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Equi-distribution over descent classes of the hyperoctahedral group ☆

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Abstract

A classical result of MacMahon shows that the length function and the major index are equi-distributed over the symmetric group. Foata and Schützenberger gave a remarkable refinement and proved that these parameters are equi-distributed over inverse descent classes, implying bivariate equi-distribution identities. Type B analogues of these results, refinements and consequences are given in this paper.

Keywords: Permutation statistics; Signed permutations; Major index; Length; Descent class

1. Introduction

Many combinatorial identities on groups are motivated by the fundamental works of MacMahon [24]. Let S_n be the symmetric group acting on $1, \dots, n$. We are interested in a refined enumeration of permutations according to (non-negative, integer valued) combinatorial parameters. Two parameters that have the same generating function are said to be *equi-distributed*. MacMahon [24] has shown, about a hundred years ago, that the inversion number and the major index statistics are equi-distributed on S_n (Theorem 2.2 below). In the last three decades MacMahon's theorem has received far-reaching refinements and generalizations. Bivariate distributions were first studied by Carlitz [11]. Foata [16] gave a bijective proof of MacMahon's theorem; then

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Foata and Schützenberger [19] applied this bijection to refine MacMahon's identity, proving that the inversion number and the major index are equi-distributed over subsets of S_n with prescribed descent set of the inverse permutation (Theorem 2.3 below). Garsia and Gessel [20] extended the analysis to multivariate distributions. In particular, they gave an independent proof of the Foata–Schützenberger theorem, relying on an explicit and simple generating function (see Theorem 2.6 below). Further refinements and analogues of the Foata–Schützenberger theorem were later found, involving left-to-right minima and maxima [9] and pattern-avoiding permutations [6, 25]. For a representation theoretic application of Theorem 2.3 see [28].

Since the length and descent parameters may be defined via the Coxeter structure of the symmetric group, it is very natural to look for analogues of the above theorems in other Coxeter groups. This is a challenging open problem. In this paper we focus on the hyperoctahedral group B_n, namely the classical Weyl group of type B. Our goal is to find a type B analogue of the Foata–Schützenberger theorem (Theorem 2.3). To solve this we have to choose an appropriate type B extension of the major index among the many candidates, which were introduced and studied in [12–14,18,26,27,31]. It turns out that the *flag-major index*, which was introduced in [5] and further studied in [1,3,4,7,15,21], has the desired property: the flag-major index and the length function are equi-distributed on inverse descent classes of B_n. In fact, we obtain a slight refinement of this result, involving the "last digit" parameter. This parameter is involved in several closely related identities on S_n, see, e.g., [4,6,25]. Our refinement also implies a MacMahon type theorem for the classical Weyl group of type D, which has recently been proved in [7]. A summary of the results of this paper appeared in [2].

The rest of the paper is organized as follows. Definitions, notation and necessary background are given in Section 2. The main results are listed in Section 3. Proofs of the main theorems are given in Section 4. Section 5 contains open problems, an application to Weyl groups of type D and remarks regarding different versions of the flag major index. Finally, in Appendix A, we give an alternative proof of our type B analogue of the Foata–Schützenberger theorem.

2. Background and notation

2.1. Notation

Let $\mathbf{P} := \{1, 2, 3, \ldots\}$, $\mathbf{N} := \mathbf{P} \cup \{0\}$, and $\mathbf{Z}$ be the ring of integers. For $n \in \mathbf{P}$ let $[n] := \{1, 2, \ldots, n\}$ and also $[0] := \emptyset$. Given $m, n \in \mathbf{Z}$, $m \leqslant n$, let $[m, n] := \{m, m+1, \ldots, n\}$. For $n \in \mathbf{P}$ denote $[\pm n] := [-n, n] \setminus \{0\}$. For $S \subset \mathbf{N}$ write $S = \{a_1, \ldots, a_r\}_<$ to mean that $S = \{a_1, \ldots, a_r\}$ and $a_1 < \cdots < a_r$. The cardinality of a set A will be denoted by $|A|$.

For $n \in \mathbf{N}$ denote

$$[n]_q := \frac{1-q^n}{1-q},$$

$$[n]_q! := \prod_{i=1}^{n} [i]_q \quad (n \geqslant 1), \qquad [0]_q! := 1.$$

For $n_1, \ldots, n_t \in \mathbf{N}$ such that $n_1 + \cdots + n_t = n$ define the *q-multinomial coefficient*

$$\begin{bmatrix} n \\ n_1, \ldots, n_t \end{bmatrix}_q := \frac{[n]_q!}{[n_1]_q! \cdots [n_t]_q!}$$

and use a shorter notation for the *q-binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{bmatrix} n \\ k,\ n-k \end{bmatrix}_q \quad (0 \leqslant k \leqslant n).$$

Given a statement P we will sometimes find it convenient to let

$$\chi(P) := \begin{cases} 1, & \text{if } P \text{ is true}, \\ 0, & \text{if } P \text{ is false}. \end{cases}$$

Given a sequence $\sigma = (a_1, \ldots, a_n) \in \mathbf{Z}^n$ we say that a pair $(i, j) \in [n] \times [n]$ is an *inversion* of σ if $i < j$ and $a_i > a_j$. We say that $i \in [n-1]$ is a *descent* of σ if $a_i > a_{i+1}$. We denote by $\mathrm{inv}(\sigma)$ (respectively, $\mathrm{des}(\sigma)$) the number of inversions (respectively, descents) of σ. We also let

$$\mathrm{maj}(\sigma) := \sum_{\{i \mid a_i > a_{i+1}\}} i$$

and call it the *major index* of σ.

For $M = \{m_1, \ldots, m_t\}_< \subseteq [0, n-1]$ denote $m_0 := 0$, $m_{t+1} := n$, and let

$$\begin{bmatrix} n \\ \Delta M \end{bmatrix}_q := \begin{bmatrix} n \\ m_1 - m_0, m_2 - m_1, \ldots, m_{t+1} - m_t \end{bmatrix}_q.$$

Let $S_1, \ldots, S_k$ be sequences of distinct integers which are pairwise disjoint as sets. A sequence S is a *shuffle* of $S_1, \ldots, S_k$ if S is a disjoint union of $S_1, \ldots, S_k$ (as sets) and the elements of each S_i appear in S in the same order as in S_i.

