The simultaneous packing and covering constants in the plane

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Abstract

In 1950, C.A. Rogers introduced and studied two simultaneous packing and covering constants for a convex body and obtained the first general upper bound. Afterwards, these constants have attracted the interests of many authors because, besides their own geometric significance, they are closely related to the packing densities and the covering densities of the convex body, especially to the Minkowski–Hlawka theorem. However, so far our knowledge about them is still very limited. In this paper we will determine the optimal upper bound of the simultaneous packing and covering constants for two-dimensional centrally symmetric convex domains, and characterize the domains attaining this upper bound.

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1. Introduction

In 1950, C.A. Rogers introduced and studied two constants $\gamma(K)$ and $\gamma^*(K)$ for an $n$-dimensional convex body $K$. Explicitly, $\gamma(K)$ is the smallest positive number $r$ such that there is...
a translative packing \( K + X \) satisfying \( E^n = rK + X \), and \( \gamma^*(K) \) is the smallest positive number \( r^* \) such that there is a lattice packing \( K + \Lambda \) satisfying \( E^n = r^*K + \Lambda \), where \( E^n \) denotes the \( n \)-dimensional Euclidean space and \( \Lambda \) denotes an \( n \)-dimensional lattice in \( E^n \). In some references, the two numbers are called the simultaneous packing and covering constants for the convex body. Clearly, these constants are closely related to the packing densities and the covering densities of the convex body, especially to the Minkowski–Hlawka theorem.

In 1970 and 1978, S.S. Ryškov [26] and L. Fejes Tóth [13] independently introduced and investigated two related numbers \( \rho(K) \) and \( \rho^*(K) \), where \( \rho(K) \) is the largest positive number \( r \) such that one can put a translate of \( rK \) into every translative packing \( K + X \), and \( \rho^*(K) \) is the largest positive number \( r^* \) such that one can put a translate of \( r^*K \) into every lattice packing \( K + \Lambda \).

Clearly, for every convex body \( K \) we have

\[
\gamma(K) \leq \gamma^*(K)
\]

and

\[
\rho(K) \leq \rho^*(K).
\]

As usual, let \( C \) denote an \( n \)-dimensional centrally symmetric convex body. Then, we also have

\[
\gamma(C) = \rho(C) + 1
\]

and

\[
\gamma^*(C) = \rho^*(C) + 1.
\]

Let \( B^n \) denote the \( n \)-dimensional unit ball. Just like the packing density problem and the covering density problem, to determine the values of \( \gamma(B^n) \) and \( \gamma^*(B^n) \) is important and interesting. However, so far our knowledge about \( \gamma(B^n) \) and \( \gamma^*(B^n) \) is very limited. We list the main known results in the following table. For the covering radii of some particular lattices, for example the Leech lattice, we refer to Conway, Parker and Sloane [7] and Conway and Sloane [8], Chapters 4 and 23.

| \( n \) | 2     | 3     | 4     | 5 \\
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<td>( \gamma^*(B^n) )</td>
<td>( \sqrt{\frac{2}{3}} )</td>
<td>( \sqrt{\frac{5}{3}} )</td>
<td>( \sqrt{2\sqrt{3}(\sqrt{3} - 1)} )</td>
<td>( \sqrt{\frac{1}{2} + \frac{\sqrt{13}}{6}} )</td>
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Let \( \delta(K) \) and \( \delta^*(K) \) denote the maximal translative packing density and the maximal lattice packing density of \( K \), respectively. A fundamental problem in Packing and Covering is to decide if

\[
\delta(K) = \delta^*(K)
\]

holds for every convex body. It is easy to see that \( \gamma^*(C) \geq 2 \) will imply

\[
\delta(C) \geq 2\delta^*(C),
\]
which will give a negative answer to the previous problem. On the other hand, if \( \gamma^*(C) \leq 2 - \mu \)
holds for a positive constant \( \mu \) and for every centrally symmetric convex body \( C \), then the
Minkowski–Hlawka theorem can be improved to

\[
\delta^*(C) \geq \frac{1}{(2 - \mu)^n}.
\]

In 1950, C.A. Rogers [23] discovered a constructive method by which he deduced

\[ \gamma^*(C) \leq 3 \]

for all \( n \)-dimensional centrally symmetric convex bodies (see W. Banaszczyk [1], J. Bourgain
[3] and M. Henk [16] for related results). In 1972, via mean value techniques developed by C.A.
Rogers [25] and C.L. Siegel [27], the above upper bound was improved by G.L. Butler [5] to

\[ \gamma^*(C) \leq 2 + o(1). \]

This result is fascinating, because it gives hopes to both (1) and (2).

In two and three dimensions, as one can imagine, the situation is much better. In 1978, based
on an ingenious idea of I. Fáry [10], J. Linhart [20] proved that

\[ \gamma^*(K) \leq \frac{3}{2} \]

holds for every two-dimensional convex domain, and the upper bound is attained by triangles
only. However, just like the packing density problem, to determine the best upper bound for
\( \gamma^*(C) \) turns out to be much more challenging. Recently C. Zong [30] and [31] obtained

\[ \gamma^*(C) \leq 1.2 \]

for all two-dimensional centrally symmetric convex domains and

\[ \gamma^*(C) \leq 1.75 \]

for all three-dimensional centrally symmetric convex bodies. Needless to say, neither of them is
optimal. In this paper we will prove the following theorem.

