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Subgraphs with Prescribed Valencies

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ABSTRACT

In this paper a generalization of the factor problem for finite undirected graphs is detailed. We prescribe certain inequalities for the valencies of a subgraph. We deduce formulas for the minimum "deviation" of this prescription and characterize the "optimally approaching" subgraphs. These results include the conditions of Tutte and Ore for the existence of a factor and the characterization of maximal independent edge-systems given in [3] and [11].

Let us associate a non-negative integer $f(x)$ with every vertex x of a graph.[1] A subgraph is said to be an *f-factor* if its valency is $f(x)$ in every vertex x. Here many problems arise: When does a graph contain an f-factor? If it does, how can these factors be characterized; if it does not, which are the subgraphs "approaching" that wanted factor optimally?

Such problems were discussed first by Petersen. Two special cases were also discussed: the case of 1-factors and the case of bipartite graphs: these special questions can be considered to be solved essentially on the base of many papers (see, e.g., references 1–5 and 7).

In the general case Tutte gave a necessary and sufficient condition for the existence of an f-factor [8]. A different condition is due to Ore [6]. These conditions can be proved by the method of alternating path or by reducing the problem to the case of a 1-factor [9].

In this paper we follow another method. We try to answer the questions asked above. We generalize the problem and prescribe for the valencies of the wanted subgraph not an integer but an interval.[2] This generalization does not contain essentially more than the original problem, but it simplifies the discussion. A number δ will be introduced to measure how

[1] We consider finite undirected graphs. Directed graphs could be investigated similarly.

[2] Later (Section 1) we shall formulate this in a more symmetric way.

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this condition can be satisfied. In Sections 1–3 we examine how δ changes if the intervals ordered to the vertices are modified. δ will have some convexity properties.

In Sections 4–5 we characterize the structure of the graph and the "optimally approaching" subgraphs, besides a special choice of the intervals. The general case will be reducible to this case easily (Section 6). From our results we can deduce among others the following theorem:

Let a, b be natural numbers and suppose that the valencies of the vertices of a graph are $\leq a + b$. Then the graph is the union of two subgraphs (i.e., every edge of the graph belongs to one of them) so that the valencies of the vertices in the subgraphs are $\leq a$ and $\leq b$, respectively.

We remark that, if the graph does not contain multiple edges, then this theorem follows from a result of Vizing [10].

In Section 7 we deduce formulas for δ. If every prescribed interval consists of a single integer, then, substituting the formulas of Theorems 7.3 and 7.4 into the left-hand side of the equality $\delta = 0$, we obtain the conditions of Tutte and Ore (disregarding trivial transformations).

In Section 8 we specify our results for the case of a 1-factor; we shall obtain the results of the mentioned papers, and, further, the following theorem:

Let T be a set consisting of some vertices of a graph. T can be covered by an independent edge-system if and only if for any k omitting any k vertices of the graph the number of those odd components of the remaining graph which contain only vertices of T is $\leq k$.

If T is the set of all vertices, then we obtain Tutte's 1-factor theorem. On the other hand, if T is an independent set then we obtain Ore's matching theorem for bipartite graphs. Note that Ore's theorem is not a special case of Tutte's result, allthough it can be deduced from it easily.

I would like to close the introduction with thanks to Professor T. Gallai for his advice and suggestions.

Notations

Throughout this paper we consider a finite undirected graph (loops and multiple edges are allowed). This graph is fixed; not even an extra notation need be introduced for it. Subgraph, vertex, edge, etc. mean subgraph, vertex, edge of this graph. S denotes the set of its vertices.

A subgraph is identified with the set of its edges. If $\mathscr{G}$ is a subgraph then $\bar{\mathscr{G}}$ denotes the set of those edges not contained in $\mathscr{G}$. If $A, B \subseteq S$ then we denote by $\mathscr{F}(A, B)$ the set of those edges, which connect a vertex of A to a vertex of B. We put briefly $\mathscr{F}(A, A) = \mathscr{F}(A)$. The graph $[X]$ spanned by the set $X \subseteq S$ means the graph the vertices and edges of which are the elements of X and $\mathscr{F}(X)$, respectively. Note that $[X]$ is not a subgraph.

The valency of the vertex x will be denoted by $\varphi(x)$ and $\varphi_{\mathscr{G}}(x)$ in the original graph and in the subgraph $\mathscr{G}$, respectively.

We shall consider integer valued functions on S. If h is such a function and $X \subseteq S$, then we put

$$h(X) = \sum_{x \in X} h(x),$$

and

$$\hat{h} = \varphi - h,$$

and let X_h denote the set of those $x \in X$ for which

$$\varphi_{\mathscr{F}}(x)(x) \leqslant h(x).$$

If h is a function defined on S, $x \in S$, and k is a number, then we define the functions h^{kx} and h_{kx} as follows:

$$h^{kx}(y) = \begin{cases} h(y) + k, & \text{if } \ y = x, \\ h(y), & \text{if } \ y \neq x; \end{cases}$$

$$h_{kx}(y) = \begin{cases} h(y) - k, & \text{if } \ y = x, \\ h(y), & \text{if } \ y \neq x. \end{cases}$$

We put, further, $h^{1x} = h^x$, $(h^x)^y = h^{xy}$, etc.

We shall consider pairs (f, g) where f and g are integer valued functions satisfying $f \geqslant 0$, $g \geqslant 0$, $f + g \geqslant \varphi$. Let $\mathscr{P}$ denote the set of all such pairs of functions.

1. As we have already mentioned in the introduction we generalize the factor problem as follows: Let a pair $(f, g) \in \mathscr{P}$ of functions be given. When does a subgraph $\mathscr{G}$ exist such that

$$\varphi_{\mathscr{G}} \leqslant f, \qquad \varphi_{\overline{\mathscr{G}}} \leqslant g? \tag{1}$$

Furthermore, if such a subgraph does not exist, which are the "optimally approaching" subgraphs and what is the measure of this "approach"?

Let $\mathscr{G}$ be a subgraph. In a vertex x, $\mathscr{G}$ deviates from condition (1) by the value[3]

$$\delta_{\mathscr{G}}(f, g; x) = |\, \varphi_{\mathscr{G}}(x) - f(x)|_+ + |\, \varphi_{\overline{\mathscr{G}}}(x) - g(x)|_+ .$$

[3] If a, b are real numbers then we put $a \cap b = \min(a, b)$, $a \cup b = \max(a, b)$, $|\, a\,|_+ = a \cup 0$.

Thus $\mathcal{G}$ deviates from (1) by

$$\delta_{\mathcal{G}}(f, g) = \delta_{\mathcal{G}}(f, g; S)$$

and we can characterize the solvability of (1) by the value

$$\delta(f, g) = \min_{\mathcal{G}} \delta_{\mathcal{G}}(f, g).$$

$\delta(f, g)$ vanishes if and only if there exists a subgraph satisfying (1).

A subgraph $\mathcal{G}$ will be called (f, g)-*optimal* if

$$\delta_{\mathcal{G}}(f, g) = \delta(f, g).$$

In the case of the pair of functions denoted by (f, g) we shall say simply "optimal" instead of "(f, g)-optimal."

Let us remark that obviously

$$\delta_{\mathcal{G}}(f, g) = \delta_{\overline{\mathcal{G}}}(g, f), \qquad \delta(f, g) = \delta(g, f).$$

From the point of view of the more detailed investigation of the "functional" δ the following obvious fact will be basic:

(1.1) *Let $(f, g) \in \mathcal{P}$. If the subgraph $\mathcal{G}$ and the vertex x satisfy*

$$\varphi_{\mathcal{G}}(x) > f(x)$$

and E is an edge of $\mathcal{G}$ incident to x, then for the subgraph $\mathcal{G}' = \mathcal{G} - \{E\}$ we have

$$\delta_{\mathcal{G}'}(f, g) \leqslant \delta_{\mathcal{G}}(f, g).$$

From this remark it is obvious that

(1.2) *If $(f, g) \in \mathcal{P}$ and $\mathcal{G}$ is a (set-theoretically) minimal optimal subgraph then*

$$\varphi_{\mathcal{G}}(x) \leqslant f(x).$$

Here we note that, because of the symmetry in f and g, all of our theorems have a "dual" one, that we obtain by substituting f, g, $\overline{\mathcal{G}}$ for $g, f, \mathcal{G}$, respectively. Thus, e.g., the "dual" of (1.2) is the following (after trivial transformations):

(1.3) *If $(f, g) \in \mathcal{P}$ is a pair of functions and $\mathcal{G}$ is a maximal optimal subgraph then*

$$\varphi_{\overline{\mathcal{G}}} \leqslant g, \qquad \textit{i.e.,} \qquad \varphi_{\mathcal{G}} \geqslant \hat{g}.$$

In what follows we shall not formulate the "dual" theorems extra.

