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Discrete Mathematics 266 (2003) 153–183
DISCRETE MATHEMATICS
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Full embeddings of (α, β)-geometries in projective spaces—Part II

Sara Cauchie

Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan 2, B-9000 Ghent, Belgium

Received 4 July 2001; received in revised form 26 April 2002; accepted 12 August 2002

Abstract

The incidence structures known as (α, β)-geometries are a generalization of partial geometries and semipartial geometries. In a previous paper, a classification of (α, β)-geometries fully embedded in $PG(n,q)$, q odd and $\alpha > 1$, assuming that every plane of $PG(n,q)$ containing an antiflag of $\mathcal{S}$ is either an α-plane or a β-plane, is given. The case that there is a so-called mixed plane and that $\beta = q+1$, is also treated there. In this paper we will treat the case $\beta = q$. This completes the classification of all proper (α, β)-geometries fully embedded in $PG(n,q)$, q odd and $\alpha > 1$, such that $PG(n,q)$ contains at least one α- or one β-plane. For q even, some partial results are obtained.

Keywords: (α, β)-geometry,; Projective embedding

1. Introduction

A *partial linear space* of order (s,t) is a connected incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, with $\mathcal{P}$ a finite non-empty set of elements called points, $\mathcal{L}$ a family of subsets of $\mathcal{P}$ called lines and I an incidence relation satisfying the following axioms.

(1) Any two distinct points are incident with at most one line.
(2) Each line is incident with exactly $s+1$ points, $s \geqslant 1$.
(3) Each point is incident with exactly $t+1$ lines, $t \geqslant 1$.

E-mail address: scauchie@cage.rug.ac.be (S. Cauchie).

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doi:10.1016/S0012-365X(02)00805-1

The *incidence number* of an antiflag (x, L) of $\mathcal{S}$ (i.e. x is a point and L is a line of $\mathcal{S}$ such that x is not incident with L) is the number, denoted by $i(x, L)$, of points collinear with the point $x \in \mathcal{P}$ and incident with the line $L \in \mathcal{L}$.

An (α, β)-*geometry* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s, t), for some s and t, such that for any $x \in \mathcal{P}$ and any $L \in \mathcal{L}$, x not incident with L, we have that $i(x, L) = \alpha$ or $i(x, L) = \beta$.

Although the concept of an (α, β)-geometry was commonly known for special values of α and β, the general definition appeared up to our knowledge for the first time in [8]. For $\alpha = \beta$ an (α, β)-geometry is a partial geometry. Partial geometries were first studied by Bose in [4].

An (α, β)-geometry $\mathcal{S}$ is called *strongly regular* if its point graph is a strongly regular graph. Strongly regular (α, β)-geometries have been studied in [10].

A *proper* (α, β)-geometry is an (α, β)-geometry with $\alpha > 0$, $\beta > 0$ and $\alpha \neq \beta$. We will assume that $\alpha < \beta$. An (α, β)-geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is said to be *fully embedded* in a projective space $\mathrm{PG}(n, q)$ if $\mathcal{P}$ is a subset of the point set of $\mathrm{PG}(n, q)$, $\mathcal{L}$ is a subset of the line set of $\mathrm{PG}(n, q)$, I is the incidence inherited from $\mathrm{PG}(n, q)$ and $s = q$. We require that the points of $\mathcal{S}$ span $\mathrm{PG}(n, q)$. Partial geometries fully embedded in $\mathrm{PG}(n, q)$ were classified in [6]. Also full embeddings of $(0, \alpha)$-geometries, with $\alpha > 1$, have been previously studied (see [7,14]).

Let $\mathcal{S}$ be an (α, β)-geometry fully embedded in $\mathrm{PG}(n, q)$, q odd and $\alpha > 1$. The restriction of $\mathcal{S}$ to a plane of $\mathrm{PG}(n, q)$ is a partial linear space, but has not necessarily an order. In case it has an order, it follows that it is a partial geometry $\mathrm{pg}(q, \alpha - 1, \alpha)$ or $\mathrm{pg}(q, \beta - 1, \beta)$ [6]. A plane in which the restriction of $\mathcal{S}$ is a partial geometry $\mathrm{pg}(q, \alpha - 1, \alpha)$, we call an α-*plane*. A plane in which the restriction of $\mathcal{S}$ is a partial geometry $\mathrm{pg}(q, \beta - 1, \beta)$, we call a β-*plane*. A plane that contains an antiflag of $\mathcal{S}$ and that is not an α-plane or a β-plane, we call a *mixed* plane. In such a mixed plane, every point of $\mathcal{S}$ in the plane is incident with either α or β lines of $\mathcal{S}$ in this plane. The points and lines of a partial geometry fully embedded in a projective plane are either all points and lines of the plane, or the points not contained in a maximal arc $\mathcal{K}$ of the plane, and the lines exterior to $\mathcal{K}$ (see for instance [2]). Now for q odd, there exists no non-trivial maximal arc in a Desarguesian projective plane [1]. So, if π is an α- or a β-plane in $\mathrm{PG}(n, q)$, then the points and lines of $\mathcal{S}$ in π are either all points and lines of π, in which case π is a $(q+1)$-plane, or all points of π except one point p and all lines of π not through p, in which case π is a q-plane. It is our aim to classify all full embeddings of proper (α, β)-geometries in $\mathrm{PG}(n, q)$, q odd and $\alpha > 1$, under the assumption that there is at least one α- or one β-plane contained in $\mathrm{PG}(n, q)$. With this assumption it follows immediately that there are three possibilities: $\alpha = q$ and $\beta = q + 1$, $\alpha < q$, in which case there are no α-planes and $\beta = q + 1$ or $\beta = q$.

In [5] we proved a classification for proper (α, β)-geometries fully embedded in $\mathrm{PG}(n, q)$, q odd, under the assumption that every plane of $\mathrm{PG}(n, q)$ that contains an antiflag of $\mathcal{S}$ is either an α- or a β-plane. We also obtained a classification under the assumption that there is a mixed plane and that $\beta = q + 1$. In particular we proved the following results.

Theorem 1. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper $(q, q+1)$-geometry fully embedded in* $\mathrm{PG}(n,q)$. *Assume that every plane of* $\mathrm{PG}(n,q)$ *that contains an antiflag of $\mathcal{S}$ is a q-plane or a $(q+1)$-plane. Then $\mathcal{P}$ is the set of points of* $\mathrm{PG}(n,q)\backslash\mathrm{PG}(m,q)$, *for some $0 \leqslant m < n-2$ and $\mathcal{L}$ is the set of the lines of* $\mathrm{PG}(n,q)$ *that are disjoint from* $\mathrm{PG}(m,q)$.

Corollary 2. *Let $\mathcal{S}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper (α, β)-geometry fully embedded in* $\mathrm{PG}(n,q)$, *with $\alpha > 1$ and q odd. Assume that every plane of* $\mathrm{PG}(n,q)$ *that contains an antiflag of $\mathcal{S}$ is an α-plane or a β-plane. Then $\mathcal{P}$ is the set of points of* $\mathrm{PG}(n,q)\backslash\mathrm{PG}(m,q)$, *for some $0 \leqslant m < n-2$ and $\mathcal{L}$ is the set of the lines of* $\mathrm{PG}(n,q)$ *that are disjoint from* $\mathrm{PG}(m,q)$.

Theorem 3. *Let $\mathcal{S}$ be a proper $(\alpha, q+1)$-geometry, $\alpha > 1$, fully embedded in* $\mathrm{PG}(n,q)$, *such that* $\mathrm{PG}(n,q)$ *contains at least one mixed plane. Then the points of $\mathcal{S}$ are the points of* $\mathrm{PG}(n,q)\backslash\mathrm{PG}(m,q)$, *with $0 \leqslant m \leqslant n-3$. Moreover there exists a partition of the points of $\mathcal{S}$ in m'-dimensional subspaces of* $\mathrm{PG}(n,q)$ *that pairwise intersect in* $\mathrm{PG}(m,q)$, *$m+2 \leqslant m' \leqslant n-2$, such that the lines of $\mathcal{S}$ are the lines that intersect $q+1$ of these m'-dimensional spaces in a point. A necessary and sufficient condition for this partition and such an $(\alpha, q+1)$-geometry to exist is that $(m'-m)|(n-m')$.*

In this paper we will consider the remaining case for a complete classification of fully embedded (α, β)-geometries in $\mathrm{PG}(n,q)$, $\alpha > 1$ and q odd, under the assumption that $\mathrm{PG}(n,q)$ contains at least one α-plane or β-plane. Through most of the rest of the paper we will assume that there exists a mixed plane, that $\beta = q$ and that $\alpha > 1$. Although we only can hope for a classification if q is odd, quite a lot of the results will also be valid for q even. For a summary of the obtained results we refer to the conclusions in Section 3.

2. Towards a full classification

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper (α, q)-geometry fully embedded in $\mathrm{PG}(n,q)$, with $\alpha > 1$ and q odd. Assume that $\mathrm{PG}(n,q)$ contains a mixed plane σ. We determine what the restriction of $\mathcal{S}$ to σ can be. Let $\mathcal{K}$ be the set of points of σ through which there are q lines of $\mathcal{S}$ in σ.

Assume first that no pair of points of $\mathcal{K}$ lie on a line of $\mathcal{S}$. Let p be a point of $\mathcal{K}$. Then all the other points of $\mathcal{K}$ in σ lie on the unique line M_p through p that does not belong to $\mathcal{S}$. In the affine plane $\sigma\backslash M_p$, there are α lines of $\mathcal{S}$ through every point. So the restriction of $\mathcal{S}$ to the affine plane $\sigma\backslash M_p$ is a partial geometry. From [13] it follows that the points and lines of $\mathcal{S}$ in $\sigma\backslash M_p$ are the points of a net, or that $(\mathcal{L} \cap \sigma) \cup \{M_p\}$ is a hyperoval of the dual of the projective closure of $\sigma\backslash M_p$. Since q is odd, the hyperoval case cannot occur. Hence the points and lines of $\mathcal{S}$ in $\sigma\backslash M_p$ are the points of a net. The points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$ in σ, which is the projective closure of $\sigma\backslash M_p$, are $q+1-\alpha$ points of M_p, while the lines of $\mathcal{S}$ in σ are all lines of σ intersecting M_p in a point of $\mathcal{S}$. We therefore call the intersection of $\mathcal{S}$ with σ the *closure of a net*.

Assume next that σ contains two points of $\mathscr{K}$ that lie on a line of $\mathscr{S}$. Then either all points of σ belong to $\mathscr{S}$, or there is exactly one point of σ that does not belong to $\mathscr{S}$.

