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Existence of Logarithmic-Type Solutions to the Kapila–Kassoy Problem in Dimensions 3 through 9

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1. INTRODUCTION

The ignition model for a high activation energy thermal explosion of a solid fuel in the n-dimensional unit sphere is given by

$$\begin{aligned} &u_t - \Delta u = \delta e^u, && (\xi, t) \in \Omega \times (0, \infty), \quad \delta > 0 \\ &u(\xi, 0) = \psi(\xi), && \xi \in \Omega \\ &u(\xi, t) = 0, && (\xi, t) \in \partial\Omega \times [0, \infty), \end{aligned} \tag{1}$$

where $u(\xi, t)$ is the temperature perturbation of the boundary temperature of Ω and where $\psi(\xi)$ is a radially decreasing function ($\psi(\xi_1) \geqslant \psi(\xi_2) \geqslant 0$ for $|\xi_1| \leqslant |\xi_2| \leqslant 1$) and $\Delta\psi + \delta \exp(\psi) \geqslant 0$ on Ω. This problem has been studied by Kapila [4] and by Kassoy–Poland [5].

Let $\psi(\xi) \equiv 0$. For each $n \geqslant 1$, there is a critical value δ^* such that if $\delta > \delta^*$, then the solution to (1) is singular at a finite time, T. In fact, solutions to (1) are radially symmetric, so $u(\xi, t) = v(r, t)$, where $r = |\xi|$. The equations in (1) can be rewritten as

$$\begin{aligned} &v_t = v_{rr} + \frac{n-1}{r} v_r + \delta e^v, && 0 < r < 1 \\ &v(r, 0) = 0, && 0 \leqslant r \leqslant 1 \\ &v_r(r, 0) = 0, && v(1, t) = 0, \quad 0 \leqslant t \leqslant T, \end{aligned} \tag{2}$$

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where $\delta > \delta^*$. Let $\tau = T - t$, $x = r\tau^{-1/2}$, and $\theta(x, \tau) = v(r, t)$. It is suggested by Kassoy and Poland [5] that the asymptotic representation of θ for each fixed x as $\tau \to 0$ is

$$\theta \sim -\ln(\delta\tau) + y(x) + \sum_{k=1}^{\infty} \tau^k y_k(x). \tag{3}$$

Formally evaluating (2) with the above expression and grouping the appropriate terms leads to the equation for $y(x)$ as

$$y'' + \left(\frac{n-1}{x} - \frac{x}{2}\right) y' + e^y - 1 = 0, \qquad 0 < x < \infty, \tag{4}$$

where $y'(0) = 0$ and matching conditions at the boundary of the hot spot yield the condition $1 + \frac{1}{2}xy'(x) \to 0$ as $x \to \infty$. An integration yields the asymptotic condition $y(x) \sim K - 2 \ln x$ as $x \to \infty$. These boundary conditions are summarized as

$$y'(0) = 0, \qquad \lim_{x \to \infty} [1 + \tfrac{1}{2}xy'(x)] = 0. \tag{5}$$

The nonexistence of solutions to (4), (5) for $n = 1$ is answered by Bebernes and Troy [1]. Although 1, 2, and 3 are the only physically relevant values for n, treating n as a continuous variable, nonexistence of solutions to (4), (5) for $1 \leq n \leq 2$ is answered by Eberly [2]. Thus, the asymptotic relationship (5) is not valid for dimensions 1 and 2.

Consider equation (4) with the initial values

$$y(0) = \alpha \in \mathbb{R}, \qquad y'(0) = 0. \tag{6}$$

Let solutions to initial value problem (4)–(6) be denoted $y(x, \alpha)$. We prove the following:

THEOREM. *For each $n \in (2, 10)$, there is an unbounded sequence of positive numbers $\{\bar{\alpha}_m(n)\}_{m=1}^{\infty}$ such that the solutions $y(x, \bar{\alpha}_m)$ to the initial value problem (4)–(6) satisfy the limit condition in (5).*

2. Preliminary Results

We will make use of a Wronskian argument throughout this paper. The argument is given in

LEMMA 1. *For $x > 0$, let $p(x)$ be a continuously differentiable positive function and let $q(x)$ be a nonnegative continuous function. Let $L(x)$ be the* *solution to* $[p(x)\,L']' + q(x)\,L = 0$, $x > 0$, $L(x_0) = 0$, $L'(x_0) \neq 0$ *for some* $x_0 > 0$.

(i) *Let* $N(x)$ *be a function defined on a right (or left) neighborhood* I *of* x_0 *such that* $N(x_0) = 0$, $N'(x_0) = L'(x_0)$, $N(x) \neq 0$ *for* $x \in I - \{x_0\}$, *and* $[p(x)\,N']' + q(x) N \leqslant 0$ *on* I. *Then there is a right (or left) neighborhood* J *of* x_0 *such that* $L(x) \neq 0$, $N(x) \neq 0$, *and* $N(x) < L(x)$ *on* $J - \{x_0\}$.

(ii) *If* $N(x)$ *satisfies all the conditions in* (i) *except that* $[p(x)\,N']' + q(x)\,N \geqslant 0$ *on* I, *then there is a right (or left) neighborhood* J *of* x_0 *such that* $L(x) \neq 0$, $N(x) \neq 0$, *and* $N(x) > L(x)$ *on* $J - \{x_0\}$.

Proof. The argument for (i) is given; the proof of (ii) is similar. Let $x \geqslant x_0$ and suppose that $N'(x_0) = L'(x_0) > 0$. Then $L(x) > 0$ and $N(x) > 0$ on a right neighborhood J of x_0. Define $w(x) = L(x)\,N'(x) - L'(x)\,N(x)$. Then $w(x_0) = 0$ and $[p(x)\,w]' = L[p(x)\,N']' - q(x)\,LN \leqslant 0$ since $L > 0$. Integrating from x_0 to x we obtain $p(x)\,w(x) \leqslant p(x_0)\,w(x_0) = 0$. Thus, $w(x) \leqslant 0$ on J and $(N/L)'(x) = w(x)/[L(x)]^2 \leqslant 0$ on J. Integrating again from x_0 to x leads to $N(x)/L(x) \leqslant N'(x_0)/L'(x_0) = 1$. Since $L > 0$ on J, $N(x) \leqslant L(x)$ on J. Equality is ruled out on $J - \{x_0\}$ by uniqueness to initial value problems.

If $x \leqslant x_0$, then $L(x) < 0$ and $N(x) < 0$ on a left neighborhood J of x_0. For $w = LN' - L'N$, $w(x_0) = 0$ and $[p(x)\,w]' = L[p(x)\,N']' - q(x)\,LN \geqslant 0$ since $L < 0$. Integrating from x to x_0, we obtain $0 = p(x_0)\,w(x_0) \leqslant p(x)\,w(x)$. Thus, $w(x) \geqslant 0$ on J and $(N/L)'(x) = w(x)/[L(x)]^2 \geqslant 0$ on J. Integrating again from x to x_0 leads to $N(x)/L(x) \geqslant N'(x_0)/L'(x_0) = 1$. Since $L < 0$ on J, $N(x) \leqslant L(x)$ on J. As before, equality is ruled out by uniqueness to initial value problems. A similar argument holds for $L'(x_0) < 0$. ▮

LEMMA 2 (Existence). *For each* $\alpha \in \mathbb{R}$, *the initial value problem* (4)–(6) *has a solution.*

Proof. The case $n = 1$ follows from standard existence results. Let $n > 1$ and make the change of variables $x = rt$, $u(t) = y(x)$. Consider

$$\ddot{u} + \frac{n-1}{t}\,\dot{u} + \lambda\left(e^u - 1 - \frac{1}{2}\,t\dot{u}\right) = 0, \qquad 0 < t < 1 \tag{7}$$

$$\dot{u}(0) = 0, \qquad u(1) = 0 \tag{8}$$

where $\lambda = r^2$. Let $B = C^1[0, 1]$ with the norm $\|u\| = \max_{t \in [0,1]} |u(t)| + \max_{t \in [0,1]} |\dot{u}(t)|$. Rewrite (7), (8) in the form

$$u = \lambda L u + F(\lambda, u), \tag{9}$$

where $Lu(t)=\int_0^1 s^{n-1}G(t,s)[u(s)-\frac{1}{2}s\dot{u}(s)]\,ds$, $F(\lambda,u)(t)=\int_0^1 s^{n-1}G(t,s)\times[e^{u(s)}-u(s)-1]\,ds$, and $G(t,s)$ is a Green's function. The function $L: B\to B$ is a linear compact operator and the function $F: \mathbb{R}\times B\to B$ is a continuous compact operator with $F(\lambda,u)=o(\|u\|)$ as $u\to 0$, uniformly for λ in bounded intervals.