2.2. The symmetric group

Let S_n be the symmetric group on $[n]$. Recall that S_n is a Coxeter group with respect to the set of Coxeter generators $S := \{s_i \mid 1 \leqslant i \leqslant n-1\}$, where s_i may be interpreted as the adjacent transposition $(i, i+1)$.

If $\pi \in S_n$ then the classical combinatorial statistics (defined in the previous subsection) of the sequence $(\pi(1), \ldots, \pi(n))$ may also be defined via the Coxeter generators: the inversion number $\mathrm{inv}(\pi)$ is equal to the *length* $\ell(\pi)$ of π with respect to the set of generators S; the *descent set* of π is

$$\mathrm{Des}(\pi) := \bigl\{1 \leqslant i < n \mid \pi(i) > \pi(i+1)\bigr\} = \bigl\{1 \leqslant i < n \mid \ell(\pi) > \ell(\pi s_i)\bigr\};$$

the *descent number* of π is $\mathrm{des}(\pi) := |\mathrm{Des}(\pi)|$; and the *major index* of π is the sum (possibly zero)

$$\mathrm{maj}(\pi) := \sum_{i \in \mathrm{Des}(\pi)} i.$$

The *inverse descent class* in S_n corresponding to $M \subseteq [n-1]$ is the set $\{\pi \in S_n \mid \mathrm{Des}(\pi^{-1}) = M\}$. Note the following relation between inverse descent classes and shuffles.

Observation 2.1. *Let $\pi \in S_n$ and $M = \{m_1, \ldots, m_t\}_< \subseteq [n-1]$. Then*: $\mathrm{Des}(\pi^{-1}) \subseteq M$ *if and only if $(\pi(1), \ldots, \pi(n))$ is a shuffle of the following increasing sequences*:

$(1, \ldots, m_1)$,

$(m_1 + 1, \ldots, m_2)$,

$$\vdots$$

$$(m_t + 1, \dots, n).$$

MacMahon's classical theorem asserts that the length function and the major index are equidistributed on S_n.

Theorem 2.2 *(MacMahon's Theorem [24]).*

$$\sum_{\pi \in S_n} q^{\ell(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = [n]_q!.$$

Foata [16] gave a bijective proof of this theorem. Foata and Schützenberger [19] applied this bijection to prove the following refinement.

Theorem 2.3 *(Foata–Schützenberger Theorem [19, Theorem 1]). For every subset $M \subseteq [n-1]$,*

$$\sum_{\{\pi \in S_n \mid \mathrm{Des}(\pi^{-1}) = M\}} q^{\ell(\pi)} = \sum_{\{\pi \in S_n \mid \mathrm{Des}(\pi^{-1}) = M\}} q^{\mathrm{maj}(\pi)}.$$

This theorem implies

Corollary 2.4.

$$(1) \quad \sum_{\pi \in S_n} q^{\ell(\pi)} t^{\mathrm{des}(\pi^{-1})} = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi^{-1})},$$

$$(2) \quad \sum_{\pi \in S_n} q^{\ell(\pi)} t^{\mathrm{maj}(\pi^{-1})} = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{maj}(\pi^{-1})}.$$

An alternative proof of Theorem 2.3 may be obtained using the following classical fact [30, Proposition 1.3.17].

Fact 2.5. *For any $M \subseteq [n-1]$,*

$$\sum_{\{\pi \in S_n \mid \mathrm{Des}(\pi^{-1}) \subseteq M\}} q^{\mathrm{inv}(\pi)} = \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q.$$

Garsia and Gessel proved that a similar identity holds for the major index.

Theorem 2.6. [20, Theorem 3.1] *For any $M \subseteq [n-1]$,*

$$\sum_{\{\pi \in S_n \mid \mathrm{Des}(\pi^{-1}) \subseteq M\}} q^{\mathrm{maj}(\pi)} = \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q.$$

Combining this theorem with Fact 2.5 implies Theorem 2.3.

2.3. The hyperoctahedral group

We denote by B_n the group of all bijections σ of the set $[\pm n]$ onto itself such that

$$\sigma(-a) = -\sigma(a) \quad \bigl(\forall a \in [\pm n]\bigr),$$

with composition as the group operation. This group is usually known as the group of "signed permutations" on $[n]$, or as the *hyperoctahedral group* of rank n. We identify S_n as a subgroup of B_n, and B_n as a subgroup of S_{2n}, in the natural ways.

If $\sigma \in B_n$ then write $\sigma = [a_1, \ldots, a_n]$ to mean that $\sigma(i) = a_i$ for $1 \leqslant i \leqslant n$, and let

$$\begin{aligned}
&\operatorname{inv}(\sigma) := \operatorname{inv}(a_1, \ldots, a_n),\\
&\operatorname{Des}_A(\sigma) := \operatorname{Des}(a_1, \ldots, a_n),\\
&\operatorname{des}_A(\sigma) := \operatorname{des}(a_1, \ldots, a_n),\\
&\operatorname{maj}_A(\sigma) := \operatorname{maj}(a_1, \ldots, a_n),\\
&\operatorname{Neg}(\sigma) := \bigl\{i \in [n] \mid a_i < 0\bigr\},\\
&\operatorname{neg}(\sigma) := \bigl|\operatorname{Neg}(\sigma)\bigr|.
\end{aligned}$$

It is well known (see, e.g., [8, Proposition 8.1.3]) that B_n is a Coxeter group with respect to the generating set $\{s_0, s_1, s_2, \ldots, s_{n-1}\}$, where

$$s_0 := [-1, 2, \ldots, n]$$

and

$$s_i := [1, 2, \ldots, i-1, i+1, i, i+2, \ldots, n] \quad (1 \leqslant i < n).$$

This gives rise to two other natural statistics on B_n (similarly definable for any Coxeter group), namely

$$\ell_B(\sigma) := \min\bigl\{r \in \mathbf{N} \mid \sigma = s_{i_1} \cdots s_{i_r} \text{ for some } i_1, \ldots, i_r \in [0, n-1]\bigr\}$$