Assume that σ contains one point y of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. There are α or q lines of $\mathscr{S}$ in σ through every point of σ different from y. Now we look at the dual plane σ^{D}. In this plane, every line different from y^{D} contains either α or q points of the form L^{D}, with $L\in\mathscr{L}$. The line y^{D} contains 0 such points. Hence the set of points $\{L^{\mathrm{D}}|L\in\mathscr{L},L\subset\sigma\}$ is an affine (α,q)-set. The complement of this set is a $(0,q-\alpha)$-set, which is a maximal arc. Since we assumed that q is odd, a non-trivial maximal arc cannot exist (see [1]). Hence the set $\{L^{\mathrm{D}}|L\in\mathscr{L},L\subset\sigma\}^{\mathrm{C}}$ is either one point or all points of the affine plane $\sigma\backslash y^{\mathrm{D}}$. It cannot be the affine plane $\sigma\backslash y^{\mathrm{D}}$, for then $\alpha=0$, a contradiction with the assumptions. Hence $\{L^{\mathrm{D}}|L\in\mathscr{L},L\subset\sigma\}^{\mathrm{C}}$ is a point and this point is contained in $\sigma\backslash y^{\mathrm{D}}$. Dualising again we get that σ contains exactly one line that does not belong to $\mathscr{S}$. This line does not contain the point y of σ that does not belong to $\mathscr{S}$, since otherwise in the dual plane σ^{D} the point $\{L^{\mathrm{D}}|L\in\mathscr{L},L\subset\sigma\}^{\mathrm{C}}$ would lie on the line y^{D}, a contradiction with the previous. It immediately follows that $\alpha=q-1$.

Assume that every point of σ belongs to $\mathscr{S}$. Then clearly there are either q or α lines of $\mathscr{S}$ in σ through every point of σ. This implies that in the dual plane σ^{D} every line contains either q or α points that are of the form L^{D}, with $L\in\mathscr{L}$. Hence the set of points $\{L^{\mathrm{D}}|L\in\mathscr{L},L\subset\sigma\}$ is an (α,q)-set. The complement of this set is a $(1,q+1-\alpha)$-set. From [11, Theorem 12.17] it follows that $\{L^{\mathrm{D}}|L\in\mathscr{L},L\subset\sigma)\}^{\mathrm{C}}$ is a unital or a Baer subplane, and hence q is a square; moreover $q+1-\alpha=\sqrt{q}+1$, so $\alpha=q-\sqrt{q}$. Taking the complement and dualising again, we get that the lines of $\mathscr{S}$ in σ are either the lines intersecting a unital in $\sqrt{q}+1$ points, or the lines tangent to a Baer subplane.

We conclude that there are three possibilities for the restriction of $\mathscr{S}$ to a mixed plane σ:

(1) it is the closure of a net, as defined above,
(2) one point of σ does not belong to $\mathscr{S}$ and one line of σ not through this point does not belong to $\mathscr{S}$. In this case $\alpha=q-1$,
(3) all points of σ belong to $\mathscr{S}$. The lines of $\mathscr{S}$ in σ are either the lines intersecting a unital in $\sqrt{q}+1$ points or the lines tangent to a Baer subplane. In this case $\alpha=q-\sqrt{q}$.

We note that a certain type of mixed planes only exists if $\alpha=q-1$, while another type only exists for $\alpha=q-\sqrt{q}$. For this reason we will treat these two cases separately.

2.1. The case $\alpha=q-1$

For $\alpha=q-1$, there are no α-planes contained in $\mathrm{PG}(n,q)$. This follows from the fact that for a maximal arc, the degree n of the maximal arc has to divide q. Since $\alpha=q-1$, it is clear that α does not divide q.

In this case there are three different types of planes that contain an antiflag of $\mathcal{S}$:

- Type I are the q-planes.
- Type II are the planes in which the restriction of $\mathcal{S}$ is the closure of a net. Note that these planes contain two points that do not belong to $\mathcal{S}$, since $\alpha = q - 1$.
- Type III are the planes that contain one point p of $\mathrm{PG}(n,q)\backslash\mathcal{S}$ and one line not through p that does not belong to $\mathcal{S}$.

Remark 4. Let $\mathcal{S}$ be a proper $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, q even and $q > 2$. Then the planes containing an antiflag of $\mathcal{S}$ are precisely the planes of types I–III as defined above. This follows from the previous paragraphs, where we determined what the restriction of $\mathcal{S}$ to a mixed plane looks like. For q even the case that the lines of $\mathcal{S}$ are $q+1$ lines of a dual hyperoval and the points of $\mathcal{S}$ are the points on these lines, also does not occur. Indeed, for this kind of planes $\alpha = 1$ and we assumed that $\alpha > 1$. For this reason, the results in this section hold for any q, $q \neq 2$.

Lemma 5. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper $(q-1,q)$-geometry fully embedded in* $\mathrm{PG}(n,q)$, *with* $q \neq 2$. *Then every line of* $\mathrm{PG}(n,q)$ *contains* 0, 1, 2 *or* $q+1$ *points of* $\mathrm{PG}(n,q)\backslash\mathcal{S}$.

Proof. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with $q \neq 2$. Assume that there exists a line M in $\mathrm{PG}(n,q)$ that contains r points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$, with $r \notin \{0,1,2,q+1\}$. Then M contains at least three points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$ and at least one point w of $\mathcal{S}$.

Assume first that $r < q$. Then every plane spanned by M and a line of $\mathcal{S}$ through w contains an antiflag of $\mathcal{S}$ and hence it is of type I, II or III. Now it is clear that in a plane of type I, II or III every line contains 0, 1 or 2 points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$. This is a contradiction, as we assumed that $r \notin \{0,1,2\}$.

Assume next that $r = q$. A plane through M cannot contain an antiflag of $\mathcal{S}$. In particular, every plane through M contains at most one line of $\mathcal{S}$ through w. Since $t+1 > 1$, there are at least two lines of $\mathcal{S}$ through w in $\mathrm{PG}(n,q)$. Hence there exists a plane π through w that contains an antiflag of $\mathcal{S}$. We denote the lines through w in π by $L_1,\dots,L_{q+1}$. At least $\alpha = q-1$ of these lines belong to $\mathcal{S}$. So we may assume that $L_1,\dots,L_{q-1}$ are lines of $\mathcal{S}$. The planes $\langle M, L_i\rangle$, for $i = 1,\dots,q-1$, contain a line of $\mathcal{S}$, but they cannot contain an antiflag of $\mathcal{S}$. Hence each of them contains q^2 points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$, namely all the points not on the line L_i, for $i = 1,\dots,q-1$.

Now assume that $\langle M, \pi\rangle$ contains a point w' of $\mathcal{S}$, with $w' \notin \pi$. Let L be a line of $\mathcal{S}$ in π not through w. Then clearly the plane $\langle w', L\rangle$ contains an antiflag of $\mathcal{S}$. Hence $\langle w', L\rangle$ is of type I, II or III. This implies that $\langle w', L\rangle$ contains at most 2 points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$. However, $\langle w', L\rangle$ intersects each of the planes $\langle M, L_i\rangle$, for $i \in \{1,\dots,q-1\}$, in a line that contains q points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$, giving a contradiction. Hence $\langle M, \pi\rangle\backslash\pi$ contains no points of $\mathcal{S}$.

If $n = 3$, then the above gives a contradiction, as the points of $\mathcal{S}$ have to span $\mathrm{PG}(n,q)$. So we may assume that $n \geqslant 4$. Let z be a point of $\mathcal{S}$ that does not belong to $\langle M, \pi\rangle$. Then there exists a line $N \in \mathcal{L}$ through z that intersects $\langle M, \pi\rangle$ in a point.

Assume that there exists a point $w'' \in \mathscr{P}$ in the four-dimensional space $\langle N, M, \pi \rangle$ that is not contained in the three-dimensional space $\langle N, \pi \rangle$. Then $\langle w'', N \rangle$ contains q points of $\mathrm{PG}(n,q) \backslash \mathscr{S}$ on its intersection line with the three-dimensional space $\langle M, \pi \rangle$. Hence $\langle w'', N \rangle$ is a plane containing an antiflag of $\mathscr{S}$ and at least q points of $\mathrm{PG}(n,q) \backslash \mathscr{S}$. This is a contradiction, because every plane containing an antiflag of $\mathscr{S}$ has to be of type I, II or III. Hence all the points of $\mathscr{S}$ in the four-dimensional space $\langle N, M, \pi \rangle$ are contained in the three-dimensional space $\langle N, \pi \rangle$. If $n = 4$, then the above gives a contradiction, since the points of $\mathscr{S}$ have to span $\mathrm{PG}(n,q)$.

Now assume that $\Pi[m]$ is an m-dimensional subspace of $\mathrm{PG}(n,q)$, $m \geqslant 5$, such that $\langle M, N, \pi \rangle \subset \Pi[m]$ and such that all points of $\mathscr{S}$ in $\Pi[m]$ are contained in a hyperplane $\Upsilon[m-1]$ of $\Pi[m]$. If $n = m$ then we have found a contradiction, since the points of $\mathscr{S}$ have to span $\mathrm{PG}(n,q)$. So we may assume that $n > m$. Let u be a point of $\mathscr{S}$, $u \notin \Pi[m]$. Since $\mathscr{S}$ is connected, there exists a line N_u through u intersecting $\Pi[m]$ in a point. Let $\Gamma[m+1]$ be the $(m+1)$-dimensional subspace spanned by $\Pi[m]$ and N_u. We will prove that all points of $\mathscr{S}$ in $\Gamma[m+1]$ are contained in the m-dimensional space $\langle u, \Upsilon[m-1] \rangle$. Assume that there would be a point $u' \in \mathscr{P}$ that is contained in $\Gamma[m+1]$ but not in $\langle u, \Upsilon[m-1] \rangle$. Then the plane $\langle u', N_u \rangle$ intersects $\Pi[m]$ in a line containing q points that do not belong to $\mathscr{S}$. However $\langle u', N_u \rangle$ contains an antiflag of $\mathscr{S}$, so it is a plane of type I, II or III. This implies that $\langle u', N_u \rangle$ contains at most 2 points that do not belong to $\mathscr{S}$. This is a contradiction. It follows that all points of $\mathscr{S}$ in $\Gamma[m+1]$ are contained in the m-dimensional subspace $\langle N_u, \Upsilon[m-1] \rangle$.

Continuing in this way, after a finite number of steps we find that all the points of $\mathscr{S}$ in $\mathrm{PG}(n,q)$ are contained in an $(n-1)$-dimensional subspace. This is a contradiction, because we assumed that the points of $\mathscr{S}$ span $\mathrm{PG}(n,q)$. Hence every line of $\mathrm{PG}(n,q)$ contains 0, 1, 2 or $q+1$ points of $\mathrm{PG}(n,q) \backslash \mathscr{S}$. □

Corollary 6. *Let $\mathscr{S} = (\mathscr{P}, \mathscr{L}, \mathrm{I})$ be a proper $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with $q \neq 2$. Then every plane that contains a line of $\mathscr{S}$ is a plane of type* I, II *or* III.