2. We consider connections of some type between subgraphs. The investigation of them will lead us to some inequality concerning the "functional" δ.

Let $(f, g) \in \mathscr{P}$ and let $\mathscr{G}$, $\mathscr{G}'$ be two subgraphs. We say that $\mathscr{G}'$ is *accessible from* $\mathscr{G}$ *concerning* (f, g) if we can find a sequence $\mathscr{G}_0 = \mathscr{G}, \mathscr{G}_1, \ldots, \mathscr{G}_n = \mathscr{G}'$ of subgraphs such that[4]

$$| \mathscr{G}_i \triangle \mathscr{G}_{i-1} | \leqslant 1, \qquad \delta_{\mathscr{G}_i}(f, g) \leqslant \delta_{\mathscr{G}_{i-1}}(f, g) \qquad (i = 1, \ldots, n).$$

Let us agree to omit "concerning (f, g)" in the case of the pair of functions denoted by (f, g).

On the other hand, we say that $\mathscr{G}'$ is *nearer* (f, g) *than* $\mathscr{G}$ if

$$| \varphi_{\mathscr{G}'} - f |_+ \leqslant | \varphi_{\mathscr{G}} - f |_+ , \qquad | \varphi_{\overline{\mathscr{G}}'} - g |_+ \leqslant | \varphi_{\overline{\mathscr{G}}} - g |_+ .$$

We can state at once that, in both cases,

$$\delta_{\mathscr{G}'}(f, g) \leqslant \delta_{\mathscr{G}}(f, g);$$

consequently if $\mathscr{G}$ is optimal then so is $\mathscr{G}'$. Furthermore these relations are transitive and reflexive, and among optimal subgraphs also symmetric.

(2.1) THEOREM. *Let* $(f_i, g_i) \in \mathscr{P}$ $(i = 1, \ldots, n)$ *and* $(f_j', g_j') \in \mathscr{P}$ $(j = 1, \ldots, m)$; *suppose that* $f_i \leqslant f_j'$ *and* $g_i \leqslant g_j'$ *hold for any* $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. *Further, let* $\mathscr{G}_1$, $\mathscr{G}_2$ *be two subgraphs. Then there exists a subgraph which is accessible from* $\mathscr{G}_1$ *concerning* (f_i, g_i) *for every* $1 \leqslant i \leqslant n$ *and which is nearer* (f_j', g_j') *than* $\mathscr{G}_2$ *for every* $1 \leqslant j \leqslant m$.

(2.2) COROLLARY. *If* $(f_i, g_i) \in \mathscr{P}$ $(i = 1, \ldots, n)$ *and* $f_1 \leqslant f_2 \leqslant \cdots \leqslant f_n$, $g_1 \leqslant g_2 \leqslant \cdots \leqslant g_n$ *then there exists a subgraph which is* (f_i, g_i)*-optimal for every* $1 \leqslant i \leqslant n$.

(2.3) COROLLARY. *Let* $(f, g) \in \mathscr{P}$, $X \subseteq S$, *and* $f(X) + g(X) = \varphi(X)$; *further, let* $\mathscr{G}_0$ *be an optimal subgraph. For any optimal subgraph* $\mathscr{G}$ *there exists a subgraph* $\mathscr{G}'$ *accessible from* $\mathscr{G}_0$ *such that*

$$\varphi_{\mathscr{G}'}(x) = \varphi_{\mathscr{G}}(x)$$

for any $x \in X$.

PROOF OF THEOREM (2.1): Let us choose a subgraph $\mathscr{G}$ which is accessible from $\mathscr{G}_1$ concerning (f_i, g_i) for every $1 \leqslant i \leqslant n$ and for which $| \mathscr{G} \triangle \mathscr{G}_2 |$ is minimal. We show that this subgraph satisfies the requirements, i.e., $\mathscr{G}$ is nearer (f_j', g_j') than $\mathscr{G}_2$ for every $1 \leqslant j \leqslant m$.

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[4] $| X |$ means the number of elements of the set X; $X \triangle Y = (X - Y) \cup (Y - X)$.

Suppose the opposite statement. Then there exist a $1 \leqslant j \leqslant m$ and a vertex x such that, e.g.,

$$|\varphi_{\mathscr{G}}(x) - f_j'(x)|_+ > |\varphi_{\mathscr{G}_2}(x) - f_j'(x)|_+,$$

i.e.,

$$\varphi_{\mathscr{G}}(x) > \varphi_{\mathscr{G}_2}(x), \qquad \varphi_{\mathscr{G}}(x) > f_j'(x); \tag{2}$$

(2) implies that there exists an $E \in \mathscr{G} - \mathscr{G}_2$ incident to x. By (2)

$$\varphi_{\mathscr{G}}(x) > f_j'(x) \geqslant f_i(x) \qquad (1 \leqslant i \leqslant n).$$

The application of (1.1) now gives that the subgraph $\mathscr{G}' = \mathscr{G} - \{E\}$ is accessible from $\mathscr{G}_1$ concerning (f_i, g_i) $(1 \leqslant i \leqslant n)$, which is a contradiction since

$$|\mathscr{G}' \triangle \mathscr{G}_2| < |\mathscr{G} \triangle \mathscr{G}_2|.$$

PROOF OF COROLLARY (2.2): We use induction on n. In case $n = 1$ the statement is obvious. Let $n > 1$ and suppose that there exists a subgraph $\mathscr{G}_1$ which is (f_i, g_i)-optimal for $1 \leqslant i \leqslant n - 1$. Take an arbitrary (f_n, g_n)-optimal subgraph $\mathscr{G}_2$. By Theorem (2.1) there exists a subgraph which is accessible from $\mathscr{G}_1$ concerning (f_i, g_i) if $1 \leqslant i \leqslant n - 1$ and which is nearer (f_n, g_n) than $\mathscr{G}_2$. This subgraph is (f_i, g_i)-optimal for every $1 \leqslant i \leqslant n$.

PROOF OF COROLLARY (2.3): Put $i = j = 1, f_1 = f_1' = f, g_1 = g_1' = g$ in Theorem (2.1).

We shall need a lemma for real numbers:

(2.4) *If a, b, c, d are real numbers, $a \leqslant b \leqslant d$ and $a + d = b + c$, then*

$$|a|_+ + |d|_+ \geqslant |b|_+ + |c|_+.$$

PROOF: The center of the interval connecting the points $(a, |a|_+)$ and $(d, |d|_+)$ in the plane has the same abscissa as the center of the interval connecting $(b, |b|_+)$ and $(c, |c|_+)$. By the convexity of the function $|x|_+$ the latter point lies lower, i.e.,

$$\frac{|a|_+ + |d|_+}{2} \geqslant \frac{|b|_+ + |c|_+}{2};$$

this proves (2.4).

(2.5) THEOREM. *For any pair $(f, g) \in \mathscr{P}$ and subgraphs $\mathscr{G}_1, \mathscr{G}_2$*

$$\delta_{\mathscr{G}_1 \cup \mathscr{G}_2}(f, g) + \delta_{\mathscr{G}_1 \cap \mathscr{G}_2}(f, g) \geqslant \delta_{\mathscr{G}_1}(f, g) + \delta_{\mathscr{G}_2}(f, g).$$

(2.6) COROLLARY. *If* $(f, g) \in \mathscr{P}$, $\mathscr{G}_1 \subseteq \mathscr{G} \subseteq \mathscr{G}_2$ *and* $\mathscr{G}_1$, $\mathscr{G}_2$ *are optimal subgraphs, then so is* $\mathscr{G}$.

(2.7) COROLLARY. *If* $(f, g) \in \mathscr{P}$ *and* $\mathscr{G}_1 \subseteq \mathscr{G}_2$ *are optimal subgraphs, then they are accessible from each other.*

PROOF OF THEOREM (2.5): We are going to apply (2.4):

$$(\varphi_{\mathscr{G}_1 \cup \mathscr{G}_2} - f) + (\varphi_{\mathscr{G}_1 \cap \mathscr{G}_2} - f) = (\varphi_{\mathscr{G}_1} - f) + (\varphi_{\mathscr{G}_2} - f),$$
$$\varphi_{\mathscr{G}_1 \cup \mathscr{G}_2} - f \geqslant \varphi_{\mathscr{G}_1} - f \geqslant \varphi_{\mathscr{G}_1 \cap \mathscr{G}_2} - f;$$

hence by (2.4)

$$| \varphi_{\mathscr{G}_1 \cup \mathscr{G}_2} - f |_+ + | \varphi_{\mathscr{G}_1 \cap \mathscr{G}_2} - f |_+ \geqslant | \varphi_{\mathscr{G}_1} - f |_+ + | \varphi_{\mathscr{G}_2} - f |_+ ; \qquad (3)$$

similarly

$$| \varphi_{\overline{\mathscr{G}_1 \cup \mathscr{G}_2}} - g |_+ + | \varphi_{\overline{\mathscr{G}_1 \cap \mathscr{G}_2}} - g |_+ \geqslant | \varphi_{\overline{\mathscr{G}}_1} - g |_+ + | \varphi_{\overline{\mathscr{G}}_2} - g |_+ . \qquad (4)$$

From (3) and (4) the statement follows by addition.