(known as the *length* of σ) and

$$\operatorname{des}_B(\sigma) := \bigl|\operatorname{Des}_B(\sigma)\bigr|,$$

where the *B-descent set* $\operatorname{Des}_B(\sigma)$ is defined as

$$\operatorname{Des}_B(\sigma) := \bigl\{i \in [0, n-1] \mid \ell_B(\sigma s_i) < \ell_B(\sigma)\bigr\}.$$

Remark 2.7. Note that for every $\sigma \in B_n$

$$\operatorname{Des}_A(\sigma) = \operatorname{Des}_B(\sigma) \setminus \{0\}.$$

There are well-known direct combinatorial ways to compute these statistics for $\sigma \in B_n$ (see, e.g., [8, Propositions 8.1.1 and 8.1.2] or [10, Proposition 3.1 and Corollary 3.2]), namely

$$\ell_B(\sigma) = \operatorname{inv}(\sigma) + \sum_{i \in \operatorname{Neg}(\sigma)} \bigl|\sigma(i)\bigr| \tag{1}$$

and

$$\operatorname{des}_B(\sigma) = \bigl|\bigl\{i \in [0, n-1] \mid \sigma(i) > \sigma(i+1)\bigr\}\bigr|, \tag{2}$$

where $\sigma(0) := 0$. For example, if $\sigma = [-3, 1, -6, 2, -4, -5] \in B_6$ then $\operatorname{inv}(\sigma) = 9$, $\operatorname{des}_A(\sigma) = 3$, $\operatorname{maj}_A(\sigma) = 11$, $\operatorname{neg}(\sigma) = 4$, $\ell_B(\sigma) = 27$, and $\operatorname{des}_B(\sigma) = 4$.

3. Main results

Definition 3.1. The *flag major index* of a signed permutation $\sigma \in B_n$ is defined by

$$\mathrm{fmaj}(\sigma) := 2 \cdot \mathrm{maj}_A(\sigma) + \mathrm{neg}(\sigma),$$

where $\mathrm{maj}_A(\sigma)$ is the major index of the sequence $(\sigma(1), \ldots, \sigma(n))$ with respect to the order $-n < \cdots < -1 < 1 < \cdots < n$.

The main theorem is

Theorem 3.2. *For every subset $M \subseteq [0, n-1]$ and $i \in [\pm n]$*

$$\sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) \subseteq M,\, \sigma(n)=i\}} q^{\ell_B(\sigma)} = \sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) \subseteq M,\, \sigma(n)=i\}} q^{\mathrm{fmaj}(\sigma)}.$$

See more details in Theorems 4.2 and 4.3 below. Forgetting $\sigma(n)$, we have

Theorem 3.3. *For every subset $M = \{m_1, \ldots, m_t\}_< \subseteq [0, n-1]$,*

$$\sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) \subseteq M\}} q^{\ell_B(\sigma)} = \sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) \subseteq M\}} q^{\mathrm{fmaj}(\sigma)} = \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} \left(1 + q^j\right).$$

We deduce a Foata–Schützenberger type theorem for B_n.

Theorem 3.4. *For every subset $M \subseteq [0, n-1]$,*

$$\sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) = M\}} q^{\ell_B(\sigma)} = \sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) = M\}} q^{\mathrm{fmaj}(\sigma)}.$$

4. Proofs

Lemma 4.1. *Let $\sigma \in B_n$ and $M = \{m_1, \ldots, m_t\}_< \subseteq [0, n-1]$. Denote $m_{t+1} := n$. Then*: $\mathrm{Des}_B(\sigma^{-1}) \subseteq M$ *if and only if there exist* (*unique*) *integers $r_1, \ldots, r_t$ satisfying $m_i \leqslant r_i \leqslant m_{i+1}$* ($\forall i$) *such that* $(\sigma(1), \ldots, \sigma(n))$ *is a shuffle of the following increasing sequences*:

$$\begin{aligned}
&(1, 2, \ldots, m_1),\\
&\big(-r_1, -r_1 + 1, \ldots, -(m_1 + 1)\big),\\
&(r_1 + 1, r_1 + 2, \ldots, m_2),\\
&\quad\vdots\\
&\big(-r_t, -r_t + 1, \ldots, -(m_t + 1)\big),\\
&\big(r_t + 1, r_t + 2, \ldots, n (= m_{t+1})\big).
\end{aligned}$$

Some of these sequences may be empty (*if $r_i = m_i$ or $r_i = m_{i+1}$ for some i, or if $m_1 = 0$*).

Proof. Assume first that $0 \in M$ (i.e., $m_1 = 0$). Then

$$\mathrm{Des}_B\big(\sigma^{-1}\big) \subseteq M \quad \Leftrightarrow \quad \mathrm{Des}_A\big(\sigma^{-1}\big) \subseteq \{m_2, \ldots, m_t\}.$$

This is equivalent to

$$\sigma^{-1}(j) < \sigma^{-1}(j+1) \quad \big(\forall j \in [n] \setminus M\big),$$

namely,

$$\sigma^{-1}(m_i + 1) < \sigma^{-1}(m_i + 2) < \cdots < \sigma^{-1}(m_{i+1}) \quad (1 \leqslant i \leqslant t).$$

Defining $m_i \leqslant r_i \leqslant m_{i+1}$ such that $\sigma^{-1}(r_i)$ is the last negative value in this sequence, we get the desired conclusion (with an empty sequence $(1, \ldots, m_1)$).