**Theorem.** For every two-dimensional centrally symmetric convex domain \( C \) we have

\[ \gamma(C) = \gamma^*(C) \leq 2(2 - \sqrt{2}) \approx 1.17157 \ldots, \]

where the second equality holds if and only if \( C \) is an affinely regular octagon.

**Remark 1.** To determine the optimal upper bound for \( \gamma^*(C) \) has been listed as an open prob-
lem by several mathematicians (see Brass, Moser and Pach [4, p. 63], Linhart [20], Zong [30]
and [29]).
Remark 2. The identity
\[ \gamma(C) = \gamma^*(C) \]
for two-dimensional centrally symmetric convex domains was independently shown by J. Linhart [20] and C. Zong [30]. We restate it here just for completeness.

Remark 3. Let \( \theta(K) \) and \( \theta^*(K) \) denote the least translative covering density and the least lattice covering density of \( K \), respectively. In the plane it was proved by L. Fejes Tóth [11] that
\[ \theta(C) = \theta^*(C) \leq \frac{2\pi}{3\sqrt{3}}, \]
where the second equality holds if and only if \( C \) is an ellipse. In 1951, it was proved by C.A. Rogers [24] that
\[ \delta(K) = \delta^*(K) \]
holds for every two-dimensional convex domain \( K \). However, to find the optimal lower bound for \( \delta(C) \) is still a challenging open problem (see Reinhardt [22], Mahler [21] or Brass, Mosser and Pach [4, p. 11]). Nevertheless, it has been proved that neither ellipses nor affinely regular octagons can attain the optimal lower bound. For more about packing and covering, we refer to [14] and [15].

2. Several basic lemmas

Let \( \partial(K) \) and \( \text{int}(K) \) denote the boundary and the interior of \( K \), respectively. As usual, we call a convex body regular if for every point \( x \in \partial(K) \) there is a unique tangent hyperplane and every tangent plane touches its boundary at a single point. For convenience, in the rest of this paper \( C \) always means a two-dimensional centrally symmetric convex domain. Now let us introduce several basic lemmas which will be useful in our proof.

Lemma 1. (See Mahler [21].) If \( \pm v_1, \pm v_2 \) and \( \pm v_3 \) are the six vertices of an affinely regular hexagon inscribed in \( C \), then \( C + \Lambda \) is a lattice packing of \( C \), where
\[ \Lambda = \{2z_1v_1 + 2z_2v_2 : z_i \in \mathbb{Z}\}. \]

Lemma 2. (See Eggleston [9].) For every convex body there is a sequence of regular convex bodies which converges to the convex body in the sense of Hausdorff metric.

Let \( v_1, v_2, \ldots, v_6 \) be the six vertices (in anti-clock order) of a centrally symmetric hexagon \( H \) which is inscribed in \( C \) (clearly, \( v_{i+3} = -v_i \) for \( i = 1, 2, 3 \), let \( m_i \) denote the midpoint of \( v_i v_{i+1} \), and let \( m_i^* \) denote the point on the boundary of \( C \) in direction \( m_i \). Then, we define
\[ f_i(v_1) = \frac{\|o, m_i^*\|}{\|o, m_i\|}, \quad i = 1, 2, \ldots, 6, \]
where \( \|x, y\| \) denotes the Euclidean distance between \( x \) and \( y \).
Lemma 3. For any $x \in \partial(C)$, we can choose five points $x_2, x_3, x_4 = -x, x_5 = -x_2$ and $x_6 = -x_3$ from $\partial(C)$ such that they together with $x$ are the six vertices of an affinely regular hexagon. When $C$ is regular and $x$ moves along $\partial(C)$, we can choose the points such that all of $f_1(x), f_2(x)$ and $f_3(x)$ are continuous functions of $x$.

Proof. The first part of the lemma was proved by Zong in [28]. For the second part let $C$ be regular and, without loss of generality, we assume further that $x_2 x_3 x_4 x_5 x_6$ is a regular hexagon with $x = (1, 0)$, as shown in Fig. 1. Let $\epsilon$ be a positive number, let $x'$ be a point on the boundary of $C$ such that $\angle x'ox = \epsilon$, let $\Gamma_2$ and $\Gamma_3$ denote the straight lines which are parallel with $ox'$ and pass through $x_2$ and $x_3$, respectively.

Clearly, when $\epsilon$ is sufficiently small, $\Gamma_2$ intersects $\partial(C)$ at two points $x_2$ and $x_3^*$, $\Gamma_3$ intersects $\partial(C)$ at two points $x_3$ and $x_2^*$, and both $\|x_2, x_3^*\|$ and $\|x_3, x_2^*\|$ are small. In addition, then the three directions $xx', x_2x_3^*$ and $x_3x_2^*$ are approximately the tangent directions of $C$ at $x, x_2$ and $x_3$, respectively. By convexity, it is easy to see that

$$\angle x_3x_2x_3^* < \angle oxx'.$$

Thus, comparing triangle $oxx'$ with $x_3x_2x_3^*$, we get

$$\|x_3^*, x_3\| < \|o, x'\|.$$

Similarly, comparing $x_2x_3x_2^*$ with $ox_4x_4'$ where $x_4' = -x'$, we get

$$\|x_2, x_2^*\| > \|o, x_4'\| = \|o, x'\|.$$

Therefore $\partial(C)$ has two points $x_2'$ and $x_3'$ between $\Gamma_2$ and $\Gamma_3$ such that $x_3'x_2'$ is parallel with $ox'$ and

$$\|x_3', x_2'\| = \|o, x'\|.$$
Taking \(x'_4 = -x', \ x'_5 = -x'_2\) and \(x'_6 = -x'_3\), it is easy to see that \(x'_2x'_3x'_4x'_5x'_6\) is an affinely regular hexagon. Since both \(x'_2\) and \(x'_3\) continuously depend on \(x'\), all \(f_1(x), f_2(x)\) and \(f_3(x)\) are continuous functions of \(x\). The lemma is proved.