PROOF OF COROLLARY (2.6): Put

$$\mathscr{G}' = \mathscr{G}_1 \cup (\mathscr{G}_2 - \mathscr{G}).$$

Then $\mathscr{G}_1 \cup \mathscr{G}_1' = \mathscr{G}_2$, $\mathscr{G} \cap \mathscr{G}' = \mathscr{G}_1$; hence by Theorem (2.5)

$$\delta_{\mathscr{G}}(f, g) + \delta_{\mathscr{G}'}(f, g) \leqslant \delta_{\mathscr{G}_1}(f, g) + \delta_{\mathscr{G}_2}(f, g) = 2\delta(f, g).$$

This implies

$$\delta_{\mathscr{G}}(f, g) = \delta_{\mathscr{G}'}(f, g) = \delta(f, g).$$

Corollary (2.7) follows from corollary (2.6) immediately.

3. In this section we investigate how $\delta(f, g)$ changes if we vary f and g. First it is obvious that, if f or g increases, then $\delta(f, g)$ decreases (not necessarily strictly). A further trivial observation is the following:

(3.1) *Let* $(f, g) \in \mathscr{P}$, $x \in S$.

(a) $\delta(f^x, g) < \delta(f, g)$ *if and only if there exists an* (f, g)*-optimal subgraph* $\mathscr{G}$ *such that* $\varphi_{\mathscr{G}}(x) > f(x)$.

(b) *Suppose* $f(x) + g(x) > \varphi(x)$. *Then* $\delta(f_x, g) > \delta(f, g)$ *if and only if every* (f, g)*-optimal subgraph* $\mathscr{G}$ *satisfies* $\varphi_{\mathscr{G}}(x) \geqslant f(x)$.

(3.2) THEOREM. *If* $(f_i, g_i) \in \mathscr{P}$ $(i = 1, 2, 3, 4)$ *and* $f_1 \leqslant f_2 \leqslant f_4$, $g_1 \leqslant g_2 \leqslant g_4$, $f_1 + f_4 = f_2 + f_3$, $g_1 + g_4 = g_2 + g_3$ *then*

$$\delta(f_1, g_1) + \delta(f_4, g_4) \geqslant \delta(f_2, g_2) + \delta(f_3, g_3).$$

(3.3) COROLLARY. *If* $(f, g) \in \mathscr{P}$, $x \in S$, $f(x) + g(x) > \varphi(x)$ *and*

$$\delta(f_x, g) = \delta(f, g),$$

then

$$\delta(f^x, g) = \delta(f, g).$$

(3.4) COROLLARY. *If* $(f, g) \in \mathscr{P}$, $x \in S$, $f(x) + g(x) > \varphi(x)$ *then*

$$\delta(f, g) = \delta(f_x, g) \cap \delta(f, g_x) = \delta(f^x, g) \cup \delta(f, g^x).$$

PROOF OF THEOREM (3.2): The same argument as used in the proof of Theorem (2.5) shows that for any $\mathscr{G}$

$$\delta_{\mathscr{G}}(f_1, g_1) + \delta_{\mathscr{G}}(f_4, g_4) \geqslant \delta_{\mathscr{G}}(f_2, g_2) + \delta_{\mathscr{G}}(f_3, g_3). \tag{5}$$

Let us choose a subgraph $\mathscr{G}$ which is (f_1, g_1)-optimal and (f_4, g_4)-optimal; such a subgraph exists by corollary (2.2). Then by (5)

$$\begin{aligned}\delta(f_1, g_1) + \delta(f_4, g_4) &= \delta_{\mathscr{G}}(f_1, g_1) + \delta_{\mathscr{G}}(f_4, g_4)\\ &\geqslant \delta_{\mathscr{G}}(f_2, g_2) + \delta_{\mathscr{G}}(f_3, g_3)\\ &\geqslant \delta(f_2, g_2) + \delta(f_3, g_3).\end{aligned}$$

PROOF OF COROLLARY (3.3): By Theorem (3.2)

$$\delta(f_x, g) + \delta(f^x, g) \geqslant 2\delta(f, g);$$

hence by our conditions

$$\delta(f^x, g) \geqslant \delta(f, g).$$

This gives the conclusion because of the monotonity of δ.

PROOF OF COROLLARY (3.4): The first equality follows from (3.1). We suppose, e.g.,

$$\delta(f_x, g) = \delta(f, g);$$

hence by Corollary (3.3)

$$\delta(f^x, g) = \delta(f, g).$$

That proves the second equality.

The following lemma shows the connection between the structure of the graph and the values of $\delta(f, g)$:

(3.5) *If* $(f, g) \in \mathscr{P}$, $x, y \in S$, *and*

$$\delta(f^x, g) = \delta(f, g^x) = \delta(f, g), \qquad \delta(f_x, g) = \delta(f, g_x) = \delta(f, g) - 1,$$

then x *and* y *are not joined.*

PROOF: First of all we notice that by (3.1) we have for any (f, g)-optimal subgraph $\mathcal{G}$

$$\delta_{\mathcal{G}}(f, g; x) = 0 \tag{6}$$

and by Corollary (3.4)

$$f(y) + g(y) = \varphi(y). \tag{7}$$

Suppose now that indirectly there exists an edge E joining x and y. Let $\mathcal{G}$ be an optimal subgraph; suppose, e.g., $E \in \mathcal{G}$. By (3.1) there exists an optimal subgraph $\mathcal{G}'$ for which

$$\varphi_{\mathcal{G}'}(y) > f(y).$$

By Theorem (2.1) we can choose for $\mathcal{G}'$ a subgraph accessible from $\mathcal{G}$.

First we show $E \notin \mathcal{G}'$. Really, if $E \in \mathcal{G}'$, then putting $\mathcal{G}'' = \mathcal{G}' - \{E\}$, equation (6) gives

$$\delta_{\mathcal{G}''}(f, g; x) = \delta_{\mathcal{G}'}(f, g; x) = 0$$

and obviously

$$\delta_{\mathcal{G}''}(f, g; y) = \delta_{\mathcal{G}'}(f, g; y) - 1;$$

thus

$$\delta_{\mathcal{G}''}(f, g) < \delta_{\mathcal{G}'}(f, g) = \delta(f, g),$$

a contradiction. Thus $E \notin \mathcal{G}'$.

Since $\mathcal{G}'$ is accessible from $\mathcal{G}$, there exists a sequence

$$\mathcal{G}_0 = \mathcal{G}, \mathcal{G}_1, \ldots, \mathcal{G}_n = \mathcal{G}'$$

of optimal subgraphs such that $| \mathcal{G}_i \triangle \mathcal{G}_{i-1} | \leqslant 1$. There exists a last member $\mathcal{G}_i$ of this sequence containing E. Then $\mathcal{G}_{i+1} = \mathcal{G}_i - \{E\}$. But this is a contradiction since by (6) and (7)

$$\delta_{\mathcal{G}_{i+1}}(f, g) - \delta_{\mathcal{G}_i}(f, g) = \pm 1.$$

4. Throughout this section we consider a pair $(f, g) \in \mathcal{P}$ such that

$$f \leqslant \varphi, \qquad g \leqslant \varphi, \qquad \delta(f, g) = \delta_0, \tag{8}$$

where δ_0 is a fixed integer and (f, g) is maximal among all pairs of functions satisfying (8). This latter supposition can be expressed as follows: if $x \in S$ and $f(x) < \varphi(x)$, then

$$\delta(f^x, g) < \delta_0$$

and if $x \in S$, $g(x) < \varphi(x)$ then

$$\delta(f, g^x) < \delta_0 .$$

Such a pair of functions will be called simply *maximal.*

Let A, B, C, D denote the sets where in order

$$f(x) = \varphi(x),\ g(x) < \varphi(x);$$
$$f(x) < \varphi(x),\ g(x) = \varphi(x);$$
$$f(x) = g(x) = \varphi(x);$$
$$f(x) < \varphi(x),\ g(x) < \varphi(x).$$

Corollary (3.4) implies that if $x \in D$ then

$$f(x) + g(x) = \varphi(x).$$

(4.1) *If $x \in A$ and $g(x) > 0$ then (f, g_x) is also maximal and*

$$\delta(f, g_x) = \delta(f, g) + 1 = \delta_0 + 1.$$

PROOF: The second conclusion follows from Corollary (3.3). If $y \in S$ and $f(y) < \varphi(y)$ then

$$\delta(f^y, g_x) \leqslant \delta(f^y, g) + 1 < \delta_0 + 1 = \delta(f, g_x),$$

and if $g_x(y) < \varphi(y)$ then

$$\delta(f, g_x{}^y) \leqslant \delta(f, g^y) + 1 < \delta_0 + 1 = \delta(f, g_x).$$

This proves the maximality of (f, g_x).