If $0 \notin M$ (i.e., $m_1 > 0$) then we get the same conclusion for the intervals $[m_i + 1, m_{i+1}]$ with $1 \leqslant i \leqslant t$, and in addition $\sigma^{-1}(1) > 0$ so that

$$0 < \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(m_1).$$

This gives us the desired conclusion (with a nonempty sequence $(1, \ldots, m_1)$). □

Theorem 4.2. *Let $M = \{m_1, m_2, \ldots, m_t\}_< \subseteq [0, n-1]$ and $i \in [\pm n]$. Denote $m_0 := 0$ and $m_{t+1} := n$. Then*

$$\sum_{\{\sigma \in B_n \mid Des_B(\sigma^{-1}) \subseteq M,\, \sigma(n) = i\}} q^{\mathrm{fmaj}(\sigma)} = \frac{\alpha_i(M)}{q^n - q^{-n}} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} \big(1 + q^j\big),$$

where

$$\alpha_i(M) = \begin{cases} q^{m_1} - q^{-m_1}, & \text{if } i = m_1 > 0, \\ q^{-m_s} - q^{-m_{s+1}}, & \text{if } i = m_{s+1} \text{ for } s \in [t], \\ q^{m_{s+1}} - q^{m_s}, & \text{if } i = -(m_s + 1) \text{ for } s \in [t], \\ 0, & \text{otherwise.} \end{cases}$$

Note that the first case $(i = m_1 > 0)$ occurs only if $0 \notin M$.

Proof. By induction on n. It is easy to verify the result for $n \leqslant 2$. Assume that $n \geqslant 3$, and that the result holds for $n - 1$.

We shall use the induction hypothesis by "deleting" the value $\sigma(n) = i$ from $\sigma \in B_n$. Formally, for $i \in [\pm n]$, define a function $\phi_i : [-n, n] \to [-(n-1), n-1]$ by

$$a' = \phi_i(a) := \begin{cases} a, & \text{if } |a| < |i|, \\ \frac{a}{|a|}(|a| - 1), & \text{if } |a| \geqslant |i| \end{cases} \quad \big(\forall a \in [-n, n]\big).$$

When restricted to $[-n, n] \setminus \{0, i, -i\} = [\pm n] \setminus \{i, -i\}$, ϕ_i is a bijection onto $[\pm(n-1)]$. For $\sigma \in B_n$ with $\sigma(n) = i$, let

$$\tau := \big[\sigma(1)', \ldots, \sigma(n-1)'\big] \in B_{n-1}.$$

Lemma 4.1 (for σ and τ) implies that, for any $M = \{m_1, \ldots, m_t\}_< \subseteq [0, n-1]$, the map $\sigma \mapsto \tau$ is a bijection from $\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) \subseteq M\}$ onto $\{\tau \in B_{n-1} \mid \mathrm{Des}_B(\tau^{-1}) \subseteq M'\}$, where $M' := \{m'_1, \ldots, m'_t\}_\leqslant$. Note that we may have $m'_s = m'_{s+1}$ (if $m_s + 1 = i = m_{s+1}$), but this will make no difference in the sequel.

Assume that $\sigma(n-1) = j$, so that $\tau(n-1) = j'$. Then:

$$\mathrm{fmaj}(\sigma) = \mathrm{fmaj}(\tau) + 2(n-1)\chi(j > i) + \chi(i < 0).$$

Denote

$$B_n(M,i) := \bigl\{\sigma \in B_n \mid \mathrm{Des}_B\bigl(\sigma^{-1}\bigr) \subseteq M,\ \sigma(n) = i\bigr\},$$

and similarly $B_{n-1}(M', j')$. Denote also

$$M_\pm := \{m_1, \ldots, m_{t+1}\} \cup \bigl\{-(m_1+1), \ldots, -(m_t+1)\bigr\}.$$

By Lemma 4.1, if $\mathrm{Des}_B(\sigma^{-1}) \subseteq M$ then $\sigma(n) \in M_\pm$. Thus the sum in the statement of our theorem is zero whenever $i \notin M_\pm$. We shall check the values $i \in M_\pm$ case-by-case.

First note that the coefficients $\alpha_i(M)$ defined in the statement of the theorem satisfy

$$\sum_{j=-n}^{n} \alpha_j(M) = q^n - q^{-n} \tag{3}$$

and

$$\sum_{j>i} \alpha_j(M) = \begin{cases} q^{-m_s} - q^{-n}, & \text{if } i = m_s \text{ for } s \in [t+1], \\ q^{m_s} - q^{-n}, & \text{if } i = -(m_s+1) \text{ for } s \in [t]. \end{cases} \tag{4}$$

Case 1 *($i = m_{s+1}$ for $s \in [t]$).* Here

$$\mathrm{fmaj}(\sigma) = \mathrm{fmaj}(\tau) + 2(n-1)\chi(j > i),$$

so that

$$\begin{aligned}
&\sum_{\sigma \in B_n(M,i)} q^{\mathrm{fmaj}(\sigma)} \\
&\quad = \sum_{j \in [\pm n]\setminus\{i,-i\}} \ \sum_{\substack{\sigma \in B_n(M,i) \\ \sigma(n-1)=j}} q^{\mathrm{fmaj}(\sigma)} \\
&\quad = \sum_{j<i} \ \sum_{\tau \in B_{n-1}(M',j')} q^{\mathrm{fmaj}(\tau)} + \sum_{j>i} \ \sum_{\tau \in B_{n-1}(M',j')} q^{\mathrm{fmaj}(\tau)+2(n-1)} \\
&\quad = \frac{\left[\begin{smallmatrix} n-1 \\ \Delta M' \end{smallmatrix}\right]_q}{q^{n-1} - q^{-(n-1)}} \cdot \prod_{j=m'_1+1}^{n-1} \bigl(1+q^j\bigr) \cdot \Bigl[\sum_{j' \leqslant i'} \alpha_{j'}(M') + q^{2(n-1)} \sum_{j'>i'} \alpha_{j'}(M')\Bigr].
\end{aligned}$$

We used here the fact that $j < i \Leftrightarrow j' \leqslant i'$. Now, by equalities (3) and (4) (for M' instead of M, with $i' = m'_{s+1} = m_{s+1} - 1$),

$$\begin{aligned}
\sum_{j' \leqslant i'} \alpha_{j'}(M') + q^{2(n-1)} \sum_{j'>i'} \alpha_{j'}(M') &= \bigl(q^{n-1} - q^{-m'_{s+1}}\bigr) + q^{2(n-1)}\bigl(q^{-m'_{s+1}} - q^{-(n-1)}\bigr) \\
&= \bigl(q^{2(n-1)} - 1\bigr) q^{-(m_{s+1}-1)}.
\end{aligned}$$

