Ming Chang[b], Yuwen Cheng[c], Yue-Li Wang[d,a,*]

[a] *Department of Information Management, National Taiwan University of Science and Technology, Taipei, Taiwan, ROC*
[b] *Department of Information Management, National Taipei College of Business, Taipei, Taiwan, ROC*
[c] *Department of Mathematics, National Taitung University, Taitung, Taiwan, ROC*
[d] *Department of Computer Science and Information Engineering, National Chi Nan University, Nantou, Taiwan, ROC*

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Abstract

A king x in a tournament T is a player who beats any other player y directly (i.e., $x \to y$) or indirectly through a third player z (i.e., $x \to z$ and $z \to y$). For $x, y \in V(T)$, let $b(x, y)$ denote the number of third players through which x beats y indirectly. Then, a king x is strong if the following condition is fulfilled: $b(x, y) > b(y, x)$ whenever $y \to x$. In this paper, a result shows that for a tournament on n players there exist exactly k strong kings, $1 \leqslant k \leqslant n$, with the following exceptions: $k = n - 1$ when n is odd and $k = n$ when n is even. Moreover, we completely determine the uniqueness of tournaments.

Keywords: Tournaments; Kings; Strong kings; Score sequences

1. Introduction

An *n-tournament* $T = (V, E)$ is an orientation of the complete graph with n vertices, where V is the vertex set and E is the arc set of T. For convenience, we use $v \in T$ instead of $v \in V(T)$. Usually, an n-tournament can represent n players in a round robin tournament in which every two players compete exactly once to decide the winner and a tie is not permitted. The literature on tournaments is rather vast, and the reader is referred to [5,9,10] for further references.

For a tournament $T = (V, E)$, we write $x \to y$ to mean that x beats y if $xy \in E$. Further, we use $A \to B$ to denote that every player of A beats every player of B, where $A, B \subset V$ are two disjoint subsets. For simplicity, $x \to B$ stands for $\{x\} \to B$, and $A \to y$ for $A \to \{y\}$. A *king* x in a tournament is a player who beats any other player y directly (i.e., $x \to y$) or indirectly through a third player z (i.e., $x \to z$ and $z \to y$). The definition of kings in tournaments emerged from the work of the mathematical sociologist Landau in 1953 [2]. Subsequently, in [3], Maurer used the

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* Corresponding author: Department of Computer Science and Information Engineering, National Chi Nan University, No. 1 University Road, Puli, Nantou Hsien 545, Taiwan, ROC.
E-mail address: yuelwang@ncnu.edu.tw (Y.-L Wang).

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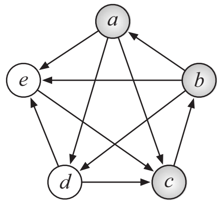

Fig. 1. A sample tournament, kings and strong kings.

idea of kings to model the dominance in flocks of chickens and presented an inductive construction to characterize the possible number of kings in a tournament. Additional results on the number of kings in tournaments can also be found in [6–8].

Recently, in [1], Ho and Chang showed that the notion of kings mentioned in [3] can be strengthened by using strong kings which represent a strong sense of dominance in tournaments. In a tournament T, we define $b_T(x, y) = |\{z \in T \backslash \{x, y\}: x \rightarrow z \text{ and } z \rightarrow y\}|$ for each pair $x, y \in T$. When no ambiguity arises, we drop the index T from the notation. A player x in a tournament is said to be a *strong king* if $x \rightarrow y$ or $b(x, y) > b(y, x)$ for any other player y. Obviously, it is not true that every king is a strong king. For example, Fig. 1 depicts that players a, b and c are kings in the tournament. However, a and b are strong kings but c is not since $a \rightarrow c$ and $b(c, a) = 1 < b(a, c) = 2$.

The purpose of this paper is to characterize the existence and the uniqueness of an n-tournament with exactly k strong kings for all possible k, where a tournament is said to be unique if no other tournament with the same order (barring isomorphic ones) has the same number of strong kings.