The only eigenvalue of $v=\lambda Lv$ is $\lambda_0=2n$ and the eigenspace is spanned by $v(t)=1-t^2$. Thus, λ_0 is an eigenvalue of odd multiplicity. By the results in Rabinowitz [6], there is a maximal, closed, connected set of solutions, $C(\lambda)$, of (7), (8). Since λ_0 is the only eigenvalue of the linear problem, it must be that $C(\lambda)$ is unbounded in $\mathbb{R}\times B$.

Each pair $(\lambda,u)\in C(\lambda)$ produces a pair $(\lambda,\alpha)\in\mathbb{R}^2$ where $\alpha=u(0)=y(0)$. We claim that the set $P=\{(\lambda,\alpha)\in\mathbb{R}^2\colon(\lambda,u)\in C(\lambda)\}$ is unbounded. For $\alpha>0$, $y(x,\alpha)$ has the property that $y(0)=\alpha$ and $y(\sqrt{\lambda})=0$ with $y(x)>0$ on $[0,\sqrt{\lambda})$. Let $p(x)=x^{n-1}e^{-(1/4)x^2}$ and $q(x)=1$. Let $v(x)=\alpha(1-x^2/2n)$. Then $[p(x)v']'+q(x)v=0$ and $[p(x)y']'+q(x)y\leqslant 0$ for $x\geqslant 0$. By Lemma 1, $y(x)\leqslant v(x)$ while $v(x)\geqslant 0$. Thus, $\lambda<2n$ for all $\alpha>0$.

Since $C(\lambda)$ is unbounded and λ is bounded, either $\alpha=\max\{|u(t)|\colon t\in[0,1]\}$ or $\beta=\max\{|\dot{u}(t)|\colon t\in[0,1]\}$ is unbounded. By an integration of (4), it can be demonstrated that the boundedness of α implies the boundedness of β. Thus, α cannot be bounded and (4)–(6) has a solution for each $\alpha>0$.

For $\alpha<0$, other arguments can be used to show that there are pairs $(\lambda,\alpha)\in P$, but the existence of solutions to (4)–(6) for $\alpha<0$ is not relevant to the development in the remainder of the paper. ∎

LEMMA 3 (Uniqueness). *For each* $\alpha\in\mathbb{R}$, *the initial value problem* (4)–(6) *has a unique solution.*

Proof. We give an outline of the proof. Suppose that $y_1(x)$ and $y_2(x)$ are two solutions to (4)–(6) for a given α. Define $\Delta(x)=y_1(x)-y_2(x)$. Then Δ satisfies

$$\Delta''+\left(\frac{n-1}{x}-\frac{x}{2}\right)\Delta'+\left(\frac{e^{y_1}-e^{y_2}}{y_1-y_2}\right)\Delta=0,\qquad 0<x<\infty \tag{10}$$

$$\Delta(0)=0,\qquad \Delta'=0. \tag{11}$$

Consider the equation $L''+[(n-1)/x-x/2]L'+e^{\alpha}L=0$ for $x>0$, $L(x_0)=0$, and $L'(x_0)\neq 0$, for x sufficiently small. It can be shown that the solution $L(x)\neq 0$ on $(0,x_0)$.

For $\alpha>0$, there is a δ sufficiently small such that $y_i(x)>0$ and $y_i'(x)<0$ on $(0,\delta)$, $i=1,2$. Consequently, $(e^{y_1}-e^{y_2})/(y_1-y_2)\leqslant e^{\alpha}$ on $(0,\delta)$. If $\Delta(x_0)=0$ and $\Delta'(x_0)<0$ (otherwise rename y_1 and y_2) for some $x_0\in(0,\delta)$, then while $\Delta>0$ on a left neighborhood J of x_0, $[p(x)\Delta']'+p(x)e^{\alpha}\Delta\geqslant 0$, where $p(x) = x^{n-1} e^{-(1/4)x^2}$. By Lemma 1, $\Delta(x) > L(x) > 0$ on J. Consequently, $J = (0, x_0)$ and $\Delta > 0$ on J.

Equation (10) implies that Δ cannot have a local minimum on $(0, x_0)$. Since $\Delta(0) = 0$ and $\Delta > 0$ on $(0, x_0)$, it must be that $\Delta' > 0$ on $(0, \sigma)$ for some $\sigma \in (0, x_0)$. But then (10) implies that $\Delta'' \leqslant 0$ on $(0, \sigma)$ and so $\Delta'(x) \leqslant \Delta'(0) = 0$ on $(0, \sigma)$. This is a contradiction, so $\Delta(x) \equiv 0$. Similar arguments work for $\alpha \leqslant 0$. ▮

LEMMA 4 (Continuous dependence). *Let $y(x, \alpha)$ be the unique solution to the initial value problem* (4)–(6). *Then $y(x, \alpha)$ and $y'(x, \alpha)$ are continuous on compact subsets of* $[0, \infty) \times \mathbb{R}$.

Proof. The results on existence and uniqueness combined with the fact that $C(\lambda)$ is closed and connected immediately imply the continuous dependence of $y(x, \alpha)$ and $y'(x, \alpha)$ on compact subsets of their domain. ▮

3. The Main Results for $2 < n < 10$

From the results in [2], if $\alpha < 0$, then the solution $y(x, \alpha)$ to the initial value problem (4)–(6) has the property that $|y'(x, \alpha)| \to \infty$ as $x \to \infty$. Such a solution cannot satisfy the boundary conditions (5). It is sufficient to consider only the values $\alpha > 0$.

Equation (4) has a singular solution $S(x) = \ln[2(n-2)/x^2]$. Define $h(x, \alpha) = y(x, \alpha) - S(x)$. Then h satisfies the differential equation

$$h'' + \left(\frac{n-1}{x} - \frac{x}{2}\right) h' + \frac{2(n-2)}{x^2}(e^h - 1) = 0, \qquad 0 < x < \infty. \tag{12}$$

For x sufficiently close to 0, a linearized version of Eq. (12) is $L'' + [(n-1)/x]\, L' + [2(n-2)/x^2]\, L = 0$. For each $n \in (2, 10)$, this equation has solutions which have zeros that accumulate at $x = 0$. We use this idea to show that there is an unbounded increasing sequence $\{\alpha_k\}_1^\infty$ such that the number of zeros of $h(x)$ on $(0, \sqrt{2(n-2)})$ increases as $\alpha_k \to \infty$. More precisely, we show that the sets $Z_m = \{\alpha \in [0, \infty) : h(x, \alpha)$ has at least $2m + 1$ zeros on $(0, \infty)\}$, $m = 1, 2, \ldots$, are nonempty and bounded below (by $\alpha = 1$). The values $\bar{\alpha}_m = \inf Z_m$ provide solutions $y(x, \bar{\alpha}_m)$ to (4)–(6) which satisfy condition (5).