Thus (using $m'_1 = m_1$)

$$\begin{aligned}
\sum_{\sigma \in B_n(M,i)} q^{\mathrm{fmaj}(\sigma)} &= \frac{q^{2(n-1)} - 1}{q^{n-1} - q^{-(n-1)}} \cdot q^{-(m_{s+1}-1)} \cdot \begin{bmatrix} n-1 \\ \Delta M' \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n-1} \bigl(1+q^j\bigr) \\
&= q^{(n-1)-(m_{s+1}-1)} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \frac{[m_{s+1} - m_s]_q}{[n]_q} \cdot \prod_{j=m_1+1}^{n} \bigl(1+q^j\bigr) \cdot \frac{1}{1+q^n}
\end{aligned}$$

$$= q^{n-m_{s+1}} \cdot \frac{1-q^{m_{s+1}-m_s}}{1-q^{2n}} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right)$$

$$= \frac{q^{-m_s}-q^{-m_{s+1}}}{q^n-q^{-n}} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right),$$

as claimed.

Case 2 *($i = m_1 > 0$)*. The computations are as in the previous case, except that $m_1' = m_1 - 1$:

$$\sum_{\sigma \in B_n(M,i)} q^{\mathrm{fmaj}(\sigma)} = \frac{q^{2(n-1)}-1}{q^{n-1}-q^{-(n-1)}} \cdot q^{-(m_1-1)} \cdot \begin{bmatrix} n-1 \\ \Delta M' \end{bmatrix}_q \cdot \prod_{j=m_1}^{n-1} \left(1+q^j\right)$$

$$= q^{(n-1)-(m_1-1)} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \frac{[m_1]_q}{[n]_q} \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right) \cdot \frac{1+q^{m_1}}{1+q^n}$$

$$= q^{n-m_1} \cdot \frac{1-q^{2m_1}}{1-q^{2n}} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right)$$

$$= \frac{q^{m_1}-q^{-m_1}}{q^n-q^{-n}} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right),$$

as claimed.

Case 3 *($i = -(m_s+1)$ for $s \in [t]$)*. Here $\chi(i<0) = 1$, so that

$$\sum_{\sigma \in B_n(M,i)} q^{\mathrm{fmaj}(\sigma)}$$

$$= \sum_{j<i} \sum_{\tau \in B_{n-1}(M',j')} q^{\mathrm{fmaj}(\tau)+1} + \sum_{j>i} \sum_{\tau \in B_{n-1}(M',j')} q^{\mathrm{fmaj}(\tau)+2(n-1)+1}$$

$$= \frac{q \cdot \left[\begin{smallmatrix} n-1 \\ \Delta M' \end{smallmatrix}\right]_q}{q^{n-1}-q^{-(n-1)}} \cdot \prod_{j=m_1'+1}^{n-1} \left(1+q^j\right) \cdot \left[\sum_{j' \leqslant i'} \alpha_{j'}(M') + q^{2(n-1)} \sum_{j'>i'} \alpha_{j'}(M')\right].$$

By equalities (3) and (4) (for M' instead of M, with $i' = -(m_s+1)' = -m_s$),

$$\sum_{j' \leqslant i'} \alpha_{j'}(M') + q^{2(n-1)} \sum_{j'>i'} \alpha_{j'}(M') = \left(q^{n-1} - q^{m_s'}\right) + q^{2(n-1)}\left(q^{m_s'} - q^{-(n-1)}\right)$$

$$= \left(q^{2(n-1)} - 1\right) q^{m_s}.$$

Thus (using $m_1' = m_1$)

$$\sum_{\sigma \in B_n(M,i)} q^{\mathrm{fmaj}(\sigma)} = \frac{q(q^{2(n-1)}-1)}{q^{n-1}-q^{-(n-1)}} \cdot q^{m_s} \cdot \begin{bmatrix} n-1 \\ \Delta M' \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n-1} \left(1+q^j\right)$$

$$= q^{n+m_s} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \frac{[m_{s+1}-m_s]_q}{[n]_q} \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right) \cdot \frac{1}{1+q^n}$$

$$= q^{n+m_s} \cdot \frac{1-q^{m_{s+1}-m_s}}{1-q^{2n}} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right)$$

$$= \frac{q^{m_{s+1}} - q^{m_s}}{q^n - q^{-n}} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right),$$

as claimed. $\square$

To prove the corresponding result for ℓ_B we will find it useful to use the fact that

$$\ell_B(\sigma) = \frac{\operatorname{inv}(\bar{\sigma}) + \operatorname{neg}(\sigma)}{2} \tag{5}$$

for all $\sigma \in B_n$, where $\bar{\sigma} := (\sigma(-n), \ldots, \sigma(-1), \sigma(1), \ldots, \sigma(n))$. This formula, first observed by Incitti in [23], is easily seen to be equivalent to (1). So, for example, if $\sigma = [-3, 1, -6, 2, -4, -5]$ then $\operatorname{inv}(\bar{\sigma}) = 50$ and $\ell_B(\sigma) = (50+4)/2 = 27$.

Theorem 4.3. *Let $M = \{m_1, m_2, \ldots, m_t\}_< \subseteq [0, n-1]$ and $i \in [\pm n]$. Then $q^{\ell_B(\sigma)}$ satisfies exactly the same formula as does $q^{\operatorname{fmaj}(\sigma)}$ in Theorem* 4.2.

Proof. By induction on n. As in the previous proof, the result is easy to verify for $n \leqslant 2$. We assume that $n \geqslant 3$, and that the result holds for $n-1$. Again, we may assume that

$$i \in M_\pm := \{m_1, \ldots, m_{t+1}\} \cup \bigl\{-(m_1+1), \ldots, -(m_t+1)\bigr\}.$$

These values will be checked case-by-case.