Remark 4. If \(C\) is regular, it is known in convex geometry that the corresponding points \(x_2, x_3, x_4, x_5\) and \(x_6\) in Lemma 3 are uniquely determined by \(x\). On the other hand, without the regularity assumption the second part of the lemma would not be true. For example, when \(C = \{(x,y) : |x| \leq 1, |y| \leq 1\}\) and \(x = (1,0)\), we could not find \(x_2, x_3, x_4, x_5\) and \(x_6\) on the boundary of \(C\) such that \(xx_2x_3x_4x_5x_6\) is affinely regular and the corresponding function \(f_1(x)\) is continuous at \(x\).

Lemma 4. (See Zong [30].) Let \(v_1v_2v_3v_4v_5v_6\) be a centrally symmetric hexagon inscribed in \(C\). Then we have

\[
\gamma^*(C) \leq \max\{f_1(v_1), f_2(v_1), f_3(v_1)\}.
\]

3. Proof of the theorem

For convenience, let \(L(x,y)\) denote the straight line passing through two points \(x\) and \(y\), and write

\[
\alpha = 2(2 - \sqrt{2}).
\]

To make the complicated proof more transparent, we divide it into three parts.

Assertion I. For every two-dimensional centrally symmetric convex domain \(C\), there is a corresponding inscribed affinely regular hexagon \(v_1v_2\cdots v_6\) satisfying

\[
f_1(v_1) \geq f_2(v_1) = f_3(v_1).
\]

Proof. First let us consider the case that \(C\) is regular. By Lemma 3, all \(f_1(x), f_2(x)\) and \(f_3(x)\) are continuous functions of \(x \in \partial(C)\). Therefore,

\[
f(x) = \min_{i=1,2} \{f_i(x)\} - f_3(x)
\]

is also a continuous function of \(x \in \partial(C)\). If, without loss of generality, \(x_1x_2x_3x_4x_5x_6\) is an affinely regular hexagon inscribed in \(C\) satisfying

\[
f_i(x_1) > f_3(x_1), \quad i = 1, 2,
\]

then we get

\[
f(x_1) = \min_{i=1,2} \{f_i(x_1)\} - f_3(x_1) > 0.
\]

On the other hand, by definition it follows that
\[ f_i(x_2) = f_{i+1}(x_1), \]
\[ f_i(x_1) = f_{i+3}(x_1) \]

and thus
\[ f(x_2) = \min_{i=1,2} \{ f_i(x_2) \} - f_3(x_2) = f_3(x_1) - f_1(x_1) < 0. \]

Therefore, there is a suitable point \( v \in \partial(C) \) satisfying
\[ f(v) = 0. \]

In other words, there is an affinely regular hexagon \( v_1v_2v_3v_4v_5v_6 \) inscribed in \( C \) satisfying
\[ f_1(v_1) \geq f_2(v_1) = f_3(v_1). \]

By Lemma 2 and Blaschke’s selection theorem, Assertion I is proved. \( \Box \)

**Assertion II.** For each two-dimensional centrally symmetric convex domain \( C \) there is a corresponding lattice \( \Lambda \) such that \( C + \Lambda \) is a packing and \( \alpha C + \Lambda \) is a covering in \( E^2 \).

**Proof.** Let \( v_1v_2 \cdots v_6 \) be the hexagon obtained in Assertion I. For convenience, we write
\[ \kappa = f_1(v_1), \]
\[ \lambda = f_2(v_1), \]

and define
\[ \Lambda_1 = \{ 2z_2v_2 + 2z_3v_3 : z_i \in \mathbb{Z} \}. \]

By Lemma 1, it follows that \( C + \Lambda_1 \) is a lattice packing.

If \( \kappa < \alpha \), then by Lemma 4 we can get
\[ \gamma^*(C) \leq \kappa < \alpha. \]

Thus, from now on we assume that \( \kappa \geq \alpha \).

As shown in Fig. 2, without loss of generality, we assume that \( v_1v_2v_3v_4v_5v_6 \) is a regular hexagon with \( v_2 = (\sqrt{3}/2, 1/2) \) and \( v_3 = (0, 1) \). Then, we have \( \alpha v_3 = (0, \alpha), \)
\[ m_2^* = \lambda m_2 = \left( \frac{\sqrt{3}\lambda}{4}, \frac{3\lambda}{4} \right) \]

and
\[ \alpha m_2^* = \left( \frac{\sqrt{3}\lambda \alpha}{4}, \frac{3\lambda \alpha}{4} \right). \]
Therefore, the equation of $L(\alpha v_3, \alpha m^*_2)$ is

$$y - \alpha = \frac{3\lambda - 4}{\sqrt{3}\lambda} x,$$

and $L(\alpha v_3, \alpha m^*_2)$ intersects $L(v_3, 2v_2)$ at

$$L(\alpha v_3, \alpha m^*_2) \cap L(v_3, 2v_2) = \left(\frac{\sqrt{3}\lambda(\alpha - 1)}{4 - 3\lambda}, 1\right).$$