By (3.5),

(4.2) $\mathscr{F}(C, D) = \varnothing$.

We are going to investigate the vertices of D in more detail.

(4.3) *Let $x \in D$, $k, l \geqslant 0$, $k + l > 0$. Then*

$$\delta(f^{kx}, g^{lx}) = \delta_0 - 1.$$

PROOF: It is clear that

$$\delta(f^{kx}, g^{lx}) \leqslant \delta_0 - 1.$$

On the other hand by Corollary (3.4)

$$\delta(f^x, g^x) = \delta(f^x, g) \cap \delta(f, g^x) = \delta_0 - 1.$$

Suppose, e.g., $k > 0$. Then

$$\delta(f, g^x) = \delta(f^x, g^x)$$

and hence by Corollary (3.3)

$$\delta(f^{kx}, g^x) = \delta_0 - 1.$$

Thus

$$\delta(f^{kx}, g) = \delta(f^{kx}, g^x) = \delta_0 - 1$$

and using Corollary (3.3) again we obtain what we wanted to prove.

(4.4) *Let $x, y \in D$. Then*

$$\delta(f^{xy}, g^{xy}) = \delta(f^{xy}, g) = \delta(f, g^{xy}) = \delta(f^x, g^y) = \delta(f^y, g^x).$$

PROOF: It is clear that $\delta(f^{xy}, g^{xy})$ is not greater than the other values. We show that it is equal to any one of them. It is enough to prove the following two inequalities:

(a) $\delta(f^{xy}, g^{xy}) \geqslant \delta(f^{xy}, g)$.

(b) $\delta(f^{xy}, g^{xy}) \geqslant \delta(f^x, g^y)$.

PROOF OF (a): Using Theorem (3.2) twice and (4.3)

$$\begin{aligned}\delta(f^{xy}, g^{xy}) &\geqslant \delta(f^{xy}, g^y) + \delta(f^x, g^x) - \delta(f^x, g)\\ &= \delta(f^{xy}, g^y) \geqslant \delta(f^{xy}, g) + \delta(f^y, g^y) - \delta(f^y, g) = \delta(f^{xy}, g).\end{aligned}$$

PROOF OF (b): Similarly:

$$\begin{aligned}\delta(f^{xy}, g^{xy}) \geqslant \delta(f^{xy}, g^y) &\geqslant \delta(f^x, g^y)\\ &+ \delta(f^y, g^y) - \delta(f, g^y) = \delta(f^x, g^y).\end{aligned}$$

Let $x, y \in D$. We write $x \sim y$ if

$$\delta(f^{xy}, g^{xy}) = \delta_0 - 1.$$

Obviously $x \not\sim y$ means

$$\delta(f^{xy}, g^{xy}) = \delta_0 - 2.$$

(4.5) *The relation $x \sim y$ is an equivalence-relation.*

PROOF: (a) $x \sim x$ by (4.3); (b) if $x \sim y$ then obviously $y \sim x$; (c) suppose $x \sim y$ and $y \sim z$, then by (4.3) and (3.2) $x \sim z$:

$$\begin{aligned}\delta(f^{xz}, g^{xz}) &\geqslant \delta(f^{xyz}, g^{xyz})\\ &\geqslant \delta(f^{xy}, g^{xy}) + \delta(f^{yz}, g^{yz}) - \delta(f^y, g^y) = \delta_0 - 1.\end{aligned}$$

Let us remark here that (4.3), (4.4), and (3.1) give that for any equivalence-class D_0 and for any optimal subgraph $\mathscr{G}$

$$\delta_{\mathscr{G}}(f, g, D_0) \leqslant 1. \tag{9}$$

(4.6) *If $x, y \in D$ and x and y are joined by an edge then $x \sim y$.*

PROOF: Suppose indirectly that $x \not\sim y$. By (4.3).

$$\delta(f^{xx}, g) = \delta(f^x, g^x) = \delta(f^x, g)$$

and by (4.4)

$$\delta(f^{xy}, g) = \delta(f^x, g^y) = \delta(f^x, g) - 1.$$

This contradicts Lemma (3.5) if we apply it on (f^x, g).

This lemma shows that the classification induced by the equivalence-relation $x \sim y$ is rougher than the decomposition of the graph $[D]$ into connected components. Under further conditions (detailed in Section 5 below) these classifications come to be identical.

5. We say that a maximal pair of functions (f, g) is *simple* if $g(A) = f(B) = 0$. Throughout this section we suppose that the considered pair of functions (f, g) is simple.

Let $\mathscr{G}$ be an (f, g)-optimal subgraph. Obviously

$$\mathscr{F}(A) \subseteq \mathscr{G}, \qquad \mathscr{F}(B) \subseteq \bar{\mathscr{G}}, \qquad \mathscr{F}(A, C) \subseteq \mathscr{G}, \qquad \mathscr{F}(B, C) \subseteq \bar{\mathscr{G}}.$$

If moreover

$$\mathscr{F}(C) \subseteq \mathscr{G}, \qquad \mathscr{F}(A, B) \subseteq \mathscr{G}, \qquad \mathscr{F}(A, D) \subseteq \mathscr{G}, \qquad \mathscr{F}(B, D) \subseteq \bar{\mathscr{G}},$$

then we say that $\mathscr{G}$ is simple.

If $\mathscr{G}$ is any optimal subgraph, then by (1.1)

$$\tilde{\mathscr{G}} = [\mathscr{G} - \mathscr{F}(B, D)] \cup \mathscr{F}(A, D) \cup \mathscr{F}(A, B) \cup \mathscr{F}(C)$$

is optimal too; $\tilde{\mathscr{G}}$ is obviously simple.

Consider now the connected components of the graph $[D]$; let $[D_1], [D_2], \ldots, [D_\tau]$ be these components. By (4.2) these are connected components of $[C \cup D]$ too.

(5.1) *Let $\mathscr{G}$ be a simple subgraph. Then*

$$\delta_{\mathscr{G}}(f, g; D_i) \equiv f(D_i) + |\mathscr{F}(A, D_i)| \qquad (\text{mod } 2).$$

PROOF:

$$\delta_{\mathscr{G}}(f, g; D_i) = \sum_{x \in D_i} |f(x) - \varphi_{\mathscr{G}}(x)|$$

$$\equiv f(D_i) + \varphi_{\mathscr{G}}(D_i) \ (\text{mod } 2).$$

Put $\mathscr{G}' = \mathscr{G} \cap \mathscr{F}(D_i)$. We know that D_i is not joined to C, hence by our condition that $\mathscr{G}$ is simple

$$\begin{aligned} f(D_i) + \varphi_{\mathscr{G}}(D_i) &= f(D_i) + |\mathscr{F}(A, D_i)| + \varphi_{\mathscr{G}'}(D_i) \\ &\equiv f(D_i) + |\mathscr{F}(A, D_i)| \ (\text{mod } 2). \end{aligned}$$

Let $x \in D_i$ and let $\mathscr{G}$ be an (f^x, g)-optimal subgraph. By (1.1), $\tilde{\mathscr{G}}$ is (f^x, g)-optimal again; on the other hand it is trivially (f, g)-optimal too. Since by (3.1)

$$\varphi_{\tilde{\mathscr{G}}}(x) = f(x) + 1,$$

we obtain

$$\delta_{\tilde{\mathscr{G}}}(f, g; D_i) \geqslant 1,$$

and thus by (9)

$$\delta_{\tilde{\mathscr{G}}}(f, g; D_i) = 1.$$

Lemma (5.1) now gives

(5.2) *If* $[D_i]$ *is a connected component of* $[D]$ *then*

$$f(D_i) + |\mathscr{F}(A, D_i)| \equiv 1 \pmod 2.$$

By combining (5.1), (5.2), and (9) we obtain that in (9) the equality holds, if $\mathscr{G}$ is simple:

(5.3) *If* $\mathscr{G}$ *is a simple subgraph then* $\varphi_{\mathscr{G}}$ *differs from* f *in just one vertex of* D_i *and here by* 1.