Case 1 *($i = m_{s+1}$ for $s \in [t]$).* By (5)

$$\ell_B(\sigma) = \ell_B(\tau) + \frac{2(n-i)+0}{2},$$

so that

$$\sum_{\sigma \in B_n(M,i)} q^{\ell_B(\sigma)} = \sum_{j \in [\pm n] \setminus \{i,-i\}} \sum_{\substack{\sigma \in B_n(M,i) \\ \sigma(n-1)=j}} q^{\ell_B(\sigma)} = \sum_{j \neq \pm i} \sum_{\tau \in B_{n-1}(M',j')} q^{\ell_B(\tau)+(n-i)}$$

$$= \frac{q^{n-i} \cdot \left[\begin{smallmatrix} n-1 \\ \Delta M' \end{smallmatrix}\right]_q}{q^{n-1} - q^{-(n-1)}} \cdot \prod_{j=m_1'+1}^{n-1} \left(1+q^j\right) \cdot \sum_{j'=-(n-1)}^{n-1} \alpha_{j'}(M')$$

$$= q^{n-m_{s+1}} \cdot \begin{bmatrix} n-1 \\ \Delta M' \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n-1} \left(1+q^j\right)$$

$$= q^{n-m_{s+1}} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \frac{[m_{s+1}-m_s]_q}{[n]_q} \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right) \cdot \frac{1}{1+q^n}$$

$$= \frac{q^{-m_s} - q^{-m_{s+1}}}{q^n - q^{-n}} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right),$$

as claimed.

Case 2 *($i = m_1 > 0$).* The computations are as in the previous case, except that $m_1' = m_1 - 1$:

$$\sum_{\sigma\in B_n(M,i)} q^{\ell_B(\sigma)} = q^{n-m_1} \cdot \begin{bmatrix} n-1 \\ \Delta M' \end{bmatrix}_q \cdot \prod_{j=m_1}^{n-1} \left(1+q^j\right)$$
$$= q^{n-m_1} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \frac{[m_1]_q}{[n]_q} \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right) \cdot \frac{1+q^{m_1}}{1+q^n}$$
$$= \frac{q^{m_1} - q^{-m_1}}{q^n - q^{-n}} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right),$$

as claimed.

Case 3 *($i = -(m_s + 1)$ for $s \in [t]$).* Here

$$\ell_B(\sigma) = \ell_B(\tau) + \frac{(4(|i|-1) + 2(n-|i|) + 1) + 1}{2}.$$

Thus, using $m_1' = m_1$:

$$\sum_{\sigma\in B_n(M,i)} q^{\ell_B(\sigma)} = \sum_{j\neq\pm i} \sum_{\tau\in B_{n-1}(M',j')} q^{\ell_B(\tau)+(n+|i|-1)}$$
$$= \frac{q^{n+|i|-1} \cdot \left[\begin{smallmatrix} n-1 \\ \Delta M' \end{smallmatrix}\right]_q}{q^{n-1} - q^{-(n-1)}} \cdot \prod_{j=m_1'+1}^{n-1} \left(1+q^j\right) \cdot \sum_{j'=-(n-1)}^{n-1} \alpha_{j'}(M')$$
$$= q^{n+m_s} \cdot \begin{bmatrix} n-1 \\ \Delta M' \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n-1} \left(1+q^j\right)$$
$$= q^{n+m_s} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \frac{[m_{s+1} - m_s]_q}{[n]_q} \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right) \cdot \frac{1}{1+q^n}$$
$$= \frac{q^{m_{s+1}} - q^{m_s}}{q^n - q^{-n}} \cdot \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} \left(1+q^j\right),$$

as claimed. □

From Theorems 4.2 and 4.3 we immediately deduce Theorem 3.2.

By summing Theorems 4.2 and 4.3 over $i \in [\pm n]$ we obtain Theorem 3.3, which was the original motivation for this work, and which is the analogue, for the hyperoctahedral group, of Theorem 2.6.

It would be interesting to have combinatorial (bijective) proofs of these results.

Added in proof. A bijective proof of Theorem 3.4 has been found by Foata and Han [17].

5. Final remarks

5.1. Open problems

Numerical evidence suggests that the following holds.

Conjecture 5.1. *For every subset $M \subseteq [0, n-1]$ and $i \in [\pm n]$ the polynomial*

$$\sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) \subseteq M,\, \sigma(n)=i\}} q^{\ell_B(\sigma)} = \sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) \subseteq M,\, \sigma(n)=i\}} q^{\mathrm{fmaj}(\sigma)}$$

is (*symmetric and*) *unimodal.*

Conjecture 5.2. *For every subset $M \subseteq [0, n-1]$ the polynomial*

$$\sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) \subseteq M\}} q^{\ell_B(\sigma)} = \sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) \subseteq M\}} q^{\mathrm{fmaj}(\sigma)}$$

is (*symmetric and*) *unimodal.*

Using well-known results (see, e.g., [29, Proposition 1 and Theorem 11]) and Theorems 3.3, 4.2 and 4.3, it is easy to see that the above conjectures are equivalent to the following.

Conjecture 5.3. *For $0 \leqslant k < n$ the polynomial*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \cdot \prod_{j=k+1}^{n} \left(1 + q^j\right)$$

is (*symmetric and*) *unimodal.*

We have verified these conjectures for $n \leqslant 15$. Conjecture 5.3 clearly holds for $k = n-1$ and is known to be true for $k = 0$ (see, e.g., [29, p. 510]). We have checked that the polynomials

$$\sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) = M,\, \sigma(n)=i\}} q^{\ell_B(\sigma)} = \sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) = M,\, \sigma(n)=i\}} q^{\mathrm{fmaj}(\sigma)}$$

and

$$\sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) = M\}} q^{\ell_B(\sigma)} = \sum_{\{\sigma \in B_n \mid \mathrm{Des}_B(\sigma^{-1}) = M\}} q^{\mathrm{fmaj}(\sigma)}$$

are unimodal for all $M \subseteq [0, n-1]$ and $i \in [\pm n]$ if $n \leqslant 5$. In general, these polynomials are not symmetric.