Proof. From Lemma 5 we know that $\mathrm{PG}(n,q)$ contains no lines on which there are q points of $\mathrm{PG}(n,q) \backslash \mathscr{S}$. It follows that every plane through a line of $\mathscr{S}$ contains an antiflag of $\mathscr{S}$. So every plane through a line of $\mathscr{S}$ is of type I, II or III. □

Theorem 7. *Let $\mathscr{S} = (\mathscr{P}, \mathscr{L}, \mathrm{I})$ be a proper $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with $q \neq 2$. Assume that all the planes containing an antiflag of $\mathscr{S}$ are of type* I *or of type* III. *Then the points of $\mathrm{PG}(n,q) \backslash \mathscr{S}$ are the points of an $(n-2)$-dimensional subspace $\Pi[n-2]$. The lines of $\mathscr{S}$ are the lines skew to $\Pi[n-2]$ that do not belong to a partition Σ of the points of $\mathrm{PG}(n,q) \backslash \Pi[n-2]$ in r-dimensional spaces meeting $\Pi[n-2]$ in subspaces of dimension $r-2$, with $1 \leqslant r \leqslant n-2$. Further, such a partition exists for every $1 \leqslant r \leqslant n-2$, and gives a $(q-1,q)$-geometry.*

Proof. Let $\mathscr{S} = (\mathscr{P}, \mathscr{L}, \mathrm{I})$ be a proper $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with $q \neq 2$. Assume that all the planes containing an antiflag of $\mathscr{S}$ are of type I or of type III. From Lemma 5 we know that every line of $\mathrm{PG}(n,q)$ contains 0, 1, 2 or

$q+1$ points of $PG(n,q)\backslash\mathscr{S}$. Since we assumed that there are no planes of type II, there are no lines in $PG(n,q)$ that contain two points of $PG(n,q)\backslash\mathscr{S}$. It follows that the points of $PG(n,q)\backslash\mathscr{S}$ are the points of a subspace $\Pi[m]$ of $PG(n,q)$ of dimension m, $m\leqslant n-2$. Now let L be a line of $\mathscr{S}$. Then all points of $\mathscr{S}$ are contained in $\langle L,\Pi[m]\rangle$. Indeed, if a point $x\in\mathscr{P}$ would not be contained in $\langle L,\Pi[m]\rangle$, then the plane $\langle x,L\rangle$ would contain an antiflag of $\mathscr{S}$ and no point of $PG(n,q)\backslash\mathscr{S}$, a contradiction since every plane containing an antiflag of $\mathscr{S}$ is of type I or III. Hence all the points of $\mathscr{S}$ are contained in the $(m+2)$-dimensional space $\langle L,\Pi[m]\rangle$. Since the points of $\mathscr{S}$ span $PG(n,q)$, this proves that $m=n-2$.

Let $\mathscr{B}$ be the set of lines of $PG(n,q)$ that contain $q+1$ points of $\mathscr{S}$ but that do not belong to $\mathscr{S}$. There are $t+1$ lines of $\mathscr{S}$ through every point of $\mathscr{S}$ and there are $(q^{n-1}-1)/(q-1)$ lines through a point of $\mathscr{S}$ that contain a point of $\Pi[n-2]$. So the number of elements of $\mathscr{B}$ through a point of $\mathscr{S}$ equals $(q^n-q^{n-1})/(q-1)-t-1$ and hence it is constant.

If through every point of $\mathscr{S}$ in $PG(n,q)$ there is exactly one line of $\mathscr{B}$, then the elements of $\mathscr{B}$ are the lines of a line spread of $PG(n,q)\backslash\Pi[n-2]$.

If through a point z of $\mathscr{S}$ there are two lines N_1 and N_2 of $\mathscr{B}$, then the plane spanned by N_1 and N_2 cannot contain a line of $\mathscr{S}$, for otherwise this plane would contain an antiflag of $\mathscr{S}$, and a plane containing an antiflag of $\mathscr{S}$ cannot contain two lines of $\mathscr{B}$. So all lines of $\langle N_1,N_2\rangle$ not through the point $\langle N_1,N_2\rangle\cap\Pi[n-2]$, belong to $\mathscr{B}$. If all elements of $\mathscr{B}$ through z are contained in $\langle N_1,N_2\rangle$, then through every point of $PG(n,q)\backslash\mathscr{S}$ there has to be a plane containing all elements of $\mathscr{B}$ through this point and all these planes are disjoint or their intersection belongs to $\Pi[n-2]$. Hence the lines of $\mathscr{S}$ are the lines not contained in a partition Σ of the points of $PG(n,q)\backslash\Pi[n-2]$ in planes intersecting $\Pi[n-2]$ in a point.

If there is a line N_3 of $\mathscr{B}$ through z not contained in $\langle N_1,N_2\rangle$, then the three-dimensional space spanned by N_1, N_2 and N_3 contains no lines of $\mathscr{S}$. Indeed, assume that there would be a line L_z of $\mathscr{S}$ through z in $\langle N_1,N_2,N_3\rangle$. The plane $\langle N_1,N_2\rangle$ contains no lines of $\mathscr{S}$, so the plane $\langle L_z,N_3\rangle$ intersects $\langle N_1,N_2\rangle$ in a line of $\mathscr{B}$ through z. It follows that the plane $\langle L_z,N_3\rangle$ contains an antiflag of $\mathscr{S}$ and at least two lines of $\mathscr{B}$, a contradiction since every plane containing an antiflag of $\mathscr{S}$ is of type I or III. This implies that all lines through z in $\langle N_1,N_2,N_3\rangle$ are lines of $\mathscr{B}$. If there would be a line M of $\mathscr{S}$ in $\langle N_1,N_2,N_3\rangle$ not through z, then there would be 0 lines through z that intersect the line M, a contradiction since $\mathscr{S}$ is a $(q-1,q)$-geometry. This proves that $\langle N_1,N_2,N_3\rangle$ contains no lines of $\mathscr{S}$. If all elements of $\mathscr{B}$ through z are contained in $\langle N_1,N_2,N_3\rangle$ then through every point of $PG(n,q)\backslash\mathscr{S}$ there has to be a three-dimensional space that contains all line of $\mathscr{B}$ through this point, and all these three-dimensional spaces are disjoint or their intersection is contained in $\Pi[n-2]$. Hence the lines of $\mathscr{S}$ are the lines not contained in a partition Σ of the points of $PG(n,q)\backslash\Pi[n-2]$ into three-dimensional spaces intersecting $\Pi[n-2]$ in a line.

Now assume that $\Gamma[d]$ is a d-dimensional subspace of $PG(n,q)$ through z, $d\geqslant 4$, that contains no lines of $\mathscr{S}$. If all lines of $\mathscr{B}$ through z are contained in $\Gamma[d]$, then through every point of $PG(n,q)\backslash\mathscr{S}$ there has to be a d-dimensional space containing all elements of $\mathscr{B}$ through this point and all such d-dimensional spaces are disjoint or their intersection belongs to $\Pi[n-2]$. If there is a line N_d of $\mathscr{B}$ through z, N_d not contained in $\Gamma[d]$, then $\langle \Gamma[d], N_d \rangle$ contains no lines of $\mathscr{S}$. Indeed, by assumption $\Gamma[d]$ contains no lines of $\mathscr{S}$. If there would be a line L'_z of $\mathscr{S}$ through z, $L'_z \subset \langle \Gamma[d], N_d \rangle$, then the plane $\langle L'_z, N_d \rangle$ would intersect $\Gamma[d]$ in a line of $\mathscr{B}$. So the plane $\langle L'_z, N_d \rangle$ would contain a line of $\mathscr{S}$ and two lines of $\mathscr{B}$ through z. This is a contradiction, since every plane containing an antiflag of $\mathscr{S}$ is of type I or III. If there would be a line M' of $\mathscr{S}$ in $\langle \Gamma[d], N_d \rangle$, $z \notin M'$, then it would follow that $i(z, M') = 0$, a contradiction since $\mathscr{S}$ is a $(q-1, q)$-geometry. So $\langle \Gamma[d], N_d \rangle$ is a $(d+1)$-dimensional space through z that contain no lines of $\mathscr{S}$.

Continuing in this way, we get that all elements of $\mathscr{B}$ through z are contained in an r-dimensional space through z and that this space does not contain lines of $\mathscr{S}$, for $1 \leqslant r \leqslant n-1$. Since the number of elements of $\mathscr{B}$ through a point of $\mathscr{S}$ is a constant, it follows that the lines of $\mathscr{S}$ are the lines that do not belong to a partition Σ of the points of $\mathrm{PG}(n,q) \backslash \Pi[n-2]$ in r-dimensional spaces intersecting $\Pi[n-2]$ in $(r-2)$-spaces, for $1 \leqslant r \leqslant n-1$.

It remains to prove that such a partition Σ of r-dimensional spaces exists for each r, with $1 \leqslant r \leqslant n-2$.

If $r = n-1$, then Σ is the set of the $q+1$ $(n-1)$-dimensional spaces on $\Pi[n-2]$. So every line of $\mathrm{PG}(n,q)$ not belonging to $\mathscr{S}$ contains a point of $\mathrm{PG}(n,q) \backslash \mathscr{S}$. It follows that $\mathscr{S}$ is the partial geometry H_q^n [6,8], a contradiction since we assumed $\mathscr{S}$ to be proper. This proves that $r < n-1$.

If $r = 1$, then Σ is a partition of $\mathrm{PG}(n,q) \backslash \Pi[n-2]$ in lines. In other words, Σ is a partial spread of $\mathrm{PG}(n,q)$. In [3] a partial spread S of lines of $\mathrm{PG}(n,q)$ such that each line of S is skew to a given $(n-2)$-dimensional space is constructed as follows. Embed $\mathrm{PG}(n,q)$ in a $(2n-3)$-dimensional space $\mathrm{PG}(2n-3,q)$. In $\mathrm{PG}(2n-3,q)$ one can take a spread S' of $(n-2)$-dimensional spaces such that $\Pi[n-2] \in S'$. The elements of $S' \backslash \{\Pi[n-2]\}$ intersect $\mathrm{PG}(n,q)$ in a partial spread S of lines such that every point of $\mathrm{PG}(n,q) \backslash \Pi[n-2]$ is contained in a line of S and such that every element of S is skew to $\Pi[n-2]$. So for $r = 1$, the partition Σ is this partial spread S.