(4.3) and (3.1) now give that, if $1 \leqslant i < j \leqslant \tau$, then there exist an $x \in D_i$ and an $y \in D_j$ such that $x \sim y$, and thus

(5.4) *The classification of the set* D *induced by the equivalence-relation* $x \sim y$ *coincides with the decomposition of* $[D]$ *into connected components.*

Calculate now the value of $\delta(f, g)$. Let $\mathscr{G}$ be a simple subgraph. By (5.2) and (5.3),

$$\delta_{\mathscr{G}}(f, g; D) = \tau.$$

Furthermore

$$\delta_{\mathscr{G}}(f, g; C) = \delta_{\mathscr{G}}(f, g; B) = 0,$$

and

$$\delta_{\mathscr{G}}(f, g; A) = |\mathscr{F}(A, B)|;$$

hence

$$\delta(f, g) = \tau + |\mathscr{F}(A, B)|. \tag{10}$$

We call a graph *f-critical*, if it contains no f-factor, but, prescribing any vertex x of it and $\epsilon = \pm 1$ in an arbitrary way, there exists a subgraph of it the valency of which is $f(x) + \epsilon$ in the vertex x and $f(y)$ in any other vertex y. The argument before (5.2) gives

(5.5) *The graph* $[D_i]$ *is* $(f(x) - |\mathscr{F}(A, x)|)$*-critical.*

Now we are able to characterize the optimal subgraphs. We have to determine which edges of $\mathscr{F}(A)$, $\mathscr{F}(B)$, $\mathscr{F}(C)$, $\mathscr{F}(D)$, $\mathscr{F}(A, B)$, $\mathscr{F}(A, C)$, $\mathscr{F}(B, C)$, $\mathscr{F}(A, D)$, $\mathscr{F}(B, D)$ such a subgraph $\mathscr{G}$ contains.

By (5.3), the valency of the subgraph $\mathscr{G} \cap \mathscr{F}(D) = \mathscr{G}_D$ differs from $f(x) - |\mathscr{F}(A, x)|$ in just one vertex u_i of D_i and here by 1. Say $i \in \mathscr{I}_1$, if

$$\varphi_{\mathscr{G}_D}(u_i) = f(u_i) - |\mathscr{F}(A, u_i)| - 1$$

and $i \in \mathscr{I}_2$ if

$$\varphi_{\mathscr{G}_D}(u_i) = f(u_i) - |\mathscr{F}(A, u_i)| + 1.$$

The removal of an edge of the subgraph $\mathscr{H}_1 = \mathscr{G} \cap \mathscr{F}(B, D)$ does not decrease the value of $\delta_{\mathscr{G}}(f, g)$. Hence the edges of $\mathscr{H}_1$ join vertices of B to vertices of form u_i $(i \in \mathscr{I}_1)$, and every u_i is the end-point of at most one edge of $\mathscr{H}_1$.

Similarly, $\mathscr{H}_2 = \mathscr{G} \cap \mathscr{F}(A, D)$ contains all edges of $\mathscr{F}(A, D)$ except at most one edge joining D to u_i for any $i \in \mathscr{I}_2$.

As we have already mentioned, $\mathscr{F}(A) \subseteq \mathscr{G}$, $\mathscr{F}(A, C) \subseteq \mathscr{G}$, $\mathscr{F}(B) \subseteq \bar{\mathscr{G}}$, and $\mathscr{F}(B, C) \subseteq \bar{\mathscr{G}}$. From $\mathscr{F}(A, B)$ and $\mathscr{F}(C)$, $\mathscr{G}$ may contain an arbitrary system of edges.

(5.6) THEOREM. *The subgraphs characterized above and only these subgraphs are optimal.*

As we have seen, the properties listed are really necessary. Their sufficiency can be verified by a simple computation on the basis of (10).

We close this section with a property of simple pairs of functions which does not belong closely to our discussion but seems to complete it:

(5.7) THEOREM. *If (f, g) is a simple pair of functions then any two (f, g)-optimal subgraphs are accessible from each other concerning (f, g).*

Here we note that, if $(f, g) \in \mathscr{P}$, $f + g = \varphi$, and our graph is f-critical, then (f, g) is a simple pair of functions.

PROOF: Let two optimal subgraphs $\mathscr{G}$, $\mathscr{G}'$ be given. Take two subgraphs $\mathscr{G}_1$, $\mathscr{G}_1'$ such that $\mathscr{G}_1$ and $\mathscr{G}_1'$ are accessible from $\mathscr{G}$ and $\mathscr{G}'$, respectively, and $|\mathscr{G}_1 \triangle \mathscr{G}_1'|$ is minimal. We shall show that $\mathscr{G}_1 = \mathscr{G}_1'$.

Since $\tilde{\mathscr{G}}_1$ and $\tilde{\mathscr{G}}_1'$ are also accessible from $\mathscr{G}$ and $\mathscr{G}'$, respectively, by Corollary (2.7) (or by 1.1),

$$|\tilde{\mathscr{G}}_1 \triangle \tilde{\mathscr{G}}_1'| \leqslant |\mathscr{G}_1 \triangle \mathscr{G}_1'|,$$

we may suppose that $\mathscr{G}_1$, $\mathscr{G}_1'$ are simple.

We say that the vertex y is normal if no edge of $\mathscr{G}_1 \triangle \mathscr{G}'$ is incident to y. To show $\mathscr{G}_1 = \mathscr{G}_1'$ we have to prove that every vertex is normal. This is trivial for the vertices of $A \cup B \cup C$.

Let $y \in D_i$.

The characterization (5.6) of the optimal subgraphs and (5.5) show that there exists a simple subgraph $\mathscr{G}_0$ such that

$$\varphi_{\mathscr{G}_0}(y) \neq f(y).$$

Let now $\mathscr{G}_2$ and $\mathscr{G}_2'$ be simple subgraphs accessible from $\mathscr{G}$ and $\mathscr{G}'$, respectively, such that

$$\mathscr{G}_2 \triangle \mathscr{G}_2' = \mathscr{G}_1 \triangle \mathscr{G}_1'$$

and $|\mathscr{G}_0 \triangle \mathscr{G}_2|$ is minimal. (5.3) now gives that there exists an $x \in D_i$ such that

$$\varphi_{\mathscr{G}_2}(x) = f(x) \pm 1.$$

We prove first that x is normal. We confine ourselves to the case

$$\varphi_{\mathscr{G}_2}(x) = f(x) - 1.$$

The other case could be detailed similarly.

Suppose indirectly that there exists an $E \in \mathscr{G}_2 \triangle \mathscr{G}_2'$ incident to x. Since

$$\varphi_{\mathscr{G}_2'}(x) \geqslant f(x) - 1 = \varphi_{\mathscr{G}_2}(x),$$

we may suppose $E \in \mathscr{G}_2' - \mathscr{G}_2$. By (1.1) $\mathscr{G}_2'' = \mathscr{G}_2 \cup \{E\}$ is optimal again; since obviously $E \in \mathscr{F}(D_i)$, $\mathscr{G}_2''$ is even simple, and accessible from $\mathscr{G}$. But this is a contradiction since

$$|\mathscr{G}_2'' \triangle \mathscr{G}_2'| < |\mathscr{G}_2 \triangle \mathscr{G}_2'| = |\mathscr{G}_1 \triangle \mathscr{G}_1'|.$$

To complete our proof it is enough to show $x = y$. Suppose indirectly $x \neq y$. Then (5.3) shows that

$$\varphi_{\mathscr{G}_0}(x) = f(x).$$

Consequently there exists an $E \in \mathscr{G}_0 - \mathscr{G}_2$ incident to x; obviously $E \in \mathscr{F}(D_i)$. By (1.1) $\mathscr{G}_3 = \mathscr{G}_2 \cup \{E\}$ and $\mathscr{G}_3' = \mathscr{G}_2' \cup \{E\}$ are simple subgraphs accessible from $\mathscr{G}$ and $\mathscr{G}'$, respectively, which is a contradiction since x being normal

$$\mathscr{G}_3 \triangle \mathscr{G}_3' = \mathscr{G}_2 \triangle \mathscr{G}_2' = \mathscr{G}_1 \triangle \mathscr{G}_1'$$

and

$$|\mathscr{G}_3 \triangle \mathscr{G}_0| < |\mathscr{G}_2 \triangle \mathscr{G}_0|.$$

6. Now we deal with an arbitrary pair of functions. We may suppose that $f \leqslant \varphi$, $g \leqslant \varphi$, since

$$\delta(f \cap \varphi, g \cap \varphi) = \delta(f, g).$$

(6.1) *There exists just one maximal pair* $(\bar{f}, \bar{g})$ *of functions such that* $f \leqslant \bar{f}$, $g \leqslant \bar{g}$ *and*

$$\delta(\bar{f}, \bar{g}) = \delta(f, g).$$

PROOF: The interesting part of this lemma is naturally the statement that only one such pair exists. Suppose that $(\bar{f}, \bar{g})$ and (f^*, g^*) are two maximal pairs of the properties mentioned in the lemma. By Theorem (3.2)

$$\delta(\bar{f} \cap f^*, \bar{g} \cap g^*) + \delta(\bar{f} \cup f^*, \bar{g} \cup \bar{g}^*) \geqslant \delta(\bar{f}, \bar{g}) + \delta(f^*, g^*) = 2\delta(f, g). \tag{11}$$

Since $f \leqslant \bar{f} \cap f^* \leqslant \bar{f}$, $g \leqslant \bar{g} \cap g^* \leqslant \bar{g}$, the monotonity of δ implies

$$\delta(\bar{f} \cap f^*, \bar{g} \cap g^*) = \delta(f, g),$$

and thus by (11) we obtain

$$\delta(\bar{f} \cup f^*, \bar{g} \cup g^*) \geqslant \delta(f, g).$$

Because of the monotonity of δ here the equality holds. This contradicts the maximality of $(\bar{f}, \bar{g})$ and (f^*, g^*).