Define $g(x) = \frac{1}{2} x y'(x) + 1$ where y is any solution to (4). Then g satisfies the equation

$$g'' + \left(\frac{n-1}{x} - \frac{x}{2}\right) g' + (e^y - 1)\, g = 0, \qquad 0 < x < \infty \tag{13}$$

LEMMA 5. *Let $0<\alpha<1$ and let $y(x,\alpha)$ be the solution to* (4)–(6). *Then $g(x)$ cannot have a zero before $y(x)$ does.*

Proof. Let $w=gy'-g'y$. Then (4) and (13) imply that $[x^{n-1}e^{-(1/4)x^2}w]'=x^{n-1}e^{-(1/4)x^2}g(y-1)(e^y-1)$, $w(0)=0$. For $0<\alpha<1$, $y(x)\leqq\alpha<1$ while $y>0$, $y'<0$. So $[x^{n-1}e^{-(1/4)x^2}w]'\leqq 0$ while $y>0$ and $g>0$. An integration and the standard Wronskian argument yields $y(x)\leqq\alpha g(x)$ while $y>0$ and $g>0$. Thus, while $y>0$, $g(x)$ cannot have a zero.

LEMMA 6. *The function $h(x,\alpha)$ has at most two zeros on the interval $(\sqrt{2(n-2)},\infty)$. Moreover, if $h(\bar{x})=0$, $h'(\bar{x})<0$ at the first zero $\bar{x}>\sqrt{2(n-2)}$, then h has exactly one zero for $x>\sqrt{2(n-2)}$. As a consequence, if $0<\alpha<1$, then $h(x)$ cannot have more than two zeros on $(0,\infty)$.*

Proof. Define $w=Sy'-S'y$ where S is the singular solution given earlier. Then w satisfies the equation $[x^{n-1}e^{-(1/4)x^2}w]'=x^{n-1}e^{-(1/4)x^2}yS[F(S)-F(y)]$, where $F(u)=(e^u-1)/u$. Let r_1 be the first zero for $y(x)$.

Suppose $r_1>\sqrt{2(n-2)}$. If $y(\bar{x})=S(\bar{x})$ at some first $\bar{x}>r_1$, then $y'(\bar{x})<S'(\bar{x})$. While $0>S(x)>y(x)$, $[x^{n-1}e^{-(1/4)x^2}w]'\geqq 0$, $w(\bar{x})>0$. By integrating, we have $x^{n-1}e^{-(1/4)x^2}w(x)\geqq(\bar{x})^{n-1}e^{-(1/4)\bar{x}^2}w(\bar{x})=p>0$, and so $w(x)\geqq px^{1-n}e^{(1/4)x^2}$. This implies that $(y/S)'(x)\geqq px^{1-n}e^{(1/4)x^2}/[S(x)]^2$ and $(y/S)(x)\geqq p\int_{\bar{x}}^{x}t^{1-n}e^{(1/4)t^2}/S^2(t)\,dt+1\geqslant 1$. So $y(x)<S(x)$ for $x>\bar{x}$ and y has at most one point of intersection with S for $x>\sqrt{2(n-1)}$.

Suppose $r_1<\sqrt{2(n-2)}$. If $y(\bar{x})=S(\bar{x})$ at some first $\bar{x}>r_1$, then $y(\bar{x})<0$ and $y'(\bar{x})>S'(\bar{x})$. Thus, $y>S$ to the immediate right of $\bar{x}$. If $y=S$ at some first $\hat{x}>\bar{x}$, then $y(\hat{x})<0$ and $y'(\hat{x})<S'(\hat{x})$. A repetition of the previous argument shows that $y<S$ for $x>\hat{x}$.

Suppose that $r_1=\sqrt{2(n-2)}$. Then the arguments used for $r_1>\sqrt{2(n-2)}$ or $r_1<\sqrt{2(n-2)}$ are valid depending on whether $y'>S'$ or $y'<S'$, respectively, at $\sqrt{2(n-2)}$.

If $0<\alpha<1$, and if $h(x_i)=0$ for two numbers $x_1,x_2<\sqrt{2(n-2)}$, then there is a number $\bar{x}$ between x_1 and x_2 where $0=h'(\bar{x})=(2/\bar{x})\,g(\bar{x})$. This forces g to have a zero before y does, a contradiction to lemma 5. So for this range of α, h can have at most one zero before $\sqrt{2(n-2)}$. By the earlier work in this lemma, one can see that h has at most two zeros on $(0,\infty)$. ∎

These last two lemmas show that the set Z_m, $m\geqq 1$, is bounded below by $\alpha=1$. We need to show that each of these sets is nonempty.

LEMMA 7. *Let $n\in(2,10)$. Let $\sigma\in(0,1)$ be any number such that $n<8+2\sigma$. If $u(x)\not\equiv 0$ is a solution to the differential equation*

$$u'' + \left(\frac{n-1}{x} + \frac{x}{2}\right) u' + \frac{2\sigma(n-2)}{x^2} u = 0, \qquad 0 < x < \infty \tag{14}$$

then there is a decreasing sequence of zeros of u, say $\{r_k\}_1^\infty$ *such that* $r_k \to 0$ *as* $k \to \infty$. *Moreover, if* $\theta = \exp[2\pi/\sqrt{(n-2)(8+2\sigma-n)}]$ *and if* $\phi = \exp[\{2\tan^{-1}(\sqrt{8+2\sigma-n}/\sqrt{n-2})\}/\sqrt{(n-2)(8+2\sigma-n)}]$, *then* $1/\theta < r_{k+1}/r_k < 1/\phi$.

Proof. Let $z(x) = x^{(1/2)(n-2)} e^{-(1/8)x^2} u(x)$. Then z is a solution to the equation $z'' + [\{8\sigma(n-2) - (n-1)(n-3)\}/4x^2 + (4n - x^2)/16]\, z = 0$. Let $r_{k+1} > 0$ be such that $u(r_{k+1}) = 0$ and $u'(r_{k+1}) > 0$. Let $v(x) = A\sqrt{x}\sin[(\pi/\ln\theta)\ln(x/r_{k+1})]$ where A is chosen so that $v'(r_{k+1}) = u'(r_{k+1})$. Then $v(x)$ satisfies $v'' + [\{8\sigma(n-2) - (n-1)(n-3)\}/4x^2]\, v = 0$ and $v(\theta r_{k+1}) = 0$, $v(x) > 0$ on the interval $(r_{k+1}, \theta r_{k+1})$. On a right neighborhood of r_{k+1}, z satisfies $z'' + [\{8\sigma(n-2) - (n-1)(n-3)\}/4x^2]\, z \leqq 0$. By Lemma 2, $z(x) \leqq v(x)$ on this neighborhood. Thus, $z(x)$ must have another zero $r_k \in (r_{k+1}, \theta r_{k+1})$. Similar arguments show that $z(x)$ has a zero $r_{k+2} \in (r_{k+1}/\theta, r_{k+1})$. Repetition of the argument shows the existence of a sequence of zeros converging to zero.