Assume that $2 \leqslant r \leqslant n-2$. Then every element of Σ intersects $\Pi[n-2]$ in an $(r-2)$-dimensional space. Indeed, every element of Σ contains elements of $\mathscr{B}$ which are skew to $\Pi[n-2]$. Now let $\Upsilon[r-2]$ be an $(r-2)$-dimensional subspace of $\Pi[n-2]$. Let $\Omega[n-r+1]$ be an $(n-r+1)$-dimensional space skew to $\Upsilon[r-2]$. Then $\Omega[n-r+1] \cap \Pi[n-2]$ is an $(n-r-1)$-dimensional space. In the same way as in the previous paragraph, we can take a partial spread S of lines of $\Omega[n-r+1]$ such that every element of S is skew to $\Omega[n-r+1] \cap \Pi[n-2]$ and such that every point of $\Omega[n-r+1] \backslash \Pi[n-2]$ belongs to an element of S. Now the set $\Sigma := \{\langle \Upsilon[r-2], M \rangle | M \in S\}$ is a partition of the points of $\mathrm{PG}(n,q) \backslash \Pi[n-2]$ in r-dimensional spaces. Hence also for $2 \leqslant r \leqslant n-2$, the partition Σ exists.

That such a partition gives rise to a $(q-1,q)$-geometry is easy to show. □

Remark 8. If $r \neq 1$, then the elements of the partition Σ of the points of $\mathrm{PG}(n,q) \backslash \Pi[n-2]$ are not necessarily disjoint. Indeed, it is possible that two different elements of Σ both contain a point p of $\Pi[n-2]$. Note that in the example given in the proof of the theorem, all elements of Σ intersect $\Pi[n-2]$ in the same $(r-2)$-dimensional space. If $r = 1$, then the elements of Σ are lines that are skew to $\Pi[n-2]$. Hence in the case $r=1$, the elements of Σ are two by two disjoint.

Lemma 9. *Let $\mathscr{S}$ be a $(q-1,q)$-geometry fully embedded in* $\mathrm{PG}(n,q)$, *for* $q \neq 2$. *Assume that* $\mathrm{PG}(n,q)$ *contains a plane of type* II. *Then the points of* $\mathrm{PG}(n,q)\backslash\mathscr{S}$ *are the points of two subspaces of* $\mathrm{PG}(n,q)$. *One of these subspaces has dimension $n-2$, the other one has dimension less than or equal to $n-2$.*

Proof. Let $\mathscr{S}$ be a $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, for $q \neq 2$. Let π be a plane of type II. Let y_1 and y_2 be the two points of π that do not belong to $\mathscr{S}$. Since $t+1$ is a constant, we know that $n>2$. The points of $\mathscr{S}$ span $\mathrm{PG}(n,q)$, so there is a point of $\mathscr{S}$ not contained in π. Let ρ be a plane through this point and through a line of $\mathscr{S}$ in π. Then ρ contains an antiflag of $\mathscr{S}$. Hence ρ is of type I, II or III. Now we will prove that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in the three-dimensional space $\langle\pi,\rho\rangle$ are the points of two subspaces of $\langle\pi,\rho\rangle$.

We denote the intersection point of the line $\langle y_1,y_2\rangle$ with the plane ρ by w. Since ρ intersects π in a line of $\mathscr{S}$, w is a point of $\mathscr{S}$. Let L be a line of $\mathscr{S}$ in ρ through w. By Corollary 6 we know that all planes through L contain an antiflag of $\mathscr{S}$. Hence in every plane through L there are one or two points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. This implies that $q+2 \leqslant |(\mathrm{PG}(n,q)\backslash\mathscr{S}) \cap \langle\pi,\rho\rangle| \leqslant 2q+2$.

Through the line $\langle y_1,y_2\rangle$ there are at least $q-1$ planes that intersect ρ in a line of $\mathscr{S}$. Hence at least $q-1$ planes through $\langle y_1,y_2\rangle$ are of type II. This implies that all the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle\pi,\rho\rangle$ are contained in at most two planes through $\langle y_1,y_2\rangle$.

Assume first that all the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle\pi,\rho\rangle$ are contained in one plane τ_1. From Lemma 5 it follows that every line of $\mathrm{PG}(n,q)$ contains 0, 1, 2 or $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. We will prove that τ_1 contains a line on which there are no points of $\mathscr{S}$. Assume therefore that every line of τ_1 contains a point of $\mathscr{S}$. Then every line of τ_1 contains 0, 1 or 2 points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. We proved above that there are at least $q+2$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle\pi,\rho\rangle$. It follows that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in τ_1 are the points of a hyperoval. Let L_{τ_1} be a line of τ_1 that contains no points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Then L_{τ_1} intersects π in a point of $\mathscr{S}$. Hence there exists a plane through L_{τ_1} and a line of $\mathscr{S}$ in π that contains an antiflag of $\mathscr{S}$ and no point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, a contradiction. Hence we may assume that τ_1 contains a line on which there are $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Now since on every line there are 0, 1, 2 or $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, and since $q+2 \leqslant |(\mathrm{PG}(n,q)\backslash\mathscr{S}) \cap \tau_1| \leqslant 2q+2$, we can conclude that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in τ_1 are either the points on one line together with an extra point, or the points on two intersecting lines.

Assume next that all points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle\pi,\rho\rangle$ are contained in two different planes τ_1 and τ_2, but not in one plane. We will prove that either τ_1 or τ_2 contains a line on which there is no point of $\mathscr{S}$. Assume that τ_1 contains no such line. Then from Lemma 5 it follows that every line in τ_1 contains 0, 1 or 2 points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. It follows that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in τ_1 are the points of a (subset of a) conic or a hyperoval. Hence τ_1 contains a line N on which there are $q+1$ points of $\mathscr{S}$. Clearly $N \notin \mathscr{L}$. All planes through N different from τ_1 intersect π in a line of $\mathscr{S}$. Hence all these planes contain an antiflag of $\mathscr{S}$. They have to be of type III, which means that they contain exactly one point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Since all points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ are contained in τ_1 and τ_2, it follows that τ_2 contains exactly $q+2$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Since every line in τ_2 contains 0, 1, 2 or $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, it follows that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in τ_2 are either the points of one line together with one extra point or are the points of a hyperoval. Now assume the latter case. Let $N\cap\tau_2$ be the point a. Then through a we can take a line N_a in τ_2 that contains no point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. It follows that the plane $\langle N,N_a\rangle$ contains no point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. However, in the three-dimensional space $\langle\pi,\rho\rangle$ the plane $\langle N,N_a\rangle$ has to intersect π in a line N_π. Since $\langle N,N_a\rangle$ does not contain points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, N_π belongs to $\mathscr{S}$. Hence $\langle N,N_a\rangle$ contains an antiflag of $\mathscr{S}$ and no point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, a contradiction since every plane containing an antiflag of $\mathscr{S}$ is of type I, II or III. Hence the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in τ_2 are the points on one line together with one extra point. This proves that either τ_1 or τ_2 contains a line on which there is no point of $\mathscr{S}$. Hence we may assume that the line $\langle y_1,y_3\rangle$ contains no points of $\mathscr{S}$, with $y_3\in\tau_1$, $y_1\neq y_3\neq y_2$. Now let y_4 be a point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in τ_2, $y_1\neq y_4\neq y_2$. Then every plane in $\langle\pi,\rho\rangle$ through the line $\langle y_2,y_4\rangle$ contains at least 3 points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, hence every such plane contains no lines of $\mathscr{S}$. Now let π_L be a plane through L that does not contain y_2 or y_4, with L again a line of $\mathscr{S}$ through w in ρ. Then π_L intersects the line $\langle y_2,y_4\rangle$ in a point z. All the lines through z in π_L are contained in some plane through $\langle y_2,y_4\rangle$, so there are no lines of $\mathscr{S}$ through z in π_L. This implies that $z\notin\mathscr{P}$, for if $z\in\mathscr{P}$, then there are 0 lines through z that intersect L in the plane π_L, a contradiction since $\mathscr{S}$ is a $(q-1,q)$-geometry. Hence the line $\langle y_2,y_4\rangle$ contains at least 3 points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. From Lemma 5 it follows that $\langle y_2,y_4\rangle$ contains $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. We conclude that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ are the points of two disjoint lines in $\langle\pi,\rho\rangle$. (Remark that there cannot be another point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle\pi,\rho\rangle$, since we proved that $q+2\leqslant|(\mathrm{PG}(n,q)\backslash\mathscr{S})\cap\langle\pi,\rho\rangle|\leqslant 2q+2$.)

Hence it follows that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in the three-dimensional space $\langle\pi,\rho\rangle$ are the points of two subspaces of $\mathrm{PG}(n,q)$ of dimension less than or equal to 1 and that at least one of these subspaces is a line. Since π was an arbitrary chosen type II plane and $\langle\pi,\rho\rangle$ an arbitrary chosen three-dimensional space through π and a point of $\mathscr{S}$ not in π, we proved that for every three-dimensional subspace of $\mathrm{PG}(n,q)$ through a plane of type II and a point of $\mathscr{S}$ not in this plane, the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in this three-dimensional subspace are the points of two subspaces of dimension less than or equal to 1, and that at least one of these subspaces has dimension 1.

Let $\Pi[m]$ be an m-dimensional subspace of $\mathrm{PG}(n,q)$, $m\geqslant 3$, such that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\Pi[m]$ are the points of two subspaces $\Upsilon[m-2]$ and $\Omega[r]$ of dimension resp. $m-2$ and r, with $0\leqslant r\leqslant m-2$, and such that $\langle\pi,\rho\rangle\subseteq\Pi[m]$. Let $\Pi'[m+1]$ be an $(m+1)$-dimensional subspace of $\mathrm{PG}(n,q)$ through $\Pi[m]$ and a point of $\mathscr{S}$ in $\mathrm{PG}(n,q)\backslash\Pi[m]$. Such a point exists because the points of $\mathscr{S}$ span $\mathrm{PG}(n,q)$. We will prove that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ contained in $\Pi'[m+1]$ are the points of two subspaces of $\Pi'[m+1]$ of dimension resp. $m-1$ and r, with $0\leqslant r\leqslant m-1$.

It immediately follows that $\Pi'[m+1]\backslash\Pi[m]$ contains a point y of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Indeed, assume that all the points of $\Pi'[m+1]\backslash\Pi[m]$ would belong to $\mathscr{S}$. Then a plane in $\Pi'[m+1]$ that intersects $\Pi[m]$ in a line of $\mathscr{S}$ contains no point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. This is a contradiction because we know that every plane that contains an antiflag of $\mathscr{S}$ is of type I, II or III. So not all points of $\Pi'[m+1]\backslash\Pi[m]$ belong to $\mathscr{S}$.