Let $w$ denote the midpoint of the segment $[2v_3, 2v_2]$, let $p$ denote the midpoint of $v_2 w$ and let $p^*$ denote the boundary point of $C + 2v_2$ in the direction from $2v_2$ to $p$. By symmetry we have

$$\frac{\|2v_2, p^*\|}{\|2v_2, p\|} = \frac{\|o, m^*_1\|}{\|o, m_1\|} = \kappa.$$

Therefore, by a routine computation it can be deduced that the segment $[v_3, 2v_2]$ intersects the boundary of $\alpha C + 2v_2$ at

$$[v_3, 2v_2] \cap \partial(\alpha C + 2v_2) = \left(\frac{\sqrt{3}}{2} - \frac{\alpha\kappa}{2}, 1\right).$$

Then, by (3) and (4) it follows that $\alpha C + \Lambda_1$ will be a covering of $E^2$ and therefore

$$\gamma^*(C) < \alpha$$

if

$$\frac{\sqrt{3}}{2} - \frac{\alpha\kappa}{2} < \frac{\sqrt{3}\lambda(\alpha - 1)}{4 - 3\lambda}.$$
Please note that

\[
[\alpha v_3, \alpha m^*_2] \cap L(v_3, 2v_2) \neq L(\alpha v_3, \alpha m^*_2) \cap L(v_3, 2v_2)
\]

implies (5).

By convexity it is easy to see from Fig. 2 that \( \lambda \leq 4/3 \). Then it can be easily verified that (5) holds whenever \( \kappa > \sqrt{2} \) or \( \kappa \geq \lambda > \alpha \). Thus, in the rest of the proof we assume that

\[
1 \leq \lambda \leq \alpha, \quad \alpha \leq \kappa \leq \sqrt{2},
\]

and

\[
\frac{4 - 3\lambda}{\lambda} \geq \frac{\alpha - 1}{1 - \frac{1}{2} \alpha \kappa}.
\]

As shown in Fig. 3, without loss of generality, we assume that \( v_1v_2v_3v_4v_5v_6 \) is a regular hexagon with \( v_2 = (\sqrt{3}/2, 1/2) \) and \( v_3 = (0, 1) \). Then we have \( m_1 = (\sqrt{3}/2, 0), m^*_1 = (\sqrt{3} \kappa/2, 0), m_2 = (\sqrt{3}/4, 3/4) \) and \( m^*_2 = (\sqrt{3} \lambda/4, 3\lambda/4) \). Therefore, the equations of \( L(v_3, m^*_2) \) and \( L(v_2, m^*_1) \) are

\[
y - 1 = \frac{3\lambda - 4}{\sqrt{3} \lambda} x
\]

and

\[
y = \frac{x - \frac{\sqrt{3}}{2} \kappa}{\sqrt{3}(1 - \kappa)},
\]

respectively.
Since $\|o, m_1^*\|/\|o, m_1\| \geq \alpha$, there is a point $p \in [m_1, m_1^*]$ satisfying
\[
\frac{\|o, m_1^*\|}{\|o, p\|} = \alpha. \tag{11}
\]

In fact, the point is $p = (\sqrt{3}/2\alpha, 0)$. Then, there are two points $q_1 \in L(v_1, m_1^*)$ and $q_2 \in L(v_2, m_1^*)$ such that $p \in L(q_1, q_2)$ and $L(q_1, q_2)$ is parallel with $L(v_1, v_2)$. By (10) (the equation of $L(v_2, m_1^*)$) and the $x$-coordinate of $p$ we get
\[
q_2 = \left(\frac{\sqrt{3} \kappa}{2 \alpha}, \frac{\kappa (\alpha - 1)}{2(\kappa - 1) \alpha}\right).
\]

Let $u_2$ denote the midpoint of $v_3 q_2$. It is easy to see that
\[
u_2 = \left(\frac{\sqrt{3} \kappa}{4 \alpha}, \frac{\kappa (\alpha - 1)}{4(\kappa - 1) \alpha} + \frac{1}{2}\right),
\]
and the equation of $L(o, u_2)$ is
\[
y = \left(\frac{\kappa (\alpha - 1)}{4(\kappa - 1) \alpha} + \frac{1}{2}\right) \frac{4 \alpha}{\sqrt{3} \kappa} x. \tag{12}
\]

Let $u_2^*, u_2^\circ$ and $u_2^\triangle$ denote the points on $\partial(C)$, $L(v_3, m_2^*)$ and $L(v_2, m_1^*)$, respectively, all in direction $u_2$. By (9), (10) and (12), we get
\[
u_2^* = \left(\left(\frac{\alpha - 1}{\sqrt{3} (\kappa - 1)} + \frac{2 \alpha}{\sqrt{3} \kappa} + \frac{4 - 3 \lambda}{\sqrt{3} \lambda}\right)^{-1}, y^*\right),
\]
and
\[
u_2^\circ = \left(\frac{\sqrt{3} \kappa^2}{2 \alpha (3 \kappa - 2)}, y^\circ\right),
\]
where the $y$-coordinates of both $u_2^*$ and $u_2^\circ$ are not necessary for our purpose. Thus, we get
\[
\frac{\|o, u_2^\circ\|}{\|o, u_2\|} = \frac{4 \alpha}{\kappa} \left(\frac{\alpha - 1}{\kappa - 1} + \frac{2 \alpha}{\kappa} + \frac{4 - 3 \lambda}{\lambda}\right)^{-1}
\]
and
\[
\frac{\|o, u_2^\circ\|}{\|o, u_2\|} = \frac{2 \kappa}{3 \kappa - 2}.
\]