Let $v(x) = Ax^{-(1/2)(n-2)}\sin[(\pi/\ln\theta)\ln(x/r_{k+1})]$ where A is chosen so that $v'(r_{k+1}) = u'(r_{k+1})$. Then $v'' + [(n-1)/x]\, v' + [2\sigma(n-2)/x^2]\, v = 0$ and $v'(\phi r_{k+1}) = 0$, $v'(x) > 0$ on $(r_{k+1}, \phi r_{k+1})$. While $u' > 0$, (14) implies that $u'' + [(n-1)/x]\, u' + [2\sigma(n-2)/x^2]\, u \geqq 0$ and Lemma 2 implies that $u(x) \geqq v(x)$. Also, $(u'/u)(x) \geqq (v'/v)(x)$ and so v' must become zero before u' does. Thus, the second zero of u occurs after ϕr_{k+1} and we have $r_k \in (\phi r_{k+1}, \theta r_{k+1})$.

LEMMA 8. *There is an unbounded increasing sequence of values* $\{\alpha_k\}_1^\infty$ *such that* $h(x, \alpha_k)$ *has a first zero* $x_1(k)$ *and* $x_1(k) \to 0$ *as* $k \to \infty$.

Proof. Let $x_1 \in (0, \sqrt{2(n-2)})$. Let $I = [-2/x_1, 0]$ and consider (4) with $y(x_1) = S(x_1)$, $y'(x_1) = \beta \in I$. Denote such solutions as $Y(x, \beta)$. If $Y'(x_1) = 0$, then Y has a local maximum at x_1. Suppose that $Y(x) > 0$ on $(0, x_1)$. Then $[x^{n-1}e^{-(1/4)x^2} Y'(x)]' = -x^{n-1}e^{-(1/4)x^2}(e^Y - 1) \leqq 0$, and $0 < p = T^{n-1}e^{-(1/4)T^2} Y'(T) \leqq x^{n-1}e^{-(1/4)x^2} Y'(x) \leqq x^{n-1} Y'(x)$ for $0 < x \leqq T < x_1$. Thus, $Y'(x) \geqq px^{1-n}$. An integration leads to $Y(T) + (p/(n-2))\, T^{2-n} \geqq Y(x) + (p/(n-2))\, x^{2-n} > (p/(n-2))\, x^{2-n}$ for $0 < x \leqq T$. As $x \to 0$, the right-hand side of the inequality tends to ∞ while the left-hand side is constant. This is a contradiction, so there must be a number $r > 0$ such that $Y(r) = 0$ and $Y'(r) > 0$. Let $u(x)$ be the solution to $u'' + [(n-1)/x - x/2]\, u' + u = 0$, $u(r) = 0$, $u'(r) = Y'(r)$. Then $u(x) < 0$ on $(0, r)$ and $u(x) \to -\infty$ as $x \to 0$. Also, $Y'' + [(n-1)/x - x/2]\, Y' + Y = -(e^Y - Y - 1) \leqq 0$. By Lemma 1, $Y(x) \leqq u(x)$ on $(0, r)$. Also, $Y'(x) > 0$ on this interval.

By continuous dependence, there is an interval $I_0 = (\beta_0, 0]$ such that the solutions $Y(x, \beta)$ have a local maximum for some $x_0(\beta) \in (0, x_1)$ and such that $Y(x, \beta) < S(x)$ for $x \in (0, x_1)$. The last property is true since $|Y'(x, 0) - S'(x)| \geqq \delta > 0$ for $x \in [\eta, x_1]$ ($\eta > 0$ and small), so by continuous dependence, $|Y'(x, \beta) - S'(x)| \geqq \frac{1}{2}\delta$ for β close to 0.

Since $Y(x, -2/x_1) = S(x)$, I_0 is bounded below. Define $\beta_0 = \inf I_0$. In fact, $\beta_0 = (-2+\varepsilon)/x_1$ for some $\varepsilon > 0$ since for β close to $-2/x_1$, the function $\Delta(x) = Y(x) - S(x)$ must have a zero $x_2 < x_1$. That is, on $[\eta, x_1]$ ($\eta > 0$ and small), $Y(x, \beta) \to S(x)$ as $\beta \to -2/x_1$. Thus, $(e^{\Delta} - 1)/\Delta \geqq \sigma$ on $[\eta, x_1]$ for $\sigma \in (0, 1)$ such that $8 + 2\sigma > n$ (and for β close to $-2/x_1$). But Δ is a solution to (12), so $0 \leqq \Delta'' + [(n-1)/x - x/2]\,\Delta' + [2\sigma(n-2)/x^2]\,\Delta$. Let $u(x)$ be the solution to (14) with $u(x_1) = 0$ and $u'(x_1) = \Delta'(x_1)$. By Lemma 7, $u(x)$ has a zero $\bar{x} < x_1$. By Lemma 1, $\Delta(x) \geqq u(x)$ on $(\bar{x}, x_1)$, so $\Delta(x)$ has a zero $x_2 < x_1$. By definition of I_0, it must be that β_0 is bounded away from $-2/x_1$.

If $\beta \in I_0$, then $Y'(x, \beta) > -2/x$ on $(0, x_1)$, or else there is a number $x_2 < x_1$ such that $\Delta'(x_2) = Y'(x_2) - S'(x_2) = 0$. Since Δ is a solution to (12), this would force Δ to have another zero $\bar{x} < x_2 < x_1$, contrary to the definition of the set I_0.

Let $g(x, \beta) = \frac{1}{2}xY'(x, \beta) + 1$. Then g satisfies Eq. (13) and for $\beta \in I_0$, $g(x, \beta) > 0$ on $(0, x_1)$. While $g(x) < 1$, if $g'(\bar{x}) = 0$ for some $\bar{x} < x_1$, then (13) implies that g has a local maximum at $\bar{x}$. Before g can have a local minimum on a left neighborhood of $\bar{x}$, g must become 0 first. This cannot happen for $\beta \in I_0$. Thus, $g'(x) < 0$ while $g(x) < 1$ on a left neighborhood of x_1. At $x_0(\beta)$, $Y'(x_0) = 0$ implies that $g(x_0) = 1$. From our earlier arguments, for $x \in (0, x_0)$, $Y'(x) > 0$ and so $g(x) > 1$.

Consequently, $xY'(x, \beta) \geqq x_1 Y'(x_1, \beta) = x_1\beta > x_1\beta_0 > -2+\varepsilon$ for all $x \in (0, x_1)$, $\beta \in I_0$. Integrating from x to x_1, we have $Y(x, \beta) \leqq [Y(x_1, \beta) + (2-\varepsilon)\ln x_1] - (2-\varepsilon)\ln x$ for $0 < x \leqq x_1$, $\beta \in I_0$. By continuous dependence at x_1, $Y(x_1, \beta)$ is bounded on compact subsets of β. Thus,

$$Y(x, \beta) \leqq M - (2-\varepsilon)\ln x, \tag{15}$$

where both M and ε depend only on I_0 and where $x \in (0, x_1]$. Integrating Eq. (4) from x_0 to x (where $Y'(x_0) = 0$) yields $Y'(x) = -x^{1-n}e^{-(1/4)x^2} \int_{x_0}^{x} s^{n-1}e^{-(1/4)s^2}[e^{Y(s)} - 1]\,ds$ and then integrating from x_0 to x_1, we have

$$\begin{aligned} Y(x_0, \beta) &= Y(x_1, \beta) + \int_{x_0}^{x_1} t^{1-n}e^{-(1/4)t^2} \int_{x_0}^{t} s^{n-1}e^{-(1/4)s^2}[e^{Y(s)} - 1]\,ds\,dt \\ &\leqq K_1 + \int_{x_0}^{x_1} t^{1-n}e^{-(1/4)t^2} \int_{x_0}^{t} s^{n-1}e^{-(1/4)s^2}e^{M-(2-\varepsilon)\ln s}\,ds\,dt \end{aligned}$$

using Eq. (15) and where $K_1 = M + (2-\varepsilon)\ln x_1$. Thus,

$$\begin{aligned} Y(x_0, \beta) &\leqq K_1 + K_2 \int_{x_0}^{x_1} t^{1-n} \int_{x_0}^{t} s^{n-3+\varepsilon}\, ds\, dt \\ &\leqq K_1 + K_3 \int_{x_0}^{x_1} t^{-1+\varepsilon}\, dt \\ &= K_1 + \frac{1}{\varepsilon} K_3 (x_1^\varepsilon - x_0^\varepsilon) \\ &\leqq K_4, \end{aligned}$$

where K_i are independent of β, $i = 1, 2, 3, 4$.