We will prove that there is an $(m-1)$-dimensional space through $\Upsilon[m-2]$ that contains no point of $\mathscr{S}$. Assume that the space $\langle y, \Upsilon[m-2]\rangle$ does contain a point v of $\mathscr{S}$. Then we will construct another point y_v such that $\langle y_v, \Upsilon[m-2]\rangle \neq \langle y, \Upsilon[m-2]\rangle$ and such that $\langle y_v, \Upsilon[m-2]\rangle$ does not contain a point of $\mathscr{S}$. Let $\langle y, v\rangle \cap \Upsilon[m-2]$ be the point $\tilde{y}$. Let L_v be a line of $\mathscr{S}$ in $\Pi[m]$. The plane $\langle v, L_v\rangle$ is a plane containing an antiflag of $\mathscr{S}$. So in $\langle v, L_v\rangle$ there are at least $q-1$ lines of $\mathscr{S}$ through v that intersect $\Pi[m]$. We call these lines $L_1,\ldots,L_{q-1}$. By Corollary 6, the plane $\langle L_1, y\rangle$ contains an antiflag of $\mathscr{S}$. It contains the points y and $\tilde{y}$ of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, so it is of type II. Hence $\langle y, v, L_v\rangle$ is a three-dimensional space that contains a type II plane. By the above, the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle y, v, L_v\rangle$ are either the points of 2 lines or one point and one line. The plane $\langle v, L_v\rangle$ contains one or two points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Let y_v be a point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle v, L_v\rangle$. Then either $\langle y, y_v\rangle$ or $\langle \tilde{y}, y_v\rangle$ is a line containing $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Remark that if $\langle v, L_v\rangle$ is of type II, then through both y and $\tilde{y}$ there is a line containing $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. The three-dimensional space $\langle y, v, L_v\rangle$ intersects $\Pi[m]$ in a plane containing an antiflag of $\mathscr{S}$. Now there are two possibilities.

The first possibility is that the three-dimensional space $\langle y, v, L_v\rangle$ is disjoint from $\Omega[r]\backslash\Upsilon[m-2]$. Then clearly the point $\{u\} = \langle y, y_v\rangle \cap \Pi[m]$ belongs to $\mathscr{S}$. Hence the line $\langle \tilde{y}, y_v\rangle$ is a line containing $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. There is a line L_u of $\mathscr{S}$ through u in $\Pi[m]$. Indeed, $\Pi[m]$ contains lines of $\mathscr{S}$, and a plane through u and a line of $\mathscr{S}$ in $\Pi[m]$ contains an antiflag of $\mathscr{S}$. In this plane there are lines of $\mathscr{S}$ through u. The plane $\langle y, L_u\rangle$ contains two points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, namely y and y_v. By Corollary 6 every plane through L_u in $\Pi[m]$ contains an antiflag of $\mathscr{S}$. So the plane $\langle y, L_u\rangle$ is a type II plane. Hence by the result of a previous paragraph, the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in every three-dimensional space through $\langle y, L_u\rangle$ and a point of $\Upsilon[m-2]$ are either the points of two lines or are the points of one line together with an extra point. Since $\langle y, y_v\rangle$ does not contain $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, this implies that for each point $\tilde{z}$ of $\Upsilon[m-2]$ either $\langle \tilde{z}, y\rangle$ or $\langle \tilde{z}, y_v\rangle$ is a line containing no points of $\mathscr{S}$.

Now we will prove that for every $\tilde{z} \in \Upsilon[m-2]$, the line $\langle y, \tilde{z}\rangle$ is not a line that contains no points of $\mathscr{S}$. Let $N_{\tilde{y}}$ be a line through $\tilde{y}$ in $\Upsilon[m-2]$. We look at the plane $\langle y, N_{\tilde{y}}\rangle$. Since $\langle y, \tilde{y}\rangle$ contains points of $\mathscr{S}$, Lemma 5 implies that there can be at most one line through y in $\langle y, N_{\tilde{y}}\rangle$ that contains no points of $\mathscr{S}$. Hence there are at least q points $z_1 = \tilde{y}, z_2, \ldots, z_q$ on $N_{\tilde{y}}$ for which the lines $\langle y, z_1\rangle, \ldots, \langle y, z_q\rangle$ contain points of $\mathscr{S}$. By the previous paragraph, we know that for every $\tilde{z} \in \Upsilon[m-2]$ either $\langle y, \tilde{z}\rangle$ or $\langle y_v, \tilde{z}\rangle$ is a line that does not contain points of $\mathscr{S}$. Since $\langle y, z_1\rangle, \ldots, \langle y, z_q\rangle$ are all lines that contain points of $\mathscr{S}$, we may conclude that $\langle y_v, z_1\rangle, \ldots, \langle y_v, z_q\rangle$ are lines that contain no points of $\mathscr{S}$. Now we look at the plane $\langle y_v, N_{\tilde{y}}\rangle$. It contains the lines $\langle y_v, z_1\rangle, \ldots, \langle y_v, z_q\rangle$. By Lemma 5, this implies that $\langle y_v, N_{\tilde{y}}\rangle$ contains no points of $\mathscr{S}$. Since $N_{\tilde{y}}$ was an arbitrary line through $\tilde{y}$ in $\Upsilon[m-2]$, it follows that every point of $\langle y_v, \Upsilon[m-2]\rangle$ does not belong to $\mathscr{S}$. Hence $\langle y_v, \Upsilon[m-2]\rangle$ is an $(m-1)$-dimensional subspace of $\Pi'[m+1]$ that contains no points of $\mathscr{S}$.

The second possibility is that the three-dimensional space $\langle y, v, L_v\rangle$ contains a point $\tilde{p}$ of $\Omega[r]\backslash\Upsilon[m-2]$. In this case $\tilde{y}$ is a point of $\Upsilon[m-2]\backslash\Omega[r]$. Since $\langle y, \tilde{y}\rangle$ and $\langle \tilde{y}, \tilde{p}\rangle$

both contain points of $\mathscr{S}$, the line $\langle y, \tilde{p}\rangle$ has to be a line that contains $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Now assume that $r<m-2$. Let $\rho_{\tilde{y}}$ be a plane through $\tilde{y}$ that is disjoint from $\Omega[r]$ and that contains points of $\mathscr{S}$. We may assume that $\rho_{\tilde{y}}$ contains no line of $\mathscr{L}$. Indeed, if $\rho_{\tilde{y}}$ would contain a line of $\mathscr{S}$, then the proof would follow from the proof of the first possibility above, after replacing $\langle v, L_v\rangle$ by $\rho_{\tilde{y}}$. Through a point of $\mathscr{S}$ in $\rho_{\tilde{y}}$ we can take a line $L_{\tilde{y}}$ of $\mathscr{S}$, with $L_{\tilde{y}} \subset \Pi[m]$. Then either the three-dimensional space spanned by $\rho_{\tilde{y}}$ and $L_{\tilde{y}}$ intersects $\Omega[r]$ in one point p, or $\langle\rho_{\tilde{y}}, L_{\tilde{y}}\rangle$ is disjoint from $\Omega[r]$. Now we will prove that there exists a plane ρ' through $\tilde{y}$ that contains a line of $\mathscr{S}$ and no point of $\Omega[r]$. Either the plane $\langle L_{\tilde{y}}, \tilde{y}\rangle$ does not contain a point of $\Omega[r]$ or it contains the unique point p of $\Omega[r]$ contained in the three-dimensional space spanned by $\rho_{\tilde{y}}$ and $L_{\tilde{y}}$. Assume that $\langle L_{\tilde{y}}, \tilde{y}\rangle$ does contain p. Let $\langle p, \tilde{y}\rangle \cap L_{\tilde{y}}$ be the point x. Let π_L be a plane through $L_{\tilde{y}}$ in the three-dimensional space spanned by $\rho_{\tilde{y}}$ and $L_{\tilde{y}}$ such that $\tilde{y}\notin\pi_L$. By Lemma 5, π_L is a plane that contains an antiflag of $\mathscr{S}$. Hence, through every point of $L_{\tilde{y}}$, there are at least $q-1$ lines of $\mathscr{S}$ contained in π_L. Let L_1 be a line of $\mathscr{S}$ in π_L that does not contain the point x. Then the plane spanned by L_1 and $\tilde{y}$ clearly does not contain the point p. So in any case there exists a plane ρ' through $\tilde{y}$ that contains a line of $\mathscr{L}$ and no point of $\Omega[r]$. The three-dimensional space $\langle y, \rho'\rangle$ contains the point v, hence there is a plane through v and a line of $\mathscr{S}$ in ρ' contained in it. Now, the result follows from the proof of the first possibility, replacing the plane $\langle v, L_v\rangle$ by this plane through v and a line of $\mathscr{S}$.

Now assume that $r=m-2$. Take a line through $\tilde{y}$ and a point $\tilde{p}'$ of $\Omega[r]\backslash \Upsilon[m-2]$. Then $\langle\tilde{y}, \tilde{p}'\rangle$ contains a point w' of $\mathscr{S}$. Through w' we can take a line $L_{w'}$ of $\mathscr{L}$. The plane $\langle L_{w'}, \tilde{y}\rangle$ contains two points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, hence it is of type II. The plane $\langle L_{w'}, y\rangle$ contains an antiflag of $\mathscr{S}$ because of Corollary 6. By a result in a previous paragraph of this proof, the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in the three-dimensional space $\langle L_{w'}, \tilde{y}, y\rangle$ are either the points of two lines or are the points of one line together with an extra point. Since $\langle\tilde{y}, y\rangle$ and $\langle\tilde{y}, \tilde{p}'\rangle$ both contain points of $\mathscr{S}$, this implies that $\langle y, \tilde{p}'\rangle$ is a line containing $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Since $\tilde{p}'$ was arbitrary chosen in $\Omega[r]\backslash \Upsilon[m-2]$, every line through y and a point of $\Omega[r]\backslash \Upsilon[m-2]$ is a line containing no point of $\mathscr{S}$. Using Lemma 5, we get that the space $\langle y, \Omega[r]\rangle$ cannot contain a point of $\mathscr{S}$. Since by assumption $r=m-2$, the space $\langle y, \Omega[r]\rangle$ is $(m-1)$-dimensional.

Hence we proved that $\Pi'[m+1]$ contains an $(m-1)$-dimensional subspace $\langle\bar{y}, \Upsilon[m-2]\rangle$ of points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, with $\bar{y}\in\Pi'[m+1]\backslash\Pi[m]$. It remains to prove that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\Pi'[m+1]$ are the points of two subspaces of dimension less than or equal to $m-1$. If all points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ are contained in $\Omega[r]\cup\langle\bar{y}, \Upsilon[m-2]\rangle$, then the lemma is proved. So assume that there is a point $\bar{z}$ of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ that is not contained in $\Omega[r]\cup\langle\bar{y}, \Upsilon[m-2]\rangle$. Let $\langle\bar{y},\bar{z}\rangle\cap\Pi[m]$ be the point $\bar{x}$.