2. Preliminaries

For a player x in a tournament T, let $O_T(x)$ denote the *out-set* of x (i.e., the set of players beaten by x) and $I_T(x)$ denote the *in-set* (i.e., the set of players beating x). Let $d_T^+(x) = |O_T(x)|$ and $d_T^-(x) = |I_T(x)|$, where $d_T^+(x)$ is also called the *score* of x in T. We drop the index T if no ambiguity arises.

The *score sequence* of an n-tournament T, denoted by $(s_1, s_2, \ldots, s_n)$, is the non-decreasing list of scores of players in T. Landau gave a necessary and sufficient condition for determining a non-decreasing sequence of integers to be the score sequence of some tournament [2]. There are many different approaches that can be used to prove this fundamental result, e.g., see [8] for a survey. In the following theorem, we provide a necessary and sufficient condition to recognize a strong king.

Theorem 1. *Let x be a player in a tournament T. Then, x is a strong king if and only if $x \rightarrow y$ for every player $y \in T$ with $d^+(y) > d^+(x)$.*

Proof. An easy observation shows that for every two players $x, y \in T$, if $y \rightarrow x$ then $d^+(y) - d^+(x) = b(y, x) - b(x, y) + 1$, and thus $b(x, y) \leqslant b(y, x) \Leftrightarrow d^+(x) < d^+(y)$. So x is not a strong king if and only if there is a player $y \in T$ such that $y \rightarrow x$ with $d^+(y) > d^+(x)$. $\square$

From the theorem, we have the following corollary.

Corollary 2 (*Ho and Chang [1]*). *If x is a player with the maximum score in a tournament T, then x is a strong king.*

3. Existence theorem

Note that if a tournament T has a player x with $d^-(x) = 0$, then x is the only (strong) king in T, which is also called an *emperor*. Moon [4] proved that every tournament without emperor has at least three kings. An example in [1] pointed

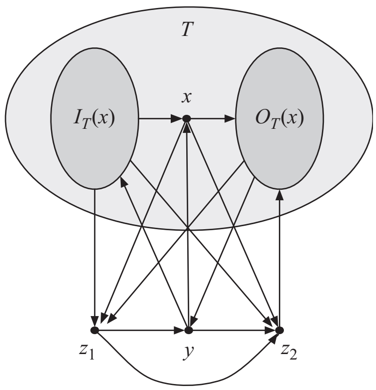

Fig. 2. An illustration for Lemma 6.

out that this does not hold for strong kings. To clarify what numbers of strong kings cannot occur in a tournament, we need the following definition.

Definition 3. For $1 \leqslant k \leqslant n$, an n-tournament is called *(n, k)-tournament* if it has exactly k strong kings.

Lemma 4. *If there exists an (n, k)-tournament, then there exists an $(n+1, k)$-tournament.*

Proof. Let T be an (n, k)-tournament. We add a new player x to T, denoted by $T + x$, such that all players of T beat x in $T + x$. Clearly, x is not a strong king. Moreover, for any player $y \in T$, the status about y to be a strong king or not does not change in $T + x$. Thus, $T + x$ is an $(n + 1, k)$-tournament. $\square$

A tournament T is called *regular* if $d^+(x) = d^-(x)$ for every player $x \in T$. Equivalently, an n-tournament is $((n-1)/2)$-regular if and only if n is odd and the score of each player is equal to $(n-1)/2$. It is well-known that regular tournaments exist when n is any odd. The following lemma directly follows from Corollary 2 and the fact that every player in a regular tournament has the maximum score.