Therefore, for $\beta \in I_0$, the local maximum values $Y(x_0, \beta)$ are bounded above. We use this to prove that $Y'(x, \beta_0) < 0$ for a right neighborhood of $x = 0$. First, $Y(x, \beta_0) < S(x)$ on $(0, x_1)$. For if there were a value x_2 such that $Y(x_2, \beta_0) = S(x_2)$, then by continuous dependence, for $\beta \in I_0$ close to β_0, there would have to be a number $x_2(\beta)$ such that $Y(x_2, \beta) = S(x_2)$, a contradiction to the definition of I_0. Second, if $Y'(x_0, \beta_0) = 0$ for some $x_0 > 0$, then on $[\eta, x_1]$ (fixed $\eta < x_0$), $|Y'(x, \beta_0) - S'(x)| \geqq \delta > 0$ (or else there is a value $x_2 < x_1$ such that $Y'(x_2) = S'(x_2)$ and, as before, there would be a number $\bar{x} < x_2$ such that $Y(\bar{x}) = S(\bar{x})$, a contradiction to the definition of β_0). By continuous dependence, $|Y'(x, \beta) - S(x)| \geqq \frac{1}{2}\delta$ for $\beta < \beta_0$ (but close) and there is an $x_0(\beta) > 0$ such that $Y'(x_0, \beta) = 0$ and $Y(x, \beta) < S(x)$ on $(0, x_1)$. This contradicts the definition of β_0. Thus, $Y'(x, \beta_0) < 0$ on $(0, x_1)$ and $Y(x_0, \beta) \leqq K_4$ for $\beta \in I_0$ imply that $Y(0, \beta_0)$ is finite.

If $xY'(x, \beta_0) \leqq -k < 0$ on $(0, x_1)$, then for x small, an integration from x to x_0 yields $Y(x, \beta_0) \geqq Y(x_1, \beta_0) + k \ln x$ which implies $Y(0)$ is not finite, a contradiction. Thus, $\lim_{x \to 0} xY'(x, \beta_0) = 0$. (The limit exists since $g(x, \beta_0) < 1$—cf. the earlier work in the lemma.) Integrating Eq. (4) yields $Y'(x, \beta_0) = -x^{1-n} e^{(1/4)x^2} \int_0^x t^{n-1} e^{-(1/4)t^2} [e^Y - 1]\, dt$. Applying L'Hopital's rule, we have $\lim_{x \to 0} Y'(x, \beta_0) = 0$. Say $Y(0, \beta_0(x_1)) = \alpha(x_1)$.

Thus, for each $x_1 \in (0, \sqrt{2(n-2)})$, there is an $\alpha(x_1) > S(x_1)$ such that $y(x, \alpha)$ is a solution to (4)–(6) and $y(x, \alpha) < S(x)$ on $(0, x_1)$, $y(x_1) = S(x_1)$. By continuous dependence, the function $x_1(\alpha)$ is continuous (but not necessarily one-to-one). Since x_1 can be picked arbitrarily close to 0, there is an unbounded sequence $\{\alpha_k\}_1^\infty$ such that $x_1(\alpha_k) =: x_1(k) \downarrow 0$ as $k \to \infty$.

LEMMA 9. *Let $\{\alpha_k\}_1^\infty$ be the sequence constructed in Lemma* 8. *Let $\theta = \exp(2\pi/\sqrt{(n-2)(10-n)})$. If k is sufficiently large, then $h(x, \alpha_k)$ has a second zero $x_2(k) \in (x_1, \theta x_1)$.*

Proof. The function h satisfies Eq. (12) where $h(x_1)=0$, $h'(x_1)>0$. Note that $h''+[(n-1)/x-x/2]\,h'+[2(n-2)/x^2]\,h=-[2(n-2)/x^2]$ $(e^h-h-1)\leqq 0$. Let $u(x)$ be the solution to (14) with $u(x_1)=0$ and $u'(x_1)=h'(x_1)$. By Lemma 7, u has another zero $\bar{x}\in(x_1, \theta x_1)$ as long as x_1 is sufficiently close to 0. But by Lemma 1, $h(x)\leqq u(x)$ on $(x_1, \bar{x})$, so h must also have a second zero $x_2(k)\in(x_1, \theta x_1)$, for k sufficiently large.

LEMMA 10. *Let $x_1(k)$ and $x_2(k)$ be the first two zeros for $h(x, \alpha_k)$ where k is sufficoently large to guarantee their existence. Then $x_i^{n-1}h'(x_i)\to 0$ as $k\to\infty$.*

Proof. At x_1, $0>y'(x_1)>-2/x_1$, so $x_1^{n-1}y'(x_1)\in(-2x_1^{n-2}, 0)$. Consequently, $x_1^{n-1}y'(x_1)\to 0$ as $k\to\infty$ (since $n>2$). But $x_1^{n-1}h'(x_1)=x_1^{n-1}y'(x_1)+2x_1^{n-2}$, so $x_1^{n-1}h'(x_1)\to 0$ as $k\to\infty$.

Integrating Eq. (4) from x_1 to x_2 yields the relationship

$$\begin{aligned}
0>x_2^{n-1}e^{-(1/4)x_2^2}y'(x_2)&=x_1^{n-1}e^{-(1/4)x_1^2}y'(x_1)-\int_{x_1}^{x_2}t^{n-1}e^{-(1/4)t^2}\left[e^{y(t)}-1\right]dt\\
&\geqq x_1^{n-1}y'(x_1)-\frac{1}{n}e^{y(x_1)}(x_2^n-x_1^n)\\
&\geqq x_1^{n-1}y'(x_1)-\frac{2(n-2)}{n}x_2^n/x_1^2\\
&\geqq x_1^{n-1}y'(x_1)-\frac{2(n-2)}{n}\theta^n x_1^{n-2}.
\end{aligned}$$

We have used the fact that $x_2\leqq\theta x_1$ from Lemma 9. The right-hand side of the inequality tends to 0 as $k\to\infty$, so $x_2^{n-1}y'(x_2)\to 0$ as $k\to\infty$. As before, $x_2^{n-1}h'(x_2)=x_2^{n-1}y'(x_2)+2x_2^{n-2}\to 0$ as $k\to\infty$.

LEMMA 11. *For k sufficiently large, there is a third zero $x_3(k)$ for $h(x, \alpha_k)$.*

Proof. It is sufficient to show that there is a number $q(k)>x_2(k)$ such that $h'(q)=0$. For if $h'(q)=0$ and $h(q)<0$, then $h''(q)>0$ and $h'(x)>0$ in a right neighborhood of q. The function h cannot have a local maximum while $h<0$, so either h has a zero $x_3\leqq 2$, or $h<0$ and $h'>0$ for $x>\sqrt{2(n-1)}$. In this last case, it must be that $h''>0$ by Eq. (12), so h must have a third zero $x_3>\sqrt{2(n-1)}$.