For convenience, we write
\[
f(\kappa, \lambda) = \frac{4 \alpha}{\kappa} \left(\frac{\alpha - 1}{\kappa - 1} + \frac{2 \alpha}{\kappa} + \frac{4 - 3 \lambda}{\lambda}\right)^{-1}
\]
and

\[ g(\kappa) = \frac{2\kappa}{3\kappa - 2}. \]

Next, we proceed to show \( f(\kappa, \lambda) \leq \alpha \) and \( g(\kappa) \geq \alpha \). It follows from (8) that

\[ f(\kappa, \lambda) \leq \frac{4\alpha}{\kappa} \left( \frac{\alpha - 1}{\kappa - 1} + \frac{2\alpha}{\kappa} + \frac{\alpha - 1}{1 - \frac{1}{2}\alpha\kappa} \right)^{-1}. \]

It is easy to see that

\[ \frac{4\alpha}{\kappa} \left( \frac{\alpha - 1}{\kappa - 1} + \frac{2\alpha}{\kappa} + \frac{\alpha - 1}{1 - \frac{1}{2}\alpha\kappa} \right)^{-1} \leq \alpha \]

if and only if

\[ \frac{2\alpha - 4}{\kappa} + \frac{\alpha - 1}{\kappa - 1} + \frac{\alpha - 1}{1 - \frac{1}{2}\alpha\kappa} \geq 0. \]

On the other hand, substituting \( \alpha \) by \( 2(2 - \sqrt{2}) \) and applying (7), we get

\[ \frac{2\alpha - 4}{\kappa} + \frac{\alpha - 1}{\kappa - 1} + \frac{\alpha - 1}{1 - \frac{1}{2}\alpha\kappa} = \frac{(\sqrt{2} - 1)[4 - 4(3 - \sqrt{2})\kappa + (3 - \sqrt{2})^2\kappa^2]}{\kappa(\kappa - 1)(1 - (2 - \sqrt{2})\kappa)} \]

\[ = \frac{(\sqrt{2} - 1)(3 - \sqrt{2})^2(\kappa - \frac{2}{3 - \sqrt{2}})^2}{\kappa(\kappa - 1)(1 - (2 - \sqrt{2})\kappa)} \geq 0, \]

where the last equality holds if and only if \( \kappa = \frac{2}{3 - \sqrt{2}} \). Thus, we get

\[ f(\kappa, \lambda) = \frac{\|o, u^*_2\|}{\|o, u_2\|} \leq \alpha, \quad (13) \]

where the equality holds if and only if \( \kappa = \frac{2}{3 - \sqrt{2}} \) and the equality in (8) holds, that is

\[ \begin{cases} 
\kappa = \frac{2}{3 - \sqrt{2}} \approx 1.261203875 \ldots, \\
\lambda = \frac{2 + 4\sqrt{2}}{7} \approx 1.09383632 \ldots. 
\end{cases} \quad (14) \]

On the other hand, it is easy to see that \( g(\kappa) \) is a decreasing function of \( \kappa \) when \( \kappa \) satisfies (7). Thus, we have

\[ g(\kappa) = \frac{\|o, u^*_2\|}{\|o, u_2\|} \geq \frac{2}{3 - \sqrt{2}} > \alpha. \quad (15) \]
As shown in Fig. 4, let $u'_2$, $u'_3$, $u'_5$ and $u'_6$ be the four points satisfying
\[ \frac{\|o, u'_i\|}{\|o, u_i\|} = \alpha, \quad i = 2, 3, 5, 6. \] (16)

Then $m^*_1u'_2u'_3m^*_4u'_5u'_6$ is an affinely regular hexagon. For convenience, we write
\[ H = \text{conv}\{q_1, q_2, v_3, q_4, q_5, v_6\}, \]
\[ C' = \text{conv}\{C, u'_2, u'_3, u'_5, u'_6\}, \]
and
\[ \Lambda_2 = \{2z_1m^*_1 + 2z_2u'_2: z_i \in Z\}. \]

By (13), (15) and convexity it follows that
\[ \{m^*_1, u'_2, u'_3, m^*_4, u'_5, u'_6\} \subset \partial(C'). \]

Thus, by Lemma 1, $C' + \Lambda_2$ is a packing and therefore $C + \Lambda_2$ is a packing too. On the other hand, it follows from (11) and (16) that $\alpha H + \Lambda_2$ is a tiling of $E^2$ and therefore $\alpha C + \Lambda_2$ is a covering of $E^2$. Hence, we get
\[ \gamma^*(C) \leq \alpha = 2(2 - \sqrt{2}). \] (17)

Assertion II is proved. \qed

**Assertion III.** The equality
\[ \gamma^*(C) = 2(2 - \sqrt{2}) \] (18)
holds if and only if $C$ is an affinely regular octagon.
Proof. Let $P_8$ denote an affinely regular octagon. It was independently proved by Linhart [20] and Zong [30] that

$$\gamma^\ast(P_8) = 2(2 - \sqrt{2}).$$

On the other hand, if $C$ is a two-dimensional centrally symmetric convex domain satisfying (18), we proceed to show that it must be an affinely regular octagon.

Claim 1. Both $\kappa$ and $\lambda$ satisfy (14).