5.2. Classical Weyl groups of type D

Let D_n be the classical Weyl group of type D and rank n. For an element $\sigma \in D_n$, let $\ell_D(\sigma)$ be the length of σ with respect to the Coxeter generators of D_n. It is well known that we may take

$$D_n = \left\{\sigma \in B_n \mid \mathrm{neg}(\sigma) \equiv 0 \bmod 2\right\}.$$

Let $\sigma = [\sigma(1), \ldots, \sigma(n)] \in D_n$. Biagioli and Caselli [7] introduced a flag major index for D_n

$$\mathrm{fmaj}_D(\sigma) := \mathrm{fmaj}\left(\sigma(1), \ldots, \sigma(n-1), \left|\sigma(n)\right|\right).$$

By definition,

$$\sum_{\sigma \in D_n} q^{\mathrm{fmaj}_D(\sigma)} = \sum_{\{\sigma \in B_n \mid \sigma(n) > 0\}} q^{\mathrm{fmaj}(\sigma)}. \tag{6}$$

Proposition 5.4.

$$\sum_{\sigma\in D_n} q^{\ell_D(\sigma)} = \sum_{\{\sigma\in B_n \mid \sigma(n)>0\}} q^{\ell_B(\sigma)}.$$

Proof. It is well known (see, e.g., [22, §3.15]) that

$$\sum_{\sigma\in D_n} q^{\ell_D(\sigma)} = [n]_q \cdot \prod_{i=1}^{n-1} [2i]_q.$$

On the other hand,

$$\begin{aligned}\sum_{\{\sigma\in B_n \mid \sigma(n)>0\}} q^{\ell_B(\sigma)} &= \sum_{i=1}^{n} \sum_{\{\sigma\in B_n \mid \sigma(n)=i\}} q^{\ell_B(\sigma)} \\ &= \sum_{i=1}^{n} \sum_{\{\sigma\in B_n \mid \mathrm{Des}_B(\sigma^{-1})\subseteq[0,n-1],\, \sigma(n)=i\}} q^{\ell_B(\sigma)}.\end{aligned}$$

By Theorem 4.3, this is equal to

$$\frac{q^0-q^{-n}}{q^n-q^{-n}} \cdot \begin{bmatrix} n \\ 1,\dots,1 \end{bmatrix}_q \cdot \prod_{j=1}^{n}\left(1+q^j\right) = [n]_q! \cdot \prod_{j=1}^{n-1}\left(1+q^j\right) = [n]_q \cdot \prod_{j=1}^{n-1} [2j]_q,$$

completing the proof. $\square$

We deduce the following type D analogue (first proved in [7]) of MacMahon's theorem.

Corollary 5.5.

$$\sum_{\sigma\in D_n} q^{\mathrm{fmaj}_D(\sigma)} = \sum_{\sigma\in D_n} q^{\ell_D(\sigma)}.$$

Proof. Combine (6) and Proposition 5.4 with Theorem 3.2. $\square$

Problem 5.6. Find an analogue of the Foata–Schützenberger theorem for D_n.

The obvious candidate for such an analogue does not work.

5.3. Two versions of the flag major index

The flag-major index of $\sigma \in B_n$, flag-major(σ), was originally defined as the length of a distinguished canonical expression for σ. In [5] this length was shown to be equal to $2\cdot \mathrm{maj}_A(\sigma) + \mathrm{neg}(\sigma)$, where the major index of the sequence $(\sigma(1),\dots,\sigma(n))$ was taken with respect to the order $-1 < \dots < -n < 1 < \dots < n$. In [1] we considered a different order: $-n < \dots < -1 < 1 < \dots < n$ (i.e., we defined fmaj as in Section 3 above).

While both versions give type B analogues of the MacMahon and Carlitz identities, only the second one gives an analogue of the Foata–Schützenberger theorem. On the other hand, the first one has the alternative natural interpretation as length, as mentioned above, and also produces a natural analogue of the signed Mahonian formula of Gessel and Simion, see [4]. The relation between these two versions and their (possibly different) algebraic roles requires further study.

Appendix A

In this appendix we give an alternative proof of Theorems 3.3 and 3.4 (but not Theorem 3.2), using q-binomial identities.

A.1. Binomial identities

In this subsection we recall several q-binomial identities.

Lemma A.1. *For every positive integer n*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^k = \prod_{i=1}^{n} (1+q^i).$$

This identity may be easily proved by induction on n. The following lemma is a multinomial extension of it.

Lemma A.2. *For every subset $M = \{m_1, \dots, m_t\}_< \subseteq [0, n-1]$*

$$\sum_{r_1,\dots,r_t} \begin{bmatrix} n \\ \Delta M_r \end{bmatrix}_{q^2} q^{\sum_{i=1}^{t}(r_i - m_i)} = \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} (1+q^j),$$

where $m_{t+1} := n$, the sum on the left-hand side is over all $r_1, \dots, r_t$ such that $m_i \leqslant r_i \leqslant m_{i+1}$ $(\forall i)$, and

$$\begin{bmatrix} n \\ \Delta M_r \end{bmatrix}_{q^2} := \begin{bmatrix} n \\ m_1, r_1 - m_1, m_2 - r_1, \dots, r_t - m_t, m_{t+1} - r_t \end{bmatrix}_{q^2}.$$

Proof. Decomposing the multinomial coefficient,

$$\sum_{r_1,\dots,r_t} \begin{bmatrix} n \\ \Delta M_r \end{bmatrix}_{q^2} q^{\sum_{i=1}^{t}(r_i - m_i)} = \begin{bmatrix} n \\ \Delta M \end{bmatrix}_{q^2} \cdot \prod_{i=1}^{t} \sum_{r_i=m_i}^{m_{i+1}} \begin{bmatrix} m_{i+1} - m_i \\ r_i - m_i \end{bmatrix}_{q^2} q^{r_i - m_i}.$$

By Lemma A.1 this is equal to

$$\begin{bmatrix} n \\ \Delta M \end{bmatrix}_{q^2} \cdot \prod_{i=1}^{t} \prod_{j=1}^{m_{i+1}-m_i} (1+q^j)$$

and, since

$$\begin{aligned} \begin{bmatrix} n \\ \Delta M \end{bmatrix}_{q^2} &= \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=1}^{n} (1+q^j) \cdot \left[\prod_{i=0}^{t} \prod_{j=1}^{m_{i+1}-m_i} (1+q^j) \right]^{-1} \\ &= \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} (1+q^j) \cdot \left[\prod_{i=1}^{t} \prod_{j=1}^{m_{i+1}-m_i} (1+q^j) \right]^{-1}, \end{aligned}$$

we get the desired conclusion. □

The following "q-binomial theorem" is well known.