Assume first that $\bar{x}$ is a point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Then the line $\langle\bar{y},\bar{z}\rangle$ contains 3 points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. From Corollary 6 it follows that $\langle\bar{y},\bar{z}\rangle$ contains no points of $\mathscr{S}$. This implies that every line through $\bar{z}$ and a point of $\Upsilon[m-2]\backslash\Omega[r]$ contains $q-1$ points of $\mathscr{S}$. Indeed, let $\overline{y_1}$ be a point of $\Upsilon[m-2]\backslash\Omega[r]$. Since $\bar{x}\in\Omega[r]\backslash \Upsilon[m-2]$, the line $\langle\overline{y_1},\bar{x}\rangle$ contains $q-1$ points of $\mathscr{S}$. Now we look at the plane $\langle\bar{y},\bar{x},\overline{y_1}\rangle$. It contains the lines $\langle\bar{y},\bar{x}\rangle$ and $\langle\bar{y},\overline{y_1}\rangle$, on which there are no points of $\mathscr{S}$. It contains also the line $\langle\overline{y_1},\bar{x}\rangle$ on which there are $q-1$ points of $\mathscr{S}$. By Lemma 5 we know that the plane $\langle\bar{y},\bar{x},\overline{y_1}\rangle$ cannot contain other points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Hence the line $\langle\overline{y_1},\bar{z}\rangle$ contains

$q-1$ points of $\mathscr{S}$. Since $\overline{y_1}$ was an arbitrary point of $\Upsilon[m-2]\backslash\Omega[r]$, all lines through $\bar{z}$ and a point of $\Upsilon[m-2]\backslash\Omega[r]$ contain $q-1$ points of $\mathscr{S}$. Now, let $\bar{p}$ be a point of $\Omega[r]\backslash\Upsilon[m-2]$. Let $N_{\bar{p}}$ be a line through $\bar{p}$ that intersects $\Upsilon[m-2]$. Through a point of $\mathscr{S}$ on $N_{\bar{p}}$ we can take a line $L_{\bar{p}}$ of $\mathscr{S}$ in $\Pi[m]$. The plane $\langle L_{\bar{p}}, N_{\bar{p}}\rangle$ is of type II. The plane $\langle L_{\bar{p}}, \bar{z}\rangle$ contains an antiflag of $\mathscr{S}$. By the result of a previous paragraph of this proof, we know that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in the three-dimensional space spanned by these two planes have to be either the points of two lines or the points of one line together with an extra point. We have already shown that for every $x_1\in\Upsilon[m-2]\backslash\Omega[r]$ the line $\langle\bar{z},x_1\rangle$ contains points of $\mathscr{S}$. Hence $\langle\bar{z},\bar{p}\rangle$ is a line containing no point of $\mathscr{S}$. Since $\bar{p}$ was an arbitrary point of $\Omega[r]\backslash\Upsilon[m-2]$, we get that $\langle\bar{z},\Omega[r]\rangle$ contains no point of $\mathscr{S}$ (here we also use Lemma 5). Since $\bar{y}\in\langle\bar{z},\Omega[r]\rangle$, we get that the subspaces $\langle\bar{y},\Upsilon[m-2]\rangle$ and $\langle\bar{y},\Omega[r]\rangle$ contain no point of $\mathscr{S}$. If all points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ are contained in $\langle\bar{y},\Omega[r]\rangle$ and $\langle\bar{y},\Upsilon[m-2]\rangle$, then the lemma is proved. Assume therefore that there is a point $\bar{z}'$ of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ that does not belong to $\langle\bar{y},\Upsilon[m-2]\rangle$ and $\langle\bar{y},\Omega[r]\rangle$. Clearly the line $\langle\bar{y},\bar{z}'\rangle$ intersects $\Pi[m]$ in a point $\bar{x}'$ of $\mathscr{S}$. Through $\bar{x}'$ we can take a line $L_{\bar{x}'}$ of $\mathscr{S}$ contained in $\Pi[m]$. Let $\pi_{\bar{x}'}$ be a plane through $L_{\bar{x}'}$ and a point $\bar{p}_1$ of $\Omega[r]\backslash\Upsilon[m-2]$. Then $\pi_{\bar{x}'}$ contains two points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, namely the point $\overline{p_1}$ and a point $\overline{y_2}$ of $\Upsilon[m-2]\backslash\Omega[r]$. Hence it is a plane of type II. The plane $\langle L_{\bar{x}'},\bar{y}\rangle$ is also of type II. Hence, by the result of a previous paragraph of this proof, the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in the three-dimensional space spanned by $\langle L_{\bar{x}'},\bar{y}\rangle$ and $\pi_{\bar{x}'}$ are either the points of two lines or the points of one line together with an extra point. However, this three-dimensional space contains the two lines $\langle\bar{y},\overline{y_2}\rangle$ and $\langle\bar{y},\overline{p_1}\rangle$ that contain no point of $\mathscr{S}$, and the point $\bar{z}'$. This is a contradiction. So the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\Pi'[m+1]$ are the points of two subspaces of dimension at most $m-1$.

Assume next that $\bar{x}$ is a point of $\mathscr{S}$. Let $L_{\bar{x}}$ be a line of $\mathscr{S}$ through $\bar{x}$ in $\Pi[m]$. The plane $\langle\bar{y},L_{\bar{x}}\rangle$ contains two points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, namely $\bar{y}$ and $\bar{z}$. So it is of type II. Using Corollary 6, we see that all planes through $L_{\bar{x}}$ in $\Pi[m]$ contain an antiflag of $\mathscr{S}$. Let $\bar{p}'$ be a point of $\Omega[r]\backslash\Upsilon[m-2]$. Then the plane $\langle\bar{p}',L_{\bar{x}}\rangle$ meets $\Upsilon[m-2]\backslash\Omega[r]$ in a point $\bar{w}$. Hence $\langle\bar{p}',L_{\bar{x}}\rangle$ is a plane of type II. The three-dimensional space spanned by $\bar{p}'$ and the plane $\langle\bar{y},L_{\bar{x}}\rangle$ contains two planes through $L_{\bar{x}}$ of type II, namely $\langle\bar{y},L_{\bar{x}}\rangle$ and $\langle\bar{p}',L_{\bar{x}}\rangle$. By the above, the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in this three-dimensional space are then the points of two lines or the points of one line together with an extra point. Since $\langle\bar{y},\bar{z}\rangle$ contains points of $\mathscr{S}$, and $\langle\bar{y},\bar{w}\rangle$ contains no points of $\mathscr{S}$, the line $\langle\bar{z},\bar{p}'\rangle$ has to contain $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Since $\bar{p}'$ was arbitrary chosen in $\Omega[r]\backslash\Upsilon[m-2]$, we know that every line through $\bar{z}$ and a point of $\Omega[r]\backslash\Upsilon[m-2]$ contains no point of $\mathscr{S}$. Lemma 5 then tells us that $\langle\bar{z},\Omega[r]\rangle$ contains no point of $\mathscr{S}$. If all points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ are contained in $\langle\bar{z},\Omega[r]\rangle$ or $\langle\bar{y},\Upsilon[m-2]\rangle$, then the lemma is proved. Assume therefore that there is a point $\bar{z}''$ of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ that does not belong to $\langle\bar{y},\Upsilon[m-2]\rangle$ or $\langle\bar{z},\Omega[r]\rangle$. If $\langle\bar{y},\bar{z}''\rangle$ intersects $\Pi[m]$ in a point of $\Omega[r]$, then in the same way as in the above, we find a contradiction. Hence we may assume that the line $\langle\bar{y},\bar{z}''\rangle$ intersects $\Pi[m]$ in a point $\bar{x}''\in\mathscr{P}$. In the same way as we did for $\bar{z}$, one can prove that $\langle\bar{z}'',\Omega[r]\rangle$ contains no points of $\mathscr{S}$. Now $\langle\bar{z},\Omega[r]\rangle$ and $\langle\bar{z}'',\Omega[r]\rangle$ span an $(r+2)$-dimensional space $\Omega'[r+2]$. The space $\Omega'[r+2]$ intersects $\langle\bar{y},\Upsilon[m-2]\rangle$ in a space of dimension at least r. Hence there are points of $\langle\bar{y},\Upsilon[m-2]\rangle\backslash\Omega[r]$ contained in $\Omega'[r+2]$.

Suppose first that the $(r+1)$-dimensional space $\Omega'[r+2] \cap \Pi[m]$ is disjoint from $\Upsilon[m-2]\backslash\Omega[r]$. Let $\bar{y}_3$ be a point of $\langle \bar{y}, \Upsilon[m-2]\rangle\backslash\Omega[r]$ in $\Omega'[r+2]$. Then by the assumption $\bar{y}_3$ does not belong to $\Pi[m]$. A line through $\bar{y}_3$ in $\Omega'[r+2]$ disjoint from $\Omega[r]$ contains at most q points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, since it intersects $\Pi[m]$ in a point of $\mathscr{S}$. Moreover this line contains at least 3 points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, namely the point $\bar{y}_3$ and its intersection points with the spaces $\langle \bar{z}, \Omega[r]\rangle$ and $\langle \bar{z}', \Omega[r]\rangle$. This is a contradiction because of Lemma 5.

Suppose next that the $(r+1)$-dimensional space $\Omega'[r+2] \cap \Pi[m]$ is not disjoint from $\Upsilon[m-2]\backslash\Omega[r]$. Then there exists a point $\bar{y}_4$ of $\Upsilon[m-2]\backslash\Omega[r]$ in $\Omega'[r+2]$. Then every line through $\bar{y}_4$ in the space $\Omega'[r+2]$ that does not contain a point of $\Omega[r]$, contains at least 3 points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. By Lemma 5, every such line has to contain $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Now let $M_{\bar{y}_4}$ be a line through $\bar{y}_4$ that intersects $\Omega[r]\backslash\Upsilon[m-2]$ in a point. Then $M_{\bar{y}_4}$ contains $q-1$ points of $\mathscr{S}$. Take a plane through $M_{\bar{y}_4}$ in $\Omega'[r+2]$, that intersects $\Pi[m]$ in the line $M_{\bar{y}_4}$. The points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in this plane are the points of an affine plane together with two extra points. By Lemma 5, such a plane cannot exist. This proves that all points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ have to be contained in $\langle \bar{y}, \Upsilon[m-2]\rangle$ or in $\langle \bar{z}, \Omega[r]\rangle$. So we proved that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\Pi'[m+1]$ are the points of two subspaces of dimension at most $m-1$ and that one of the subspaces has dimension $m-1$.

Next, we take an $(m+2)$-dimensional subspace $\Gamma[m+2]$ containing $\Pi'[m+1]$ and a point of $\mathscr{S}$ not in $\Pi'[m+1]$. In the same way as above, we can prove that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\Gamma[m+2]$ are the points of two subspaces, one of dimension m and the other of dimension less than or equal to m.