Lemma 5. *There exist (n, n)-tournaments for any odd integer n.*

Lemma 6. *There exist $(n, n-2)$-tournaments for any even integer $n \geqslant 4$.*

Proof. For $n = 4$, an easy examination shows that a tournament with score sequence (1, 1, 2, 2) is a (4, 2)-tournament. For any even integer $n \geqslant 6$, we will show that an $(n, n-2)$-tournament can be constructed from an $((n-4)/2)$-regular tournament. Since $n - 3$ is odd, by Lemma 5, there exists an $(n - 3, n - 3)$-tournament T such that every player has score $m = (n - 4)/2$. Let $x \in T$ be any player and add three players z_1, z_2 and y to T using the following dominance to form an n-tournament T' (see Fig. 2 for an illustration):

(1) $O_T(x) \to y \to \{x, z_2\} \cup I_T(x)$,
(2) $\{x\} \cup I_T(x) \cup O_T(x) \to z_1 \to \{y, z_2\}$,
(3) $\{x\} \cup I_T(x) \to z_2 \to O_T(x)$.

Consequently, we have the following scores in T': $d^+_{T'}(y) = |I_T(x)| + 2 = m + 2$, $d^+_{T'}(z_1) = 2$, $d^+_{T'}(z_2) = |O_T(x)| = m$, and $d^+_{T'}(x) = m + 2$. Note that the maximum score in T' is $m + 2$. By Corollary 2, any player in $T \cup \{y\}$ is a strong king. By Theorem 1, both z_1 and z_2 are not strong kings since $d^+_{T'}(x) > d^+_{T'}(z_1)$, $x \to z_1$, $d^+_{T'}(y) > d^+_{T'}(z_2)$, and $y \to z_2$. Thus, T' contains n players and exactly $n - 2$ strong kings. $\square$

Lemma 7. *There exist no $(n, n-1)$-tournaments for any odd integer $n \geqslant 3$.*

Proof. Suppose, to the contrary, that T is an n-tournament where $n \geqslant 3$ is odd, and such that T has exactly $n-1$ strong kings. Let Z be the set containing all strong kings with the maximum score and $k = |Z|$. Let Y be the other $n-1-k$ strong kings. We use Δ to denote the maximum score in T. Clearly, T is not a regular tournament and $\Delta \geqslant (n-1)/2+1$. If $k = n-1$, then $\sum_{z \in Z} d^+(z) = (n-1) \cdot \Delta > \binom{n}{2}$, a contradiction. On the other hand, if $k \leqslant n-2$, then $Y \neq \emptyset$. Since there is only one vertex that is not a strong king, any vertex in Z has a score at most k. By Theorem 1, $y \to z$ for all $y \in Y$ and $z \in Z$. Thus, $d^+(z) \leqslant k \leqslant d^+(y)$, a contradiction. □

Theorem 8. *There exist (n, k)-tournaments for all integers $1 \leqslant k \leqslant n$ with the following exceptions*:

(1) $k = n-1$ *when n is odd*, *and*
(2) $k = n$ *when n is even*.

Proof. The proof is by two separate inductions to construct (n, k)-tournaments, one is for odd integer k (by Lemmas 4 and 5) and the other is for even integer k (by Lemmas 4 and 6). The exception cases can be determined by Lemmas 7 and 4. □

4. Uniqueness theorem

Under the existence of tournaments, a natural question to ask is "Given n and k, do there exist two distinct (n, k)-tournaments?" In this section we further discuss the uniqueness of tournaments. Here is the formal definition of uniqueness.

Definition 9. A class of (n, k)-tournaments is said to be *unique* if no two non-isomorphic n-tournaments share the same number of k strong kings.

For non-unique (n, k)-tournaments T_1 and T_2, we write $T_1 \ncong T_2$ to mean that T_1 and T_2 are not isomorphic.

Lemma 10. *If the class of (n, k)-tournaments is not unique*, *then so is the class of $(n+1, k)$-tournaments*.