Let us consider now the sets A, B, C, D defined for $(\bar{f}, \bar{g})$; i.e., let

$$\begin{aligned}
A &= \{x : \bar{f}(x) = \varphi(x), \bar{g}(x) < \varphi(x)\};\\
B &= \{x : \bar{f}(x) < \varphi(x), \bar{g}(x) = \varphi(x)\};\\
C &= \{x : \bar{f}(x) = \bar{g}(x) = \varphi(x)\};\\
D &= \{x : \bar{f}(x) < \varphi(x), \bar{g}(x) < \varphi(x)\}.
\end{aligned}$$

These sets depend only on the pair $(f, g) \in \mathscr{P}$. Two other characterizations of them can be given. A, B, C, D are the sets of those vertices x for which

$$\begin{array}{lll}
\text{(I)}\quad A: & \delta(f^x, g) = \delta(f, g), & \delta(f, g^x) < \delta(f, g);\\
B: & \delta(f^x, g) < \delta(f, g), & \delta(f, g^x) = \delta(f, g);\\
C: & \delta(f^x, g) = \delta(f, g^x) = \delta(f, g); & \\
D: & \delta(f^x, g) = \delta(f, g^x) = \delta(f, g) - 1. &
\end{array}$$

On the other hand,

(II) A: for any optimal subgraph $\mathcal{G}$

$$\varphi_{\mathcal{G}}(x) \leqslant f(x),$$

but there exists an optimal $\mathcal{G}$ such that

$$\varphi_{\overline{\mathcal{G}}}(x) > g(x);$$

B: for any optimal subgraph

$$\varphi_{\overline{\mathcal{G}}}(x) \leqslant g(x),$$

but there exists an optimal $\mathcal{G}$ such that

$$\varphi_{\mathcal{G}}(x) > f(x);$$

C: for any optimal subgraph

$$\varphi_{\mathcal{G}}(x) \leqslant f(x), \qquad \varphi_{\overline{\mathcal{G}}}(x) \leqslant g(x);$$

D: there exist two optimal subgraphs $\mathcal{G}$, $\mathcal{G}'$ such that

$$\varphi_{\mathcal{G}}(x) > f(x), \qquad \varphi_{\overline{\mathcal{G}}'}(x) > g(x).$$

Characterization (I) follows from Corollary (3.3) and from the unicity of $(\bar{f}, \bar{g})$; (II) can be obtained by the application of (3.1) onto (I).

Now form the following pair of functions:

$$\tilde{f}(x) = \begin{cases} 0, & \text{if } x \in B, \\ \bar{f}(x), & \text{if } x \notin B; \end{cases} \qquad \tilde{g}(x) = \begin{cases} 0, & \text{if } x \in A, \\ \bar{g}(x), & \text{if } x \notin A. \end{cases}$$

Lemma (4.1) gives that

(6.2) *$(\tilde{f}, \tilde{g})$ is a maximal, consequently simple pair of functions and*

$$\delta(\tilde{f}, \tilde{g}) = \delta(f, g) + g(A) + f(B).$$

Note that A, B, C, D belonging to $(\tilde{f}, \tilde{g})$ are the same as A, B, C, D belonging to (f, g).

Let $[D_1],..., [D_\tau]$ denote the connected components of $[D]$ again. Substituting the value of $\delta(\tilde{f}, \tilde{g})$ from (10) we have

(6.3) Theorem. *Let $(f, g) \in \mathcal{P}$, A, B, C, D defined as above and let τ denote the number of connected components of* $[D]$. *Then*

$$\delta(f, g) = |\mathcal{F}(A, B)| + \tau - g(A) - f(B).$$

A characterization of the optimal subgraphs can be obtained similarly by reducing the problem to the case of simple pair of functions. If $\mathcal{G}$ is any subgraph, then

$$\delta_{\mathcal{G}}(f, g) \geqslant \delta_{\mathcal{G}}(\bar{f}, \bar{g}) \geqslant \delta_{\mathcal{G}}(\tilde{f}, \tilde{g}) - g(A) - f(B). \tag{12}$$

Let $\mathcal{G}$ here be an optimal subgraph; then this inequality gives that $\mathcal{G}$ is $(\tilde{f}, \tilde{g})$-optimal too. Moreover, for $\mathcal{G}$ being optimal the equality must hold in (12) in both places.

$$\delta_{\mathcal{G}}(f, g) = \delta_{\mathcal{G}}(\bar{f}, \bar{g})$$

is equivalent to the condition that

$$\varphi_{\mathcal{G}}(x) \leqslant f(x), \qquad \text{if} \quad x \in A, \tag{13}$$

$$\varphi_{\overline{\mathcal{G}}}(x) \leqslant g(x), \qquad \text{if} \quad x \in B, \tag{14}$$

$$\varphi_{\mathcal{G}}(x) \leqslant f(x), \qquad \varphi_{\overline{\mathcal{G}}}(x) \leqslant g(x), \qquad \text{if} \quad x \in C. \tag{15}$$

Similarly

$$\delta_{\mathcal{G}}(\bar{f}, \bar{g}) = \delta_{\mathcal{G}}(\tilde{f}, \tilde{g}) - f(B) - g(A)$$

holds if and only if

$$\varphi_{\overline{\mathcal{G}}}(x) \geqslant g(x), \qquad \text{if} \quad x \in A, \tag{16}$$

$$\varphi_{\mathcal{G}}(x) \geqslant f(x), \qquad \text{if} \quad x \in B. \tag{17}$$

Since (16) and (17) imply (13) and (14), we can formulate

(6.4) THEOREM. *Let* $(f, g) \in \mathcal{P}$, and let A, B, C, D *defined as above. A subgraph* $\mathcal{G}$ *is* (f, g)*-optimal if and only if it is* $(\tilde{f}, \tilde{g})$*-optimal and it satisfies* (15), (16), and (17). The $(\tilde{f}, \tilde{g})$-optimal subgraphs are characterized in Theorem (5.6).

It is to be seen that the most "unpleasant" term is D. When does D vanish? It is obvious that:

(6.5) *If* $f + g > \varphi$ *then* $D = \varnothing$.

Furthermore we prove:

(6.6) *If our graph is bipartite then* $D = \varnothing$.

PROOF: We may confine ourselves to the case in which (f, g) is simple. Suppose indirectly $D \neq \varnothing$, and let $x \in D_1$. By (5.6) there exist two simple subgraphs $\mathcal{G}$, $\mathcal{G}'$ such that

$$\varphi_{\mathcal{G}}(x) > f(x), \qquad \varphi_{\mathcal{G}'}(x) < f(x).$$

By (5.7) $\mathcal{G}$ and $\mathcal{G}'$ are accessible from each other (the use of Theorem (5.7) is not necessary here; by Theorem (2.1) we could choose for $\mathcal{G}'$ a subgraph accessible from $\mathscr{G}$). There exists a sequence $\mathscr{G}_0 = \mathscr{G}, \mathscr{G}_1, \dots, \mathscr{G}_n = \mathscr{G}'$ of (presumably, simple) subgraphs such that

$$\mathscr{G}_i \triangle \mathscr{G}_{i-1} = \{E_i\} \qquad (i = 1, \dots, n).$$

For every $1 \leqslant i \leqslant n$ there exists just one point $x_i \in D_1$ such that

$$\varphi_{\mathscr{G}_i}(x_i) \neq f(x_i).$$

Obviously

$$E_i = (x_{i-1}, x_i)$$

and

$$\varphi_{\mathscr{G}_i}(x_i) = f(x_i) + (-1)^i.$$

Since

$$\varphi_{\mathscr{G}_n}(x_n) = f(x_n) + (-1)^n < f(x_n),$$

n is odd, and thus the E_i-s form an odd cycle. This is a contradiction.