After a finite number of steps, we obtain in this way that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\mathrm{PG}(n,q)$ are the points of two subspaces of $\mathrm{PG}(n,q)$ of dimension at most $n-2$ and that one of these subspaces has dimension $n-2$. □

Lemma 10. *Let $\mathscr{S}$ be a proper $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, $q \neq 2$. Assume that $\mathrm{PG}(n,q)$ contains a plane of type* II. *Let $\Pi[n-2]$ and $\Omega[r]$ be the two subspaces of points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, for $0 < r \leqslant n-2$. Let $\mathscr{B}$ be the set of lines of $\mathrm{PG}(n,q)$ that contain $q+1$ points of $\mathscr{S}$ and that do not belong to $\mathscr{S}$. Assume that $\mathscr{B} \neq \emptyset$. Then the elements of $\mathscr{B}$ in $\mathrm{PG}(n,q)$ through a point $u \in \mathscr{P}$ are contained in an l-dimensional subspace $\Psi[l]$ of $\mathrm{PG}(n,q)$, for $r+1 \leqslant l \leqslant n-2$, such that $\langle u, \Omega[r]\rangle \subseteq \Psi[l]$. Moreover $\Psi[l]$ contains no lines of $\mathscr{S}$.*

Proof. Let $\mathscr{S}$ be a proper $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, for $q \neq 2$. Then from Lemma 9 it follows that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ are the points of two subspaces $\Pi[n-2]$ and $\Omega[r]$ of dimension $n-2$ resp. r, for $0 < r \leqslant n-2$. Let $\mathscr{B}$ be the set of lines of $\mathrm{PG}(n,q)$ that contain $q+1$ points of $\mathscr{S}$ but do not belong to $\mathscr{S}$. Assume that $\mathscr{B} \neq \emptyset$.

Let M_1 be a line of $\mathscr{B}$ in $\mathrm{PG}(n,q)$. Let $u \in M_1$. We will prove that the subspace $\langle M_1, \Omega[r]\rangle$ contains no lines of $\mathscr{S}$. Assume therefore that $\langle M_1, \Omega[r]\rangle$ contains a line $L \in \mathscr{L}$. From Corollary 6 we know that every plane through L contains an antiflag of $\mathscr{S}$. Let y be a point of $\Omega[r]\backslash\Pi[n-2]$. Then the plane $\langle L, y\rangle$ contains an antiflag of $\mathscr{S}$ and two points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, namely y and its intersection point with $\Pi[n-2]$. Hence it is of type II. This implies that it does not contain a line of $\mathscr{B}$.

Assume first that $\Omega[r] \cap \Pi[n-2]$ is non-empty. Then $\Omega[r] \cap \Pi[n-2]$ is $(r-1)$-dimensional or $(r-2)$-dimensional. In the $(r+2)$-dimensional space $\langle M_1, \Omega[r]\rangle$, the plane $\langle L, y\rangle$ intersects the space $\langle M_1, \Omega[r] \cap \Pi[n-2]\rangle$ in a point or a line. This implies that there exists a point v in $\langle L, y\rangle$, $v \neq y$ and $v \notin L$, such that the plane $\langle M_1, v\rangle$ contains a point of $\Omega[r] \backslash \Pi[n-2]$. Hence $\langle v, M_1\rangle$ contains the line M_1 of $\mathscr{B}$ and two points of $\mathrm{PG}(n,q) \backslash \mathscr{S}$. It follows that there are no lines of $\mathscr{S}$ contained in the plane $\langle v, M_1\rangle$.

Let N be a line through v in the plane $\langle v, M_1\rangle$ not through a point of $\mathrm{PG}(n,q) \backslash \mathscr{S}$. Then N is a line of $\mathscr{B}$. Now we look at the three-dimensional space spanned by N and $\langle L, y\rangle$. It intersects $\Omega[r]$ in a line through y. Since y is a point of $\Omega[r] \backslash \Pi[n-2]$, we can choose a point y' on this line that also belongs to $\Omega[r] \backslash \Pi[n-2]$. The plane $\langle y', N\rangle$ then contains two points of $\mathrm{PG}(n,q) \backslash \mathscr{S}$ and the line N of $\mathscr{B}$. Hence it cannot contain a line of $\mathscr{S}$. However, $\langle y', N\rangle$ intersects $\langle y, L\rangle$ in a line, since these two planes belong to a three-dimensional space. This line does not contain the point y, since $y \notin \langle y', N\rangle$. If this line contains no point of $\Pi[n-2]$, then it is a line of $\mathscr{L}$. In that case we found a contradiction, since $\langle y', N\rangle$ cannot contain a line of $\mathscr{S}$. If this line contains a point of $\Pi[n-2]$, then we replace y' in the previous argument by a point y'' of $\Omega[r] \backslash \Pi[n-2]$ on the line $\langle y, y'\rangle$, $y \neq y'' \neq y'$. We get a plane $\langle y'', N\rangle$ that contains two points of $\mathrm{PG}(n,q) \backslash \mathscr{S}$, the line N of $\mathscr{B}$ and a line of $\mathscr{S}$ on its intersection with the plane $\langle L, y\rangle$. This is a contradiction. We conclude that $\langle M_1, \Omega[r]\rangle$ cannot contain a line of $\mathscr{S}$.

Assume next that $\Omega[r] \cap \Pi[n-2] = \emptyset$. Then $\Omega[r]$ is a point or a line. If $\Omega[r]$ is a point, then the plane $\langle M_1, \Omega[r]\rangle$ contains two points of $\mathrm{PG}(n,q) \backslash \mathscr{S}$ and the line M_1 of $\mathscr{B}$. Hence it does not contain a line of $\mathscr{S}$. If $\Omega[r]$ is a line, then $\langle M_1, \Omega[r]\rangle$ is a three-dimensional space intersecting $\Pi[n-2]$ in a line. Hence this three-dimensional space contains two lines with $q+1$ points of $\mathrm{PG}(n,q) \backslash \mathscr{S}$. Every plane through M_1 contained in it then clearly contains 2 points of $\mathrm{PG}(n,q) \backslash \mathscr{S}$. Hence every line in $\langle M_1, \Omega[r]\rangle$ that intersects M_1 does not belong to $\mathscr{S}$. This implies that $\langle M_1, \Omega[r]\rangle$ contains no line of $\mathscr{S}$. Indeed, every plane in $\langle M_1, \Omega[r]\rangle$ that contains two points of $\mathrm{PG}(n,q) \backslash \mathscr{S}$ contains a line of $\mathscr{B}$ and hence it cannot contain a line of $\mathscr{S}$. We conclude that also in this case $\langle M_1, \Omega[r]\rangle$ cannot contain a line of $\mathscr{S}$.

So we proved that $\langle M_1, \Omega[r]\rangle$ does not contain a line of $\mathscr{S}$. Now we will use induction. Assume that $\Upsilon[d]$ is a d-dimensional subspace of $\mathrm{PG}(n,q)$ containing u and $\Omega[r]$ in which there are no lines of $\mathscr{S}$, for $r+2 \leqslant d \leqslant n-2$. If all elements of $\mathscr{B}$ through u are contained in $\Upsilon[d]$, then the lemma is proved, since u was an arbitrary point of $\mathscr{S}$. So assume that there is a line M_2 of $\mathscr{B}$ through u that is not contained in $\Upsilon[d]$. We will prove that the $(d+1)$-dimensional space $\langle \Upsilon[d], M_2\rangle$ cannot contain a line of $\mathscr{S}$.

From the first part of the proof, it follows that the $(r+2)$-dimensional space $\langle \Omega[r], M_2\rangle$ contains no lines of $\mathscr{S}$. Let $\Gamma[r+2]$ be an $(r+2)$-dimensional subspace of $\Upsilon[d]$ that contains u and $\Omega[r]$. Then $\Gamma[r+2]$ intersects $\langle M_2, \Omega[r]\rangle$ in the $(r+1)$-dimensional space $\langle u, \Omega[r]\rangle$. Define $\Lambda[r+3] = \langle \Gamma[r+2], M_2\rangle$. We will prove that $\Lambda[r+3]$ contains no lines of $\mathscr{S}$. Assume therefore that $\Lambda[r+3]$ contains a line L' of $\mathscr{S}$.

The space $\Pi[n-2]$ intersects $\Lambda[r+3]$ in an $(r+1)$-dimensional space, and it intersects $\Gamma[r+2]$ and $\langle M_2, \Omega[r]\rangle$ in an r-dimensional space. There are $(q^{r+2}-1)/(q-1)$ planes through L' in $\Lambda[r+3]$. At most $2(q^{r+1}-1)/(q-1)$ of these planes contain a point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ that is contained in $\langle M_2, \Omega[r]\rangle$ resp. in $\Gamma[r+2]$. Hence at most $3(q^{r+1}-1)/(q-1)$ planes through L' in $\Lambda[r+3]$ contain a point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ on their intersection line with $\Gamma[r+2]$ or with $\langle M_2, \Omega[r]\rangle$. It follows that there are $(q^{r+2}-1)/(q-1)-3(q^{r+1}-1)/(q-1)$ planes through L' in $\Lambda[r+3]$ that intersect both $\Gamma[r+2]$ and $\langle M_2, \Omega[r]\rangle$ in a line of $\mathscr{B}$. Since $(q^{r+2}-1)/(q-1)-3(q^{r+1}-1)/(q-1)>0$, there exists a plane containing an antiflag of $\mathscr{S}$ and two elements of $\mathscr{B}$, a contradiction. This proves that $\Lambda[r+3]$ cannot contain a line of $\mathscr{S}$.

Now let $\Gamma'[r+3]$ be an $(r+3)$-dimensional subspace of $\Upsilon[d]$ through $\Gamma[r+2]$. Then $\Gamma'[r+3]$ intersects $\Lambda[r+3]$ in an $(r+2)$-dimensional subspace. Let $\Lambda'[r+4]$ be the $(r+4)$-dimensional subspace spanned by $\Lambda[r+3]$ and $\Gamma'[r+3]$. Assume that $\Lambda'[r+4]$ contains a line L'' of $\mathscr{S}$. There are $(q^{r+3}-1)/(q-1)$ planes through L'' in $\Lambda'[r+4]$. At most $(q^{r+2}-1)/(q-1)+(q^{r+1}-1)/(q-1)$ of these planes contain a point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ that is contained in $\Lambda[r+3]$, resp. in $\Gamma'[r+3]$. Hence at most $2(q^{r+2}-1)/(q-1)+(q^{r+1}-1)/(q-1)$ planes through L'' in $\Lambda'[r+4]$ contain a point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ on their intersection line with $\Gamma'[r+3]$ or with $\Lambda[r+3]$. It follows that there are $(q^{r+3}-1)/(q-1)-2(q^{r+2}-1)/(q-1)-(q^{r+1}-1)/(q-1)$ planes through L'' in $\Lambda'[r+4]$ that intersect both $\Gamma'[r+3]$ and $\Lambda[r+3]$ in a line of $\mathscr{B}$. Since $(q^{r+3}-1)/(q-1)-2(q^{r+2}-1)/(q-1)-(q^{r+1}-1)/(q-1)>0$, there exists a plane containing an antiflag of $\mathscr{S}$ and two elements of $\mathscr{B}$, a contradiction. This proves that $\Lambda'[r+4]$ cannot contain a line of $\mathscr{S}$. Continuing in this way, we get after a finite number of steps that the $(d+1)$-dimensional space $\langle \Upsilon[d], M_2\rangle$ cannot contain a line of $\mathscr{S}$.