If this claim is false, letting $m_1, m_1^\ast, p, q_1, q_2, q_4, q_5, v_3, v_6, u_2, u_2^\ast, u_3$ and $u_3^\ast$ be the points defined in Fig. 3, by (13) we have

$$\begin{align*}
\|o, m_1^\ast\|/\|o, u_2\| < \alpha, \\
\|o, u_2^\ast\|/\|o, u_3\| < \alpha.
\end{align*}$$

(19)

Then, as shown in Fig. 5, we define points $p', q_1', q_2', q_4', q_5', p_2, p_2^\ast, p_3$ and $p_3^\ast$ as follows: First, we define $p' = (1 + \epsilon)p$ for a small positive number $\epsilon$. Second, we choose $q_1' \in q_1 m_4^\ast$ and $q_2' \in q_2 m_4^\ast$ such that $L(q_1', q_2')$ passing through $p'$ and parallel with $L(q_1, q_2)$. Third, we take $q_4' = -q_1'$ and $q_5' = -q_2'$. Fourth, let $p_2$ be the midpoint of $q_4'v_3$ and let $p_2^\ast$ be the point on the boundary of $C$ in direction $p_2$. Finally, let $p_3$ be the midpoint of $q_5'v_3$ and let $p_3^\ast$ be the point on the boundary of $C$ in direction $p_3$. It follows from (19) that

$$\begin{align*}
\|o, m_1^\ast\|/\|o, p'\| < \alpha, \\
\|o, p_2^\ast\|/\|o, p_2\| < \alpha, \\
\|o, p_3^\ast\|/\|o, p_3\| < \alpha.
\end{align*}$$
when $\epsilon$ is sufficiently small. Please note that in this case as well as in the proof of (17) we cannot simply apply Lemma 4. Defining

$$\beta = \max \left\{ \frac{\|o, m_1^*\|}{\|o, p\|}, \frac{\|o, p_1^*\|}{\|o, p_2\|}, \frac{\|o, p_3^*\|}{\|o, p_3\|} \right\},$$

$$H = \text{conv}\{q_1', q_2', v_3, q_4', q_5', v_6\},$$

and

$$A_3 = \{2z_1\beta p' + 2z_2\beta p_2: z_i \in \mathbb{Z}\},$$

it can be verified that $C + A_3$ is a packing, $\beta H + A_3$ is a tiling, $\beta C + A_3$ is a covering, and therefore

$$\gamma^*(C) \leq \beta < \alpha,$$

which contradicts (18). Thus, $\kappa$ and $\lambda$ must satisfy (14).

**Claim 2.** The whole segment $v_2m_1^*$ (as well as $v_1m_1^*$) is a part of $\partial(C)$.

If the claim is not true, we proceed to deduce a contradiction. Let $v_1, v_2, v_3, v_4, v_5, v_6, m_1, m_1^*, m_6, m_6^*, p, q_1, q_2, u_2, u_6$ and $u_6^*$ be the points defined in Fig. 3. Then $(1 + \epsilon)q_2 \in \text{int}(C)$ holds for small positive numbers $\epsilon$. For convenience, we write $q_2' = (1 + \epsilon)q_2$ and let $q_i'$ denote the point symmetric to $q_i$ with respect to $p$, as shown in Fig. 6.

Assume that $L(q_1', q_1)$ and $L(v_6, v_1)$ intersect $L(o, m_1)$ at $o'$ and $o^*$, respectively. Since $\kappa$ and $\lambda$ satisfy (14), we have

$$\angle q_1'o'o = \angle q_2'o'o' < \angle v_2oo' = \angle v_6o'o < \angle v_1m_1'o.$$

Thus, $q_i'$ and $p$ are on the same side of $L(v_1, m_1^*)$. Let $q_1'$ be the point on $L(v_1, m_1^*)$ having the same $y$-coordinate as that of $q_1'$ and let $p'$ be the midpoint of $q_1'q_2'$. It is clear that $q_1' \in (v_1, q_1)$ and $p' \in (p, m_1^*)$ when $\epsilon$ is sufficiently small. Then, we have

$$\frac{\|o, m_1^*\|}{\|o, p'\|} < \alpha. \quad (20)$$
Let $b_2$ and $b_6$ denote the midpoints of $v_3q_2'$ and $v_6q_1'$, respectively, and let $b_i^*$ be the point on the boundary of $C$ in direction $b_i$. We proceed to show that

$$\left\{ \begin{array}{l}
\|o, b_2^*\|/\|o, b_2\| < \alpha, \\
\|o, b_6^*\|/\|o, b_6\| < \alpha.
\end{array} \right. \quad (21)$$

As shown in Fig. 7, since $\kappa$ and $\lambda$ satisfy (14), $L(o, q_2)$ intersects $L(v_3, u_2^*)$ at a point $t = (t_1, t_2)$ with $t_1 > 0$. Clearly, $L(u_2, b_2)$ is parallel with $L(o, q_2)$. Thus, we get

$$\frac{\|o, b_2^*\|}{\|o, b_2\|} < \frac{\|o, u_2^*\|}{\|o, u_2\|} \leq \alpha,$$

which proves the first inequality of (21). On the other hand, we have

$$\frac{\|o, b_6^*\|}{\|o, b_6\|} < \frac{\|o, u_6^*\|}{\|o, u_6\|} \leq \alpha,$$

which proves the second inequality of (21).