Let $\sigma\in(0,1)$ be such that $n<2+8\sigma$. Let $u(x)$ be the solution to (14) with $u(x_2)=0$, $u'(x_2)=h'(x_2)<0$. Then Lemma 7 states that u has another zero $\bar{x}\in(x_2, \theta x_2)$ for x_2 sufficiently small. If $(e^h-1)/h\geqq\sigma$ on $[x_2, \theta x_2]$, then (12) implies that $h''+[(n-1)/x-x/2]\,h'+[2\sigma(n-2)/x^2]\,h\geqq 0$ while

$h \leqq 0$. Lemma 1 implies that $h(x) \geqq u(x)$ on $[x_2, \theta x_2]$ and so h must have a third zero in $(x_2, \theta x_2]$.

Otherwise, there is a number $\bar{x} \in (x_2, \theta x_2)$ such that $[e^{h(\bar{x})} - 1]/h(\bar{x}) = \sigma$ and $h'(x) < 0$ on a right neighborhood of $\bar{x}$. Let $\delta > 0$ be fixed. For k large, $\theta x_2 < \delta$. Integrating from x_2 to 2δ, we obtain

$$\begin{aligned}(2\delta)^{n-1} e^{-\delta^2} h'(2\delta) &= x_2^{n-1} e^{-(1/4)x_2^2} h'(x_2) \\ &\quad + \int_{x_2}^{2\delta} 2(n-2)\, t^{n-3} e^{-(1/4)t^2} [1 - e^{h(t)}]\, dt \\ &\geqq x_2^{n-1} h'(x_2) + \int_{\delta}^{2\delta} 2(n-2)\, t^{n-3} e^{-(1/4)t^2} [1 - e^{h(t)}]\, dt \\ &\geqq x_2^{n-1} h'(x_2) + 2e^{-\delta^2} [(2\delta)^{n-2} - \delta^{n-2}][1 - e^{h(\bar{x})}].\end{aligned}$$

As $k \to \infty$, by Lemma 10, we have $x_2^{n-1} h'(x_2) \to 0$. Since $2e^{-\delta^2}[(2\delta)^{n-2} - \delta^{n-2}][1 - e^{h(\bar{x})}]$ is positive and independent of α_k, it must be that for k large, $(2\delta)^{n-1} e^{-\delta^2} h'(2\delta) > 0$. That is, it cannot be the case that $h' < 0$ for all $x > x_2$ where k is large. Thus, there exists a number $q(k) > x_2(k)$ such that $h'(q) = 0$ and the lemma is proved. It follows immediately from this argument that $q(k) \to 0$ as $k \to \infty$ since in the first case, $q \in (x_2, \theta x_2)$, and in the second case, δ can be chosen arbitrarily small.

LEMMA 12. *Let k be sufficiently large so that the third zero of $h(x, \alpha_k)$, $x_3(k)$, exists. Then $x_3(k) \to 0$ as $k \to \infty$.*

Proof. Let x_1 and x_2 be the first two zeros of h. Let $\hat{x} \in (x_1, x_2)$ be the unique value where $h'(\hat{x}) = 0$ on that interval. On $[x_1, \hat{x}]$, $h''(x) < 0$, so $h(x) \leqq h'(x_1)(x - x_1) \leqq h'(x_1)(x_2 - x_1) \leqq (\theta - 1)\, x_1 h'(x_1)$. Since $y'(x_1) < 0$, $x_1 h'(x_1) = x_1 y'(x_1) + 2 < 2$. Thus, $h(x) \leqq h(\hat{x}) \leqq 2(\theta - 1) =: \kappa_1$ on $[x_1, x_2]$.

Define $f(x) = h'(x) + \kappa_2/x$ where $\kappa_2 = 3(e^{\kappa_1} - 1)$. Then $f(x_1) = h'(x_1) + \kappa_2/x_1 > 0$. Suppose there is a first $\bar{x} \in (x_1, x_2]$ such that $f(\bar{x}) = 0$. Then $f'(\bar{x}) \leqq 0$. However, $f'(\bar{x}) = h''(\bar{x}) - \kappa_2/\bar{x}^2 = -\frac{1}{2}\kappa_2 + ((n-2)/\bar{x}^2)[\kappa_2 + 2(1 - e^{h(\bar{x})}] \geqq -\frac{1}{2}\kappa_2 + (n-2)(e^{\kappa_1} - 1)/\bar{x}^2$ since $h(\bar{x}) \leqq \kappa_1$. For k large, since $\bar{x} \leqq x_2$ and $x_2 \to 0$ as $k \to \infty$, $-\frac{1}{2}\kappa_2 + (n-2)(e^{\kappa_1} - 1)/\bar{x}^2 > 0$. This contradicts $f'(\bar{x}) \leqq 0$. So for k large, $f(x) > 0$ on $[x_1, x_2]$. In particular, $x_2 h'(x_2) \geqq -\kappa_2$.

For $x \geqq x_2$, we have $x^{n-1} e^{-(1/4)x^2} h'(x) \geqq x_2^{n-1} e^{-(1/4)x_2^2} h'(x_2)$ while $h < 0$ since $[x^{n-1} e^{-(1/4)x^2} h']' = -[2(n-2)/x^2][e^h - 1] \geqq 0$. Thus, while $h < 0$ on $[x_2, \sqrt{2(n-2)}]$, $x^{n-1} h'(x) \geqq -\kappa_2 e^{(1/2)(n-2)} x_2^{n-2} =: -\kappa_3 x_2^{n-2}$. Integrating from x_2 to x, we have $h(x) \geqq [-\kappa_3/(n-2)][1 - x_2^{n-2} x^{2-n}] \geqq -\kappa_3/(n-2) =: -\kappa_4$. In particular, if $q \in (x_2, x_3)$ is the value where $h'(q) = 0$, then $h(x) \geqq -\kappa_4$ where κ_4 is independent of α_k.

Suppose there is a $\delta > 0$ such that $x_3(k) \geqq \delta$ for all large k. Let $\sigma \in (0, 1)$ be such that $n < 2 + 8\sigma$. If $(e^h - 1)/h \geqq \sigma$ on $[\delta/\theta, \delta]$ where θ is the number constructed in Lemma 7, then by Lemma 7, h would have to have another zero in $[\delta/\theta, \delta]$, contradicting $h < 0$ on (x_2, x_3). So h must be bounded away from zero on $[q, \delta/\theta]$ for all large k, say $(1 - e^h)\, 2e^{-(1/4)(\delta/\theta)^2} \geqq \kappa_5 > 0$. Then $[x^{n-1}e^{-(1/4)x^2}h']' = 2(n-2)\, x^{n-3}e^{-(1/4)x^2}(1 - e^h) \geqq (n-2)\, \kappa_5 x^{n-3}$. Integrating from q to x yields $x^{n-1}h'(x) \geqq x^{n-1}e^{-(1/4)x^2}h'(x) \geqq \kappa_5[x^{n-2} - q^{n-2}]$. Integrating once more from q to x yields $h(x) \geqq h(q) + \kappa_5 \ln(x/q) - [\kappa_5/(n-2)]\, q^{n-2}[q^{2-n} - x^{2-n}]$. In particular, $h(\delta/\theta) \geqq c_1 + c_2 q^{n-2} - c_3 \ln q$ where c_i are independent of α_k, $i = 1, 2, 3$. The right-hand side of this inequality tends to ∞ as $q \to 0$ $(k \to \infty)$, a contradiction to $h(\delta/\theta) < 0$. Thus, $x_3(k)$ cannot be bounded away from 0 for all k.