Proof. Suppose that both T_1 and T_2 are (n, k)-tournaments and $T_1 \ncong T_2$. We now construct two $(n+1)$-tournaments $T_1 + x$ and $T_2 + x$ from T_1 and T_2, respectively, and let all players in T_1 and T_2, respectively, beat the new player x. Clearly, $T_1 + x \ncong T_2 + x$ and both $T_1 + x$ and $T_2 + x$ are $(n+1, k)$-tournaments. □

It is well-known that there are at least two non-isomorphic regular n-tournaments for any odd integer $n \geqslant 7$. So, we can easily obtain the following lemma.

Lemma 11. *The class of (n, n)-tournaments is not unique for any odd integer $n \geqslant 7$.*

Lemma 12. *The class of $(n, n-2)$-tournaments is not unique for any even integer $n \geqslant 10$.*

Proof. Let $n \geqslant 10$ be an even integer. By Lemma 11, let T_1 and T_2 be two $(n-3, n-3)$-tournaments and $T_1 \ncong T_2$. Using the same technique as Lemma 6, we can construct two $(n, n-2)$-tournaments T_1' and T_2' from T_1 and T_2, respectively, by adding three players z_1, z_2 and y with the same dominance as Lemma 6. Since $T_1 \ncong T_2$ and the in-sets (respectively, out-sets) of z_1, z_2 and y are different in both T_1' and T_2', we conclude that $T_1' \ncong T_2'$. □

Theorem 13. *The classes of* (1, 1)-, (2, 1)-, (3, 1)-, (3, 3)-, (4, 2)-, (4, 3)-, (5, 3)-, (5, 5)-, (6, 5)-, *and* (7, 5)-*tournament are unique*.

Proof. Let n be any integer. By Lemmas 10 and 11, we can obtain at least two non-isomorphic (n, k)-tournaments for all odd integers k with $7 \leqslant k \leqslant n$. By Lemmas 10 and 12, we can obtain at least two non-isomorphic (n, k)-tournaments for all even integers k with $8 \leqslant k \leqslant n-2$. The cases not covered above are (n, k)-tournaments with $k \leqslant 6$ and $n \geqslant k$. In these cases, all the non-unique tournaments are determined by Lemma 10 and the following instances (the validity of score sequences appeared below can easily be checked by Landau's condition):

n	k	Score sequences for the (n, k)-tournaments
4	1	(0, 1, 2, 3), (1, 1, 1, 3)
5	2	(0, 2, 2, 3, 3), (1, 1, 2, 3, 3), (1, 2, 2, 2, 3)
6	3	(0, 1, 2, 4, 4, 4), (1, 1, 1, 4, 4, 4), (2, 2, 2, 3, 3, 3)
6	4	(1, 2, 3, 3, 3, 3)
8	5	(0, 1, 2, 5, 5, 5, 5, 5), (1, 1, 1, 5, 5, 5, 5, 5), (2, 3, 3, 4, 4, 4, 4, 4)
8	6	(1, 3, 4, 4, 4, 4, 4, 4), (2, 2, 4, 4, 4, 4, 4, 4)

In the above listed score sequences, (1, 2, 3, 3, 3, 3) has exactly four non-isomorphic tournaments [5]. For the other score sequences, all realized tournaments are non-isomorphic because the score sequences are distinct.

On the other hand, the uniqueness can be determined by the contrapositive of Lemma 10 and the following instances:

n	k	Score sequence for the unique (n, k)-tournaments
3	1	(0, 1, 2)
4	2	(1, 1, 2, 2)
5	3	(0, 1, 3, 3, 3)
7	5	(0, 1, 4, 4, 4, 4, 4)

From Theorem 1, if a strong king does not have the maximum score, then its score is at least the number of vertices with the maximum score. Thus, there are exactly k strong kings in the above (n, k)-tournaments. For uniqueness, all score sequences of length n without non-isomorphic tournaments for $n \leqslant 6$ can refer to [5]. For the case of (0, 1, 4, 4, 4, 4, 4), the unique (7, 5)-tournament can be constructed from the tournament of score sequence (0, 3, 3, 3, 3, 3) by adding a vertex with score zero. By summarizing all the uniqueness cases, the theorem follows. □

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