Based on (20) and (21), by a construction similar to that in the proof of Claim 1, one can deduce

$$\gamma^*(C) < \alpha,$$

which contradicts (18). Thus, Claim 2 is proved.

**Claim 3.** $\|o, u_2^*\|/\|o, u_2\| = \|o, u_6^*\|/\|o, u_6\| = \alpha$. As a consequence, the whole segments $v_3u_2^*$ and $v_6u_6^*$ are parts of $\partial(C)$.

Let $p, q_1, q_2, v_1, v_2, v_3, u_2, u_2^*$ and $m_1^*$ be the points defined in Fig. 3.

As shown in Fig. 8, let $q_2'$ be a point in $(q_2, m_1^*)$, $p'$ be the midpoint of $q_1q_2'$, $p^* = L(o, p') \cap L(v_1, m_1^*)$ and $p^o = L(o, p') \cap L(v_2, m_1^*)$. Since $L(p, p')$ is parallel with $L(v_2, m_1^*)$, we have

$$\frac{\|o, p^*\|}{\|o, p'\|} < \frac{\|o, p^o\|}{\|o, p'\|} = \frac{\|o, m_1^*\|}{\|o, p\|} = \alpha.$$
Therefore, if \( \| \mathbf{o}, \mathbf{u}_2^* \| / \| \mathbf{o}, \mathbf{u}_2 \| < \alpha \), there are two suitable points \( q_2' \in (q_2, m_1^*) \) and \( q_1' \in (v_1, q_1) \) satisfying

\[
\begin{align*}
\| \mathbf{o}, p_1^* \| / \| \mathbf{o}, p_1 \| &< \alpha, \\
\| \mathbf{o}, p_2^* \| / \| \mathbf{o}, p_2 \| &< \alpha, \\
\| \mathbf{o}, p_6^* \| / \| \mathbf{o}, p_6 \| &< \alpha,
\end{align*}
\]

where \( p_1, p_2 \) and \( p_6 \) are the midpoints of \( q_1'q_2' \), \( v_3q_2' \) and \( v_6q_1' \), respectively, and \( p_i^* \) is the point on \( \partial(C) \) in direction \( p_i \). Then, similar to the construction in Claim 1, we can get a contradiction to (18). The claim is proved.

Finally, let us complete the proof of the assertion based on Fig. 9.

Let \( v_1, m_1^*, u_1^* \) be points defined just like those in Fig. 3. Let \( w_2' \) denote the intersection of \( L(v_3, m_2^*) \) and \( L(v_2, m_1^*) \), and let \( w_4', w_6', w_8' \) be the points defined similarly as shown in Fig. 9. It is easy to verify that \( m_1^*w_2'v_3w_4'v_5w_6'v_6w_8' \) is an affinely regular octagon. In addition,
in this case we have \( v_3 = (0, 1), \ v_2 = (\sqrt{3}/2, 1/2), \ m^*_1 = (\sqrt{3}/(3-\sqrt{2}), 1), \ u^*_2 = (\sqrt{3} - 2\sqrt{2}/3, 3/2 - \sqrt{2}/2) \), and \( w'_2 = (\sqrt{3}/3 - 2\sqrt{2}/2, \sqrt{3}/2) \). We proceed to show

\[
C = m^*_1 w'_2 v'_3 m^*_4 w'_6 v_6 w'_8. \tag{22}
\]

If \( C \) is not the octagon and let \( w^*_2 \) and \( w^*_4 \) denote the two points on \( \partial(C) \) in directions \( w'_2 \) and \( w'_4 \), respectively. In addition, we write

\[
\rho = \frac{\| o, w'_2 \|}{\| o, w^*_2 \|}, \quad \sigma = \frac{\| o, w'_4 \|}{\| o, w^*_4 \|},
\]

and assume that \( \rho \geq \sigma \). Based on the coordinates of \( v_2, u^*_2 \) and \( w'_2 \), by a routine computation we get

\[
1 \leq \sigma \leq \rho \leq 4\sqrt{2} + 2/7. \tag{23}
\]

Let \( w_2 \) be the point defined by \( o w^*_2/ow_2 = \alpha \), let \( \Gamma \) denote the straight line passing through \( w_2 \) and parallel with \( L(w'_4, w'_8) \), let \( s_1 \) and \( s_2 \) denote the intersections of \( \Gamma \) with \( L(v_2, m^*_1) \) and \( L(v_3, m^*_5) \), respectively, let \( t \) denote the midpoint of \( w^*_4 s_2 \), let \( t^* \) denote the intersection of \( L(o, t) \) with \( L(v_3, w'_4) \), and finally define

\[
h(\rho, \sigma) = \frac{\| o, t^* \|}{\| o, t \|}.
\]