LEMMA 13. *Let k be sufficiently large so that $h(x, \alpha_k)$ has at least $2m - 1$ zeros on $(0, \sqrt{2(n-2)})$. Then there is a k_0 such that for all $k > k_0$, $h(x, \alpha_k)$ has at least $2m + 1$ zeros on $(0, \sqrt{2(n-2)})$.*

Proof. Since $x_1(k)$ and $x_2(k)$ tend to 0 as $k \to \infty$, there is a value $\bar{k}$ such that for $k > \bar{k}$, $h(x, \alpha_k)$ has at least one zero $(m = 1)$. By Lemmas 11 and 12, for k sufficiently large, $x_3(k)$ exists and tends to 0 as $k \to \infty$. As in Lemma 9, for k large, a linear comparison shows the existence of a fourth zero $x_4(k)$ for $h(x)$. Thus, there is a k_0 such that for all $k > k_0$, $h(x, \alpha_k)$ has at least 3 zeros on $(0, \sqrt{2(n-2)})$. So the result is true for $m = 1$.

The inductive step depends on Lemma 10 holding for $x_i(k)$ in general. That is, we need to show that $x_i^2 h'(x_i) \to 0$ as $k \to \infty$ for $i = 2m - 1, 2m$. As in Lemma 10, it is sufficient to show that $x_i^2 y'(x_i) \to 0$ as $k \to \infty$. At x_{2m-1}, $0 > y'(x_{2m-1}) > -2/x_{2m-1}$. Consequently, $(x_{2m-1})^2\, y'(x_{2m-1}) > -2x_{2m-1}$ and so $(x_{2m-1})^2\, y'(x_{2m-1}) \to 0$ as $k \to \infty$ since $x_{2m-1} \to 0$.

Integrating Eq. (4) from x_{2m-1} to x_{2m} yields

$$0 > x_{2m}^2 \exp\left(-\frac{1}{4}x_{2m}^2\right) y'(x_{2m})$$

$$= x_{2m-1}^2 \exp\left(-\frac{1}{4}x_{2m-1}^2\right) y'(x_{2m-1}) - \int_{x_{2m-1}}^{x_{2m}} t^{n-1}e^{-(1/4)t^2}[e^y - 1]\, dt$$

$$\geqq x_{2m-1}^2\, y'(x_{2m-1}) - \frac{1}{n} e^{y(x_{2m-1})}[x_{2m}^n - x_{2m-1}^n]$$

$$= x_{2m-1}^2\, y'(x_{2m-1}) - \frac{2(n-2)}{n}(x_{2m}^n/x_{2m-1}^2) + \frac{2(n-2)}{n} x_{2m-1}^{n-2}$$

$$\geqq x_{2m-1}^2\, y'(x_{2m-1}) - \frac{2(n-2)}{n}\theta^n x_{2m-1}^{n-2} + \frac{2(n-2)}{n} x_{2m-1}^{n-2}$$

using the inequality $x_{2m} \leqq \theta x_{2m-1}$ which holds whenever the even-subscripted root comes into existence. The right-hand side of the inequality tends to 0 as $k \to \infty$, so $x_{2m}^2 \, y'(x_{2m}) \to 0$ as $k \to \infty$.

This particular information was used in Lemma 11 in the construction of a number $q(k) \in (x_{2m-1}, x_{2m})$ in the case $m=1$. The proof for general m proceeds in exactly the same way. The lemma is proved. ∎

Recall that $Z_m = \{\alpha \in [0, \infty): h(x, \alpha)$ has at least $2m+1$ zeros on $(0, \infty)\}$, $m = 1, 2, \ldots$. For $m=1$, lemmas 5 and 6 showed that Z_1 is bounded below by $\alpha = 1$. Of course, then all of the sets Z_m, $m \geqq 2$, are bounded below. Lemma 13 shows that Z_m is nonempty for $m \geqq 1$. We defined $\bar{\alpha}_m = \inf Z_m$. Also note that h has a finite number of zeros on $(0, \infty)$ by lemma 6. By continuous dependence, if $h(x, \bar{\alpha})$ has a zero on $(\sqrt{2(n-2)}, \infty)$, then so does $h(x, \alpha)$ for $|\alpha - \hat{\alpha}|$ small. So lemma 6 also implies that h cannot pick up more zeros until the $2m$-th zero decreases past $\sqrt{2(n-2)}$.

THEOREM. *Let* $\bar{\alpha}_m = \inf Z_m$. *Then the solutions* $y(x, \bar{\alpha}_m)$ *to* (4)–(6) *have the property* $\lim_{x \to \infty} [1 + \frac{1}{2} x y'(x, \bar{\alpha}_m)] = 0$.

Proof. From the definition of $\bar{\alpha}_m$, $y(x, \bar{\alpha}_m) < S(x)$ for $x > \sqrt{2(n-2)}$. For if $y(\bar{x}, \bar{\alpha}_m) = S(\bar{x})$ for some $\bar{x} > \sqrt{2(n-2)}$, then by continuous dependence, for $|\alpha - \bar{\alpha}_m|$ small, $y(\hat{x}, \alpha) = S(\hat{x})$ for some $\hat{x} > \sqrt{2(n-2)}$ and $y(x, \alpha)$, $y(x, \bar{\alpha}_m)$ have at least $2m+1$ zeros, a contradiction to $\bar{\alpha}_m = \inf Z_m$.

Suppose that $y''(\bar{x}, \bar{\alpha}_m) < 0$ for some $\bar{x} > \sqrt{2(n-1)}$. Then $y'''(\bar{x}) = [\bar{x}/2 - (n-1)/\bar{x}] \, y''(\bar{x}) + [\frac{1}{2} + (n-1)/\bar{x}^2 - e^{y(\bar{x})}] \, y'(\bar{x}) \leqq [\bar{x}/2 - (n-1)/\bar{x}] \, y''(\bar{x}) + [\frac{1}{2} + (n-1)/\bar{x}^2 - 2(n-2)/\bar{x}^2] \, y'(\bar{x}) < 0$ since $y(\bar{x}) < S(\bar{x})$ and $\bar{x} > \sqrt{2(n-1)}$. Thus, $y''(x)$ must remain negative for $x > \bar{x}$. By continuous dependence, for $\alpha > \bar{\alpha}_m$ (but close), there must be a value $\bar{x}(\alpha)$ such that $y''(\bar{x}, \alpha) < 0$. Similarly, $y''(x, \alpha) < 0$ for $x > \bar{x}$. On $[\sqrt{2(n-2)}, 2\bar{x}(\bar{\alpha}_m)]$, $|y'(x, \bar{\alpha}_m) - S'(x)| \geqq \delta > 0$ (or else $y' = S'$ for some $\hat{x} > \sqrt{2(n-2)}$, and Eq. (12) implies that y must intersect S, a contradiction to the definition of $\bar{\alpha}_m$). By continuous dependence, $|y'(x, \alpha) - S'(x)| \geqq \frac{1}{2}\delta$ on this same interval for $\alpha > \bar{\alpha}_m$ (but close). Consequently, $y(x, \alpha)$ does not intersect $S(x)$ for $x > \sqrt{2(n-2)}$, $\alpha \in Z_m$, a contradiction to the definition of $\bar{\alpha}_m$. Thus, $y''(x, \bar{\alpha}_m) > 0$ for $x > \sqrt{2(n-1)}$.