Theorem A.3.

$$\prod_{i=1}^{n}\bigl(1+q^i x\bigr)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{\binom{k+1}{2}}x^k.$$

A.2. An alternative proof of Theorems 3.3 and 3.4

Proof of Theorem 3.3. Let $m_0 := 0$ and $m_{t+1} := n$. By Lemma 4.1, for each $\sigma \in B_n$ with $\mathrm{Des}_B(\sigma^{-1}) \subseteq M$ there exist $r_1, \dots, r_t$ such that $m_i \leqslant r_i \leqslant m_{i+1}$ $(\forall i)$ and $(\sigma(1), \dots, \sigma(n))$ is a shuffle of the following increasing sequences:

$$\begin{aligned}
&(1, \dots, m_1),\\
&\bigl(-r_1, \dots, -(m_1+1)\bigr),\\
&(r_1+1, \dots, m_2),\\
&\quad\vdots\\
&\bigl(-r_t, \dots, -(m_t+1)\bigr),\\
&(r_t+1, \dots, n).
\end{aligned}$$

By (1),

$$\sum_{\{\sigma\in B_n \mid \mathrm{Des}_B(\sigma^{-1})\subseteq M\}} q^{\ell_B(\sigma)} = \sum_{\{\sigma\in B_n \mid \mathrm{Des}_B(\sigma^{-1})\subseteq M\}} q^{\mathrm{inv}(\sigma)+\sum_{\sigma(i)<0}|\sigma(i)|}.$$

Note that

$$\sigma(i)<0 \quad\Leftrightarrow\quad (\exists j)\ m_j+1 \leqslant \bigl|\sigma(i)\bigr| \leqslant r_j.$$

Therefore

$$\sum_{\sigma(i)<0}\bigl|\sigma(i)\bigr| = \sum_{i=1}^{t}\bigl[(m_i+1)+\dots+r_i\bigr] = \sum_{i=1}^{t}\frac{1}{2}(r_i-m_i)(r_i+m_i+1).$$

This is a constant, once we fix $r_1, \dots, r_t$ (and M). The inversion number of a shuffle does not depend on the actual values of the elements in the shuffled sequences, but only on their order. Therefore, by Observation 2.1 and Fact 2.5,

$$\sum_{\sigma} q^{\mathrm{inv}(\sigma)} = \begin{bmatrix} n\\ m_1, r_1-m_1, m_2-r_1, \dots, r_t-m_t, m_{t+1}-r_t\end{bmatrix}_q,$$

where the sum on the left-hand side is over all $\sigma \in B_n$ with $\mathrm{Des}_B(\sigma^{-1}) \subseteq M = \{m_1, \dots, m_t\}_<$ and prescribed $r_1, \dots, r_t$.

Combining these two formulas, we get

$$\begin{aligned}
&\sum_{\{\sigma\in B_n \mid \mathrm{Des}_B(\sigma^{-1})\subseteq M\}} q^{\ell_B(\sigma)}\\
&\quad= \sum_{r_1,\dots,r_t}\begin{bmatrix} n\\ m_1, r_1-m_1, m_2-r_1, \dots, m_{t+1}-r_t\end{bmatrix}_q q^{\sum_{i=1}^{t}\frac{1}{2}(r_i-m_i)(r_i+m_i+1)}\\
&\quad= \begin{bmatrix} n\\ \Delta M\end{bmatrix}_q \cdot \prod_{i=1}^{t}\sum_{r_i=m_i}^{m_{i+1}}\begin{bmatrix} m_{i+1}-m_i\\ r_i-m_i\end{bmatrix}_q q^{\frac{1}{2}(r_i-m_i)(r_i+m_i+1)}.
\end{aligned}$$

By the q-binomial theorem (Theorem A.3), with $x = q^{m_i}$ and $n = m_{i+1} - m_i$,

$$\prod_{j=m_i+1}^{m_{i+1}} (1+q^j) = \sum_{r_i=m_i}^{m_{i+1}} \begin{bmatrix} m_{i+1}-m_i \\ r_i - m_i \end{bmatrix}_q q^{\binom{r_i-m_i+1}{2}+m_i(r_i-m_i)}$$
$$= \sum_{r_i=m_i}^{m_{i+1}} \begin{bmatrix} m_{i+1}-m_i \\ r_i - m_i \end{bmatrix}_q q^{\frac{1}{2}(r_i-m_i)(r_i+m_i+1)}.$$

Thus

$$\sum_{\{\sigma\in B_n \mid \mathrm{Des}_B(\sigma^{-1})\subseteq M\}} q^{\ell_B(\sigma)} = \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{i=1}^{t} \prod_{j=m_i+1}^{m_{i+1}} (1+q^j) = \begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{j=m_1+1}^{n} (1+q^j).$$

This completes the proof of the second equality in the theorem, computing a generating function for ℓ_B.

An analogous computation holds for fmaj: by Definition 3.1, Lemma 4.1 and Theorem 2.6,

$$\sum_{\{\sigma\in B_n \mid \mathrm{Des}_B(\sigma^{-1})\subseteq M\}} q^{\mathrm{fmaj}(\sigma)}$$
$$= \sum_{\{\sigma\in B_n \mid \mathrm{Des}_B(\sigma^{-1})\subseteq M\}} q^{2\cdot\mathrm{maj}(\sigma)+\mathrm{neg}(\sigma)}$$
$$= \sum_{r_1,\dots,r_t} \begin{bmatrix} n \\ m_1, r_1-m_1, m_2-r_1, \dots, r_t-m_t, m_{t+1}-r_t \end{bmatrix}_{q^2} q^{\sum_{i=1}^t (r_i-m_i)}.$$

By Lemma A.2 this is equal to

$$\begin{bmatrix} n \\ \Delta M \end{bmatrix}_q \cdot \prod_{i=m_1+1}^{n} (1+q^i),$$

as claimed. □

Proof of Theorem 3.4. Apply the Inclusion–Exclusion Principle to Theorem 3.3. □

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