Suppose now, that f, g are constants.

(6.7) *If $f \equiv a$, $g \equiv b$, $a + b \geqslant \varphi$ and $D = \varnothing$ then* $\delta(f, g) = 0$.

PROOF: Suppose, e.g., $|A| \leqslant |B|$. Then

$$|\mathscr{F}(A, B)| \leqslant (a + b)|A| \leqslant a|B| + b|A|;$$

hence

$$\delta(f, g) = |\mathscr{F}(A, B)| - b|A| - a|B| \leqslant 0.$$

By combining (6.7) and (6.5), (6.6) we have

(6.8) THEOREM. *If $f(x) = a$, $g(x) = b$, $a + b > \varphi(x)$ for every $x \in S$, then* (1) *is solvable.*

(6.9) THEOREM. *The chromatic index*[5] *of a bipartite graph equals to the maximum valency of its vertices.*

That latter theorem is due to König [4, p. 171].

7. We deduce formulas for $\delta(f, g)$. (f, g) is an arbitrary pair of $\mathscr{P}$ again. Put $\delta(f, g) = \delta$.

(7.1) THEOREM. $\delta = \min |\bar{\mathscr{G}}_1 \cap \bar{\mathscr{G}}_2|$ *where $\mathscr{G}_1$ and $\mathscr{G}_2$ run through all subgraphs satisfying*

$$\varphi_{\mathscr{G}_1} \leqslant f, \qquad \varphi_{\mathscr{G}_2} \leqslant g.$$

[5] In the sense of [1, p. 31].

(7.2) THEOREM.

$$\delta = \min \left\{ |\mathcal{G}_1 \cap \mathcal{G}_2| + \sum_x |\varphi_{\mathcal{G}_1}(x) - f(x)| + \sum_x |\varphi_{\mathcal{G}_2}(x) - g(x)| \right\}.$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ run independently through all subgraphs.

Let us say that the pair $(\mathcal{G}_1, \mathcal{G}_2)$ of subgraphs is optimal if it gives the maximum substituting into the right-hand side of the formula of Theorem (7.2). From a knowledge of Theorem (7.2) we shall be able to state that a subgraph $\mathcal{G}$ is optimal if and only if the pair $(\mathcal{G}, \overline{\mathcal{G}})$ is optimal.

PROOF OF THEOREMS (7.1) AND (7.2): For the moment we denote the quantities on the right-hand side of Theorems (7.1) and (7.2) by δ' and δ'', respectively. It is obvious that

$$\delta'' \leq \delta', \qquad \delta'' \leq \delta.$$

Consequently is enough to prove the following two statements:

(a) *There exists an optimal pair $(\mathcal{G}_1, \mathcal{G}_2)$ such that*

$$\mathcal{G}_1 = \overline{\mathcal{G}}_2 .$$

(b) *There exists an optimal pair $(\mathcal{G}_1, \mathcal{G}_2)$ such that*

$$\varphi_{\mathcal{G}_1} \leq f, \qquad \varphi_{\mathcal{G}_2} \leq g.$$

PROOF OF (a): Consider an optimal pair $(\mathcal{G}_1, \mathcal{G}_2)$ for which $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$ (such an optimal pair exists since the omission of $\mathcal{G}_1 \cap \mathcal{G}_2$ from, say, $\mathcal{G}_1$ does not change their optimality). Choose this pair in such a way that $|\overline{\mathcal{G}}_1 \cap \overline{\mathcal{G}}_2|$ is minimal. We have to show $\overline{\mathcal{G}}_1 \cap \overline{\mathcal{G}}_2 = \emptyset$. Suppose indirectly $E \in \overline{\mathcal{G}}_1 \cap \overline{\mathcal{G}}_2$ and let x be an end-point of E. Since

$$\varphi_{\mathcal{G}_1}(x) + \varphi_{\mathcal{G}_2}(x) < \varphi(x) \leq f(x) + g(x),$$

we may suppose, e.g.,

$$\varphi_{\mathcal{G}_1}(x) < f(x).$$

But then obviously $(\mathcal{G}_1 \cup \{E\}, \mathcal{G}_2)$ is an optimal pair; that contradicts the minimality of $|\overline{\mathcal{G}}_1 \cap \overline{\mathcal{G}}_2|$.

PROOF OF (b): Now let us choose the optimal pair $(\mathcal{G}_1, \mathcal{G}_2)$ in such a way that $|\mathcal{G}_1| + |\mathcal{G}_2|$ is minimal. If we had

$$\varphi_{\mathcal{G}_1}(x) > f(x)$$

for some vertex x, then there would exist an edge $E \in \mathscr{G}_1$ incident to x and the pair $(\mathscr{G}_1 - \{E\}, \mathscr{G}_2)$ would be optimal. That would contradict the minimality of $|\mathscr{G}_1| + |\mathscr{G}_2|$. Hence

$$\varphi_{\mathscr{G}_1} \leqslant f$$

and similarly

$$\varphi_{\mathscr{G}_2} \leqslant g.$$

Theorems (7.1) and (7.2) give δ as the minimum of some expression; hence they do not differ very much from the definition of δ. The following two formulas give δ as a maximum. To prove them we shall use the results of the former sections.

Let $X, Y \subseteq S$, $X \cap Y = \varnothing$. We denote by $\tau(X, Y)$ the number of those connected components $[Z]$ of $[S - X - Y]$ for which

$$f(Z) + g(Z) = \varphi(Z), \qquad f(Z) + |\mathscr{F}(X, Z)| \equiv 1 \qquad (\mathrm{mod}\ 2). \quad (18)$$

Further, we put $\tau(Y) = \tau(\varnothing, Y)$.

To point out the symmetry in f and g we show that a component $[Z]$ of $[S - X - Y]$ satisfies (18) if and only if it satisfies

$$f(Z) + g(Z) = \varphi(Z), \qquad g(Z) + |\mathscr{F}(Y, Z)| \equiv 1 \qquad (\mathrm{mod}\ 2). \quad (19)$$

Really, if a component $[Z]$ of $[S - X - Y]$ satisfies, e.g., (18), then obviously

$$f(z) + g(z) = \varphi(z)$$

holds for every $z \in Z$ and thus

$$\begin{aligned} g(Z) + |\mathscr{F}(Z, Y)| &\equiv f(Z) + \varphi(Z) - |\mathscr{F}(Z, Y)| \\ &= f(Z) + |\mathscr{F}(Z, X)| + 2\,|\mathscr{F}(Z)| \\ &\equiv f(Z) + |\mathscr{F}(X, Z)| \qquad (\mathrm{mod}\ 2). \end{aligned}$$

(7.3) THEOREM.

$$\delta = \max\{\tau(X, Y) + |\mathscr{F}(X, Y)| - g(X) - f(Y)\},$$

where (X, Y) runs through all disjoint pairs of subsets of S.

The maximum is arrived for $X = A$, $Y = B$. On the other hand, if (X, Y) gives the maximum then it can be seen rather easily that $A \subseteq X$, $B \subseteq Y$. The total characterization of the "optimal" pairs (X, Y) does not seem to be easy.

PROOF: We use the notations of Section 6. By (5.2),

$$\tau(A, B) \geqslant \tau$$

and thus by Theorem (6.4)

$$\delta \leqslant \tau(A, B) + |\mathscr{F}(A, B)| - g(A) - f(B).$$

Thus we have only to prove that (rather trivial) part of the theorem that for any subgraph $\mathscr{G}$ and for any $X, Y \subseteq S$, $X \cap Y = \varnothing$

$$\delta_{\mathscr{G}}(f, g) \geqslant \tau(X, Y) + |\mathscr{F}(X, Y)| - g(X) - f(Y). \tag{20}$$

To show (20), let us count the number k_1 of edges of $\mathscr{G}$ which have at least one end-point in Y plus the number k_2 of edges of $\bar{\mathscr{G}}$ which have an end-point in X. The edges of $\mathscr{F}(X, Y)$ are certainly to be counted. Further put

$$\delta_1 = \delta_{\mathscr{G}}(f, g; X), \qquad \delta_2 = \delta_{\mathscr{G}}(f, g; Y), \qquad \delta_3 = \delta_{\mathscr{G}}(f, g; S - X - Y).$$

Consider those components $[Z]$ of $[S - X - Y]$ for which

$$f(Z) + g(Z) = \varphi(Z), \qquad f(Z) + |\mathscr{F}(X, Z)| \equiv 1 \qquad (\text{mod } 2).$$

At most δ_3 of these components have

$$\delta_{\mathscr{G}}(f, g; Z) > 0.$$

Hence there are at least $\tau(X, Y) - \delta_3$ such components for which

$$\delta_{\mathscr{G}}(f, g; Z) = 0.$$

Such a Z is joined either to X by an edge of $\bar{\mathscr{G}}$ or to Y by an edge of $\mathscr{G}$, since otherwise we would have

$$f(Z) + |\mathscr{F}(X, Z)| = \varphi_{\mathscr{G}}(Z) + |\mathscr{F}(X, Z)| \equiv 0 \qquad (\text{mod } 2).$$

Hence

$$k_1 + k_2 \geqslant |\mathscr{F}(X, Y)| + \tau(X, Y) - \delta_3\,. \tag{21}$$

On the other hand, obviously

$$k_1 + k_2 \leqslant f(Y) + \delta_2 + g(X) + \delta_1\,. \tag{22}$$

By combining (21) and (22) we obtain (20).