Based on the coordinates of \( m^*_1, w'_2, v_3 \) and \( w'_4 \), by routine computations we get

\[
w^*_2 = \left( \frac{\sqrt{3}}{(3\sqrt{2} - 2)\rho}, \frac{\sqrt{2}}{2\rho} \right),
\]

\[
w_2 = \left( \frac{\sqrt{3}}{(3\sqrt{2} - 2)\rho \alpha}, \frac{\sqrt{2}}{2\rho \alpha} \right),
\]

\[
s_1 = \left( \frac{\sqrt{3}}{8 - 5\sqrt{2}} - \frac{\sqrt{3}}{(3 - \sqrt{2})\rho \alpha}, \frac{1}{(\sqrt{2} - 1)\rho \alpha} - \frac{1}{2 - \sqrt{2}} \right),
\]

\[
s_2 = \left( \frac{\sqrt{3}}{(4\sqrt{2} - 5)\rho \alpha} - \frac{\sqrt{3}}{8 - 5\sqrt{2}}, \frac{1}{2 - \sqrt{2}} - \frac{1}{\rho \alpha} \right),
\]

\[
w^*_4 = \left( \frac{\sqrt{3}}{(2 - 3\sqrt{2})\sigma}, \frac{\sqrt{2}}{2\sigma} \right),
\]

\[
t = \left( \frac{\sqrt{3}}{2(2 - 3\sqrt{2})\sigma} + \frac{\sqrt{3}}{2(4\sqrt{2} - 5)\rho \alpha}, \frac{\sqrt{3}}{2(8 - 5\sqrt{2})}, \frac{\sqrt{2}}{4\sigma} + \frac{1}{2(2 - \sqrt{2})} - \frac{1}{2\rho \alpha} \right). \tag{24}
\]
\[ t^* = \left( \left( \frac{\sqrt{3}}{2(2 - 3\sqrt{2})\sigma} + \frac{\sqrt{3}}{2(4\sqrt{2} - 5)\rho\alpha} - \frac{\sqrt{3}}{2(8 - 5\sqrt{2})} \right) \times \left( \frac{\sqrt{2} + 1}{2} - \frac{2 + \sqrt{2}}{4\rho} + \frac{1}{2\sigma} \right)^{-1}, t_2^* \right), \]  
(25)

where the second component of \( t^* \) is not needed in the following. By (23) it can be deduced that

\[
0 < \frac{1}{(\sqrt{2} - 1)\rho\alpha} - \frac{1}{2 - \sqrt{2}} < \frac{1}{2}
\]

and

\[
\frac{\sqrt{2}}{2} < \frac{1}{2 - \sqrt{2}} - \frac{1}{\rho\alpha} < 1.
\]

Thus, we have

\[
s_1 \in v_2m_1^* \subset \partial(C),
\]

\[
s_2 \in v_3u_2^* \subset \partial(C),
\]

and

\[
\| w_2, s_1 \| = \| w_2, s_2 \|.
\]

Especially, by (24) and (25) we get

\[
h(\rho, \sigma) = \left( \frac{\sqrt{2} + 1}{2} - \frac{2 + \sqrt{2}}{4\rho} + \frac{1}{2\sigma} \right)^{-1}.
\]

Then, by (23) we have

\[
h(\rho, \sigma) \leq \left( \frac{\sqrt{2} + 1}{2} - \frac{2 + \sqrt{2}}{4\rho} + \frac{1}{2\rho} \right)^{-1}
\]

\[
= \left( \frac{\sqrt{2} + 1}{2} - \frac{\sqrt{2}}{4\rho} \right)^{-1}
\]

\[
\leq 2(2 - \sqrt{2}),
\]

where the final equality holds if and only if \( \rho = \sigma = 1 \). Then, (22) follows from Lemma 4. Assertion III is proved. \( \square \)

As a conclusion of Assertions II and III the theorem is proved.
4. Three further remarks

Remark 5. Let \( \lambda_i(C, \Lambda) \) denote the \( i \)th successive minimum of \( C \) with respect to a lattice \( \Lambda \), and let \( \mu_i(C, \Lambda) \) denote the \( i \)th covering minimum of \( C \) with respect to \( \Lambda \) (see Kannan and Lovász [19]). As a corollary of the theorem we get

\[
\min_{\Lambda} \frac{\mu_2(C, \Lambda)}{\lambda_1(C, \Lambda)} \leq 2(2 - \sqrt{2}),
\]

where the equality holds if and only if \( C \) is an affinely regular octagon.

Remark 6. It is well known (see L. Fejes Tóth [12, p. 106]) that

\[
\frac{\theta^*(C)}{\delta^*(C)} \leq \frac{4}{3} \approx 1.33333\ldots
\]

holds for every two-dimensional centrally symmetric convex domain. However, although our theorem is optimal, it only can produce

\[
\frac{\theta^*(C)}{\delta^*(C)} \leq \min_{\Lambda} \left( \frac{\mu_2(C, \Lambda)}{\lambda_1(C, \Lambda)} \right)^2 \leq 8(3 - 2\sqrt{2}) \approx 1.37258\ldots.
\]

The reason for this phenomenon is that the optimal covering lattice of a regular octagon is not homothetic to its optimal packing lattice.

Remark 7. Let \( m_2(C) \) denote the Steiner ratio of the Minkowski plane determined by a two-dimensional centrally symmetric convex domain \( C \). It is known (see Cieslik [6, p. 192]) that

\[
m_2(C) \leq \frac{3}{4} \gamma^*(C).
\]

Thus, we have

\[
m_2(C) \leq \frac{3}{2 + \sqrt{2}}.
\]

Acknowledgments

In 1996, I learned this problem from Professor C.A. Rogers when I was a visitor at University College London. In 2003, when I published two papers ([30] and [31]) on this problem, I received an offprint of [20] from Professor J. Linhart. Clearly, he was not aware of Rogers and Butler’s papers on this topic when he published [20], just like my unawareness of his paper. Fortunately, our papers have almost no important overlap. I am obliged to Professor Rogers for driving my attention to this problem and to Professor Linhart for sending me his related papers. For some helpful comments on this paper, I am grateful to Prof. M. Henk, Prof. C. Song, Dr. L. Yu and the referees.
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