We have that $y'(x. \bar{\alpha}_m) < 0$ and $y''(x, \bar{\alpha}_m) > 0$ for $x > \sqrt{2(n-1)}$. The limit of $y'(x, \bar{\alpha}_m)$ as $x \to \infty$ must exist and be nonpositive. Suppose that for large x, $y'(x, \bar{\alpha}_m) \leqq -\varepsilon < 0$. From Eq. (4) we have that $0 = y'' + [(n-1)/x - x/2] \, y' + e^y - 1 \geqq y'' - \varepsilon[(n-1)/x - x/2] - 1$. So $y'' \leqq 1 + \varepsilon(n-1)/x - \varepsilon x/2$. The right-hand side tends to $-\infty$ as $x \to \infty$ which forces $y'' < 0$ somewhere. But this contradicts $y''(x, \bar{\alpha}_m) > 0$. So $y'(x, \bar{\alpha}_m) \to 0$ as $x \to \infty$.

Consider the function $xy'(x, \bar{\alpha}_m)$. Since $y(x, \bar{\alpha}_m) < S(x)$ and since $y'(\bar{x}, \bar{\alpha}_m) = S'(\bar{x})$ for some $\bar{x} > \sqrt{2(n-2)}$ implies that $y = S$ for some x (contradicting the definition for $\bar{\alpha}_m$), it must be true that $xy'(x, \bar{\alpha}_m) < -2$ for all $x > \sqrt{2(n-2)}$. Suppose that $xy'(x, \bar{\alpha}_m) \leqq -2 - \varepsilon < -2$ for all x large. Then (4) implies that $0 \geqq y'' - (2+\varepsilon)[(n-1)/x^2 - \frac{1}{2}] - 1$ and so $y'' \leqq (2+\varepsilon)(n-1)/x^2 - \frac{1}{2}\varepsilon$. This forces $y'' < 0$ which was ruled out earlier. We have shown that $\overline{\lim}_{x \to \infty} xy'(x, \bar{\alpha}_m) = -2$. Suppose that there is a sequence $\{t_k\}_1^\infty$ such that $ty'(t_k) \leqq -2 - \varepsilon < -2$ and (without loss of generality) $t_k y''(t_k) + y'(t_k) = 0$. Using Eq. (4), we have $0 = y''(t_k) + [(n-1)/t_k - t_k/2]\, y'(t_k) + e^{y(t_k)} - 1 = -y'(t_k)/t_k + [(n-1)/t_k - t_k/2]\, y'(t_k) + e^{y(t_k)} - 1$. Thus, $\frac{1}{2} t_k y'(t_k) = y'(t_k)/t_k + e^{y(t_k)} - 1$, and letting $t_k \to \infty$, we have that $-2 - \varepsilon \geqq \lim_{k \to \infty} t_k y'(t_k) = -2$, a contradiction. We have shown that $\underline{\lim}_{x \to \infty} xy'(x, \bar{\alpha}_m) = -2$. Thus, $\lim_{x \to \infty} [1 + \frac{1}{2} xy'(x, \bar{\alpha}_m)] = 0$ and the theorem is proved.

4. Observations and Conclusions

The nonexistence of solutions to (4), (5) in dimensions 1 and 2 clearly shows that the asymptotic representation (3) is not valid. However, for dimension 3, this representation may be accurate.

For $n \geqq 10$, solutions to the linearized problem $L'' + [(n-1)/x - x/2]\, L' + [2(n-2)/x^2]\, L = 0$ do not have more than one zero. We conjecture that because of this, (4), (5) does not have a solution.

The techniques discussed in this paper appear to be more general. In fact, a result by Joseph and Lundgren [3] is obtained by the procedures here. Their equation is

$$\ddot{u} + \frac{n-1}{t}\dot{u} + \lambda e^u = 0, \qquad 0 < t < 1 \tag{16}$$

$$\dot{u}(0) = 0, \qquad u(1) = 0. \tag{17}$$

There is a closed connected set $C(\lambda)$ contained in $[0, \infty) \times B$ where B is the Banach space $C^1[0, 1]$ with the C^1-norm. The set $C(\lambda)$ has boundary point $(0, 0)$ and represent solutions (λ, u) to (16), (17). Since e^u is unbounded, there is a number $\lambda^* \in (0, \infty)$ such that $\lambda \leqq \lambda^*$ is necessary for solutions to exist.

Letting $x = rt$, $r^2 = \lambda$, $u(t) = y(x)$, we have the corresponding initial value problem

$$y'' + \frac{n-1}{x} y' + e^y = 0, \qquad 0 < x < \infty \tag{18}$$

$$y(0) = \alpha > 0, \qquad y'(0) = 0. \tag{19}$$

This equation has the singular solution $S(x) = \ln[2(n-2)/x^2]$ and $h(x) = y - S$ satisfies

$$h'' + \frac{n-1}{x} h' + \frac{2(n-2)}{x^2} (e^h - 1) = 0, \qquad 0 < x < \infty. \tag{20}$$

Define $g(x) = xy'(x) + 2$. Then g satisfies

$$g'' + \frac{n-1}{x} g' + e^y g = 0, \qquad 0 < x < \infty. \tag{21}$$

Lemma 5 is valid for this function $g(x)$. Consequently, $h(x)$ can have at most one zero in $(0, \sqrt{2(n-2)})$ for $0 < \alpha < 1$. However, Lemma 6 does not follow. It appears that the absence of the term $-\frac{1}{2}xh'$ may allow $h(x)$ to have many zeros for x large since the linearized solutions to (20) have zero which accumulate at ∞ (unlike that for Eq. (14)).

The sets $Z_m = \{\alpha \in [0, \infty): h(x, \alpha)$ has at least $2m-1$ zeros on $(0, \sqrt{2(n-2)})\}$, $m \geqq 1$, are bounded below by $\alpha = 1$. To show they are nonempty, we need to show the existence of a first zero $x_1(\alpha_k)$ for some unbounded increasing sequence $\{\alpha_k\}_1^\infty$. Lemma 8 can be modified for (18), (19) with only minor changes. In fact, for each $x_1 \in (0, \infty)$, there is an $\alpha \in \mathbb{R}$ such that $h(x_1, \alpha(x_1)) = 0$. The remaining results may be slightly modified for e^y (instead of $e^y - 1$) and x^{n-1} (instead of $x^{n-1}e^{-(1/4)x^2}$). Consequently, all sets Z_m are nonempty and bounded below. The bifurcation diagrams in (α, λ) must look like those in Fig. 1 (where $2 < n < 10$). It is known that for $n \geqslant 10$, the bifurcation diagrams for (16), (17) look like that given in Fig. 2. We have indicated the conjectured diagram for (4)–(6).

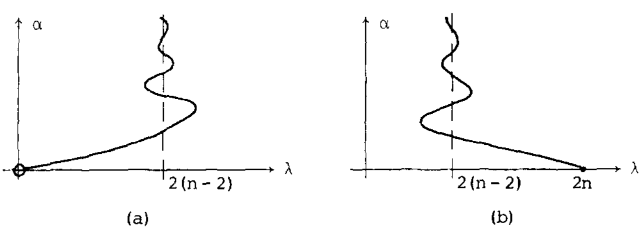

FIG. 1. (a) $\ddot{u} + ((n-1)/t)\,\dot{u} + \lambda e^u = 0$, (b) $\ddot{u} + ((n-1)/t)\,\dot{u} + \lambda(e^u - 1 - \frac{1}{2}t\dot{u}) = 0$

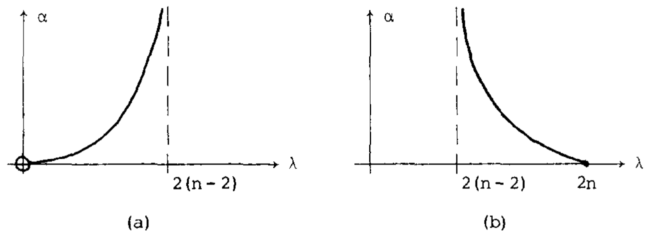

FIGURE 2

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