(7.4) THEOREM.

$$\begin{aligned}\delta = \max\{&\tau(X, (S - X)) + \tau(X_{\hat{g}}, (S - X)) + |\mathscr{F}(X_{\hat{g}}, (S - X)_{\hat{f}})| \\ &- g(X_{\hat{g}}) - f((S - X)_{\hat{f}}) - |\mathscr{F}(X - X_{\hat{g}}, S - X - (S - X)_{\hat{f}})|\},\end{aligned}$$

where X runs through all subsets of S; but it is enough to consider those subsets for which

$$\mathscr{F}(X - X_{\hat{g}}, S - X - (S - X)_{\hat{f}}) = \varnothing .$$

Note that this formula is much more difficult then (7.3) but to state the maximum only one subset must run through the subsets of S.

PROOF: We shall use the notations of Section 6 again. Let $\mathscr{G}$ be an optimal subgraph. By Theorems (6.4) and (5.6) the integers of the interval $[1, \tau]$ can be divided into two classes $\mathscr{I}_1$, $\mathscr{I}_2$ such that

$$\mathscr{F}(D_i, A) \subseteq \mathscr{G}, \qquad \text{if} \quad i \in \mathscr{I}_1,$$
$$\mathscr{F}(D_i, B) \subseteq \mathscr{G}, \qquad \text{if} \quad i \in \mathscr{I}_2.$$

Put $D_A = \bigcup_{i \in \mathscr{I}_1} D_i$, $D_B = \bigcup_{i \in \mathscr{I}_2} D_i$, $P = A \cup C \cup D_A$, $Q = B \cup D_B$. We show

$$D_A \subseteq P - P_{\hat{g}}, \qquad A \subseteq P_{\hat{g}}. \tag{23}$$

Really, if $x \in D_A$ then (5.5) shows that

$$\varphi_{[D]}(x) > f(x) - |\mathscr{F}(A, x)|,$$

i.e.,

$$\varphi_{[P]}(x) > f(x) \geqslant \hat{g}(x).$$

On the other hand, if $x \in A$, then by Theorem (6.4)

$$\varphi_{[P]}(x) \leqslant \varphi_{\mathscr{G}}(x) \leqslant \hat{g}(x).$$

From (23) it follows that

$$P_{\hat{g}} = A \cup C_1, \qquad P - P_{\hat{g}} = D_A \cup C_2, \tag{24}$$

where $C_1 \cup C_2 = C$. Similarly

$$Q_{\hat{f}} = B, \qquad Q - Q_{\hat{f}} = D_B. \tag{25}$$

Calculate now the value of

$$\begin{aligned} \nu &= \tau(P_{\hat{g}}, Q) + \tau(P, Q_{\hat{f}}) + |\mathscr{F}(P_{\hat{g}}, Q_{\hat{f}})| - g(P_{\hat{g}}) - f(Q_{\hat{f}}) \\ &\quad - |\mathscr{F}(P - P_{\hat{g}}, Q - Q_{\hat{f}})| \\ &= \tau(P_{\hat{g}}, Q) + \tau(P, Q_{\hat{f}}) + |\mathscr{F}(A, B)| + |\mathscr{F}(C_1, B)| \\ &\quad - g(A) - g(C_1) - f(B). \end{aligned}$$

By (24), (25), and (4.2)

$$\tau(P_{\hat{g}}, Q) \geqslant |\mathscr{I}_1|, \qquad \tau(P, Q_{\hat{g}}) \geqslant |\mathscr{I}_2|. \tag{26}$$

Now only members depending on C_1 are unknown. For $x \in C_1$,

$$\varphi_{[P]}(x) \leqslant \hat{g}(x).$$

On the other hand,

$$\varphi_{[P]}(x) \geqslant \varphi_{\mathscr{G}}(x) \geqslant \hat{g}(x);$$

hence here the equality must hold. Thus

$$g(C_1) = \varphi(C_1) - \varphi_{[P]}(C_1) = |\mathscr{F}(C_1, B)| \tag{27}$$

(we have used (4.2) again). (26) and (27) imply

$$\nu \geqslant \tau + |\mathscr{F}(A, B)| - g(A) - f(B) = \delta(f, g).$$

Thus we have only to prove that, for any $X \subseteq S$ and subgraph $\mathscr{G}$, putting $Y = S - X$,

$$\delta_{\mathscr{G}}(f, g) \geqslant \tau(X, Y_{\hat{f}}) + \tau(X_{\hat{g}}, Y) + |\mathscr{F}(X_{\hat{g}}, Y_{\hat{f}})| \\ - f(Y_{\hat{f}}) - g(X_{\hat{g}}) - |\mathscr{F}(X - X_{\hat{g}}, Y - Y_{\hat{f}})|.$$

The proof of this fact is just the same as the second part of the proof of Theorem (7.3); we omit it.

8. We investigate the case $f = 1$ in greater detail. Suppose that our graph has no isolated vertices. Remark (1.2) shows that in this case a minimal optimal subgraph is an independent edge-system. We may confine ourselves to the case in which $g \leqslant \varphi$, i.e., $g(x) = \varphi(x)$ or $g(x) = \varphi(x) - 1$ for any vertex x.

Consider the sets A, B, C, D.

(8.1) $\mathscr{F}(A, C) = \mathscr{F}(A, D) = \mathscr{F}(C, D) = \mathscr{F}(A) = \varnothing$. (We have already seen that $\mathscr{F}(C, D) = \varnothing$ is true for an arbitrary pair of functions.)

PROOF: Suppose indirectly that there is an edge E joining $x \in A \cup D$ to $y \in A \cup C$. By the characterization (II)[6] of A and D there exists an optimal subgraph $\mathscr{G}$ such that

$$\varphi_{\overline{\mathscr{G}}}(x) > g(x);$$

consequently

$$\varphi_{\overline{\mathscr{G}}}(x) = \varphi(x);$$

and thus $E \notin \mathscr{G}$. By (2.1) $\mathscr{G}' = \mathscr{G} \cup \{E\}$ is also optimal, and obviously

$$\varphi_{\mathscr{G}'}(y) = 2.$$

But this contradicts the characterization (II) of A and C.

[6] See Section 6.

Put $K = B \cup C$, $L = A \cup D$. Obviously $g(x) = \varphi(x) - 1$ if $x \in L$. (8.1) implies that the connected components of $[L]$ are the components of $[D]$ and the vertices of A. By (5.2) these components have an odd number of vertices; by (5.5) they even have the property that omitting any vertex of them the remaining graph contains a 1-factor. Furthermore the number of components of $[L]$ is

$$\tau(K) = |A| + \tau(A, B),$$

and thus

$$\begin{aligned}\delta(f, g) &= |\mathscr{F}(A, B)| + \tau(A, B) - g(A) - f(B)\\ &= \varphi(A) + (\tau(K) - |A|) - (\varphi(A) - |A|) - |B| = \tau(K) - |B|.\end{aligned}$$

Let us note that $\tau(K)$ can be interpreted as the number of those odd components of $[S - B]$ on which $g = \varphi - 1$. It is easy to see that these considerations give the following equivalent of the theorem mentioned in the introduction:

(8.2) THEOREM. *Let $T \subseteq S$. There exists an independent edge-system covering T if and only if for any $T' \subseteq T$ the number of the odd components of $[T']$ does not exceed the number of those vertices of $S - T'$ which are joined to T'.*

Finally let us consider the case $g \equiv \varphi - 1$. By (1.2) and (1.3), a minimal optimal subgraph is a maximal independent edge-system and a maximal optimal subgraph is a minimal edge-system covering all vertices. Denote their cardinality by e_i and e_c, respectively. Since they are optimal subgraphs,

$$\delta(f, g) = 2e_c - |S| = |S| - 2e_i,$$

and this gives the well-known formula (see [2]):

$$e_i + e_c = |S|. \tag{28}$$

(Note that to show (28) we needed only (1.2) and (1.3).)

The characterization (II) gives that L is the set of those vertices which are covered (saturated) by every maximal independent edge-system. Thus the argument before (8.2) gives the results of [3] and [11] too.

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