Using induction on the dimension d, we get that all elements of $\mathscr{B}$ through p are contained in an l-dimensional subspace $\Psi[l]$ of $\mathrm{PG}(n,q)$ through p and $\Omega[r]$, for $r+2 \leqslant l \leqslant n-2$ and that this subspace contains no lines of $\mathscr{S}$. □

Remark 11. In the previous lemma for every point u of $\mathscr{S}$ there is a subspace $\Psi[l]$ containing u. Remark that the dimension l of $\Psi[l]$ is not necessarily the same for all points u of $\mathscr{S}$.

Theorem 12. *Let $\mathscr{S}$ be a proper $(q-1,q)$-geometry fully embedded in* $\mathrm{PG}(n,q)$, *for $q \neq 2$. Assume that there is a plane of type* II. *Then the points of* $\mathrm{PG}(n,q)\backslash\mathscr{S}$ *are the points of two subspaces $\Pi[n-2]$ and $\Omega[r]$ of* $\mathrm{PG}(n,q)$, *for $1 \leqslant r \leqslant n-2$, with $\Omega[r]\cap\Pi[n-2]$ an $(r-2)$-dimensional space. The lines of $\mathscr{S}$ are either all lines of* $\mathrm{PG}(n,q)$ *that contain $q+1$ points of $\mathscr{S}$, or they are the lines not contained in a partition of the points of $\mathscr{S}$ in d-dimensional spaces pairwise intersecting in $\Omega[r]$. A necessary and sufficient condition for such a partition to exist is that $(d-r)|(n-r)$ and that $n-2 \geqslant d \geqslant r+2$. Further, if $(d-r)|(n-r)$ and $n-2 \geqslant d \geqslant r+2$, then this partition gives a $(q-1,q)$-geometry.*

Proof. Let $\mathscr{S}$ be a proper $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, for $q \neq 2$. Assume that there is a plane of type II. In Lemma 9 we proved that the points of

$\mathscr{S}$ are the points of PG(n,q) not contained in two subspaces $\Pi[n-2]$ and $\Omega[r]$ of PG(n,q), with $0 \leqslant r \leqslant n-2$. Now we want to determine which lines belong to $\mathscr{S}$. Let $\mathscr{B}$ be the set of lines of PG(n,q) that contain $q+1$ points of $\mathscr{S}$ but that do not belong to $\mathscr{S}$. We distinguish two cases.

(1) Assume that $\Omega[r]$ intersects $\Pi[n-2]$ in an $(r-1)$-dimensional space, for $r \geqslant 0$. Then $\langle \Omega[r], \Pi[n-2] \rangle$ is an $(n-1)$-dimensional space. We denote it by $\Upsilon[n-1]$. Through a point of $\mathscr{S}$ contained in $\Upsilon[n-1]$, exactly $(q^{n-1}-1)/(q-1)$ lines contain a point of PG$(n,q)\backslash\mathscr{S}$. Through a point of $\mathscr{S}$ not contained in $\Upsilon[n-1]$, there are $(q^{n-1}-1)/(q-1)+(q^{r+1}-1)/(q-1)-(q^r-1)/(q-1)=(q^{n-1}-1)/(q-1)+q^r$ lines that contain a point of PG$(n,q)\backslash\mathscr{S}$. So it is clear that $\mathscr{B} \neq \emptyset$, since otherwise the number of lines of $\mathscr{S}$ through a point would not be a constant.
Now we count the number of elements of $\mathscr{B}$ through a point $v \in \mathscr{P}$. From Lemma 10 it follows that the elements of $\mathscr{B}$ through v are contained in some d-dimensional subspace $\Psi[d]$ containing v and $\Omega[r]$. It is clear that $\Psi[d] \not\subset \Upsilon[n-1]$, because otherwise there would be no elements of $\mathscr{B}$ through the points of $\mathscr{S}$ in $\Psi[d]$, and then $t+1$ would not be constant. Hence $\Upsilon[n-1]$ meets $\Psi[d]$ in a hyperplane of $\Psi[d]$. If $v \in \Upsilon[n-1]$, then there are $(q^d-1)/(q-1)-(q^{d-1}-1)/(q-1)=q^{d-1}$ elements of $\mathscr{B}$ through v. If $v \notin \Upsilon[n-1]$, then there are $(q^d-1)/(q-1)-(q^{d-1}-1)/(q-1)-(q^{r+1}-1)/(q-1)+(q^r-1)/(q-1)=q^{d-1}-q^r$ elements of $\mathscr{B}$ through v. Now, since $t+1$ is a constant, if through a point of $\Upsilon[n-1]$ there are c elements of $\mathscr{B}$, then through a point not contained in $\Upsilon[n-1]$ there are $c-q^r$ elements of $\mathscr{B}$. This implies that the dimension of the subspace of elements of $\mathscr{B}$ has to be the same for every point of $\mathscr{S}$. Hence the subspaces containing the elements of $\mathscr{B}$ through the points of $\mathscr{S}$ are the elements of a partition Σ of the points of $\mathscr{S}$, and every element of Σ has the same dimension d. Clearly $r+2 \leqslant d \leqslant n-2$.
There are $q^n+q^{n-1}-q^r$ points of $\mathscr{S}$ in PG(n,q). Let $\Psi[d]$ be an arbitrary element of Σ. We count the number of elements of $\mathscr{P}$ that are contained in $\Psi[d]$. We know that $\Psi[d]$ is a d-dimensional subspace of PG(n,q) that intersects $\Pi[n-2]$ in a $(d-2)$-dimensional space. Indeed, $\Psi[d]$ contains lines of $\mathscr{B}$, and so the dimension of $\Pi[n-2] \cap \Psi[d]$ cannot be more than $d-2$. Hence the points of $\mathscr{S}$ in $\Psi[d]$ are all points of $\Psi[d]$ that are not contained in $\Omega[r]$ and neither in $\Pi[n-2] \cap \Psi[d]$. So we get that there are

$$\frac{q^{d+1}-1}{q-1}-\frac{q^{d-1}-1}{q-1}-\frac{q^{r+1}-1}{q-1}+\frac{q^r-1}{q-1},$$

or thus $q^d+q^{d-1}-q^r$ points of $\mathscr{S}$ in $\Psi[d]$. Since $\Psi[d]$ was an arbitrary chosen element of $\mathscr{S}$, there are $q^d+q^{d-1}-q^r$ points of $\mathscr{S}$ contained in every element of Σ. Hence

$$|\Sigma| = \frac{q^n+q^{n-1}-q^r}{q^d+q^{d-1}-q^r}. \tag{1}$$

Every element of Σ intersects $\Upsilon[n-1]$ in a $(d-1)$-dimensional space. There are $q^{n-1}-q^r$ points of $\mathscr{S}$ in $\Upsilon[n-1]$. There are $q^{d-1}-q^r$ points of $\mathscr{S}$ in the

$(d-1)$-dimensional intersection of an element of Σ with $\Upsilon[n-1]$. It follows that

$$|\Sigma| = \frac{q^{n-1}-q^r}{q^{d-1}-q^r}. \tag{2}$$

From (1) and (2) it follows that $d=n$. This implies that there are no lines of $\mathscr{S}$ in $\mathrm{PG}(n,q)$, a contradiction. Hence if $\Omega[r]$ intersects $\Pi[n-2]$ in an $(r-1)$-dimensional space, then $\mathscr{S}$ cannot be a proper $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$.

(2) Assume that $\Omega[r]$ intersects $\Pi[n-2]$ in an $(r-2)$-dimensional space, with $r \geqslant 1$. Then $\langle \Omega[r], \Pi[n-2]\rangle = \mathrm{PG}(n,q)$. Let v be a point of $\mathscr{S}$. Then we count the number of lines through v on which there are points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. The $(r+1)$-dimensional subspace $\langle v, \Omega[r]\rangle$ intersects $\Pi[n-2]$ in an $(r-1)$-dimensional space. Hence there are $(q^r-1)/(q-1)-(q^{r-1}-1)/(q-1)=q^{r-1}$ lines through v that contain 2 points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. All the other lines through v contain at most one point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. It follows that there are $(q^{n-1}-1)/(q-1)+(q^{r+1}-1)/(q-1)-(q^{r-1}-1)/(q-1)-q^{r-1}=(q^{n-1}-1)/(q-1)+q^r$ lines through v on which there are points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Since v was an arbitrary point of $\mathscr{S}$, this is valid for every point of $\mathscr{S}$. The number of elements of $\mathscr{B}$ through a point of $\mathscr{S}$ equals the total number of lines through this point minus $t+1$ minus the number of lines on which there are points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. This implies that the number of elements of $\mathscr{B}$ through a point of $\mathscr{S}$ is a constant.

If $\mathscr{B}=\emptyset$, then $t+1$ is a constant. Hence in this case $\mathscr{S}$ is a proper $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, and every line containing $q+1$ points of $\mathscr{S}$ belongs to $\mathscr{S}$.

If $\mathscr{B}\neq\emptyset$, then from Lemma 10 it follows that the elements of $\mathscr{B}$ through a point of $\mathscr{S}$ are contained in a d-dimensional subspace $\Psi[d]$ through v and $\Omega[r]$. Since we proved that the number of elements of $\mathscr{B}$ through a point of $\mathscr{S}$ is a constant, it follows that the subspaces containing the elements of $\mathscr{B}$ through the points of $\mathscr{S}$ are the elements of a partition Σ of the points of $\mathscr{S}$, and that every element of Σ has the same dimension d.

There are $q^n-q^{n-1}-q^r-q^{r-1}$ points of $\mathscr{S}$ in $\mathrm{PG}(n,q)$. As in the first part of the proof, we can count that there are $q^d+q^{d-1}-q^r-q^{r-1}$ points of $\mathscr{S}$ contained in an element of Σ. Now the remainder of the division of $q^n-q^{n-1}-q^r-q^{r-1}$ by $q^d+q^{d-1}-q^r-q^{r-1}$ equals $q^{n-cd+cr}+q^{n-cd+cr-1}-q^r-q^{r-1}$, where $c\geqslant 0$ is a positive integer. This remainder will be equal to 0 if and only if $(d-r)|(n-d)$ or thus $(d-r)|(n-r)$. Moreover if $(d-r)|(n-r)$, then the partition Σ always exists. Indeed, let $\Gamma[n-r-3]$ be an $(n-r-3)$-dimensional subspace of $\Pi[n-2]$ disjoint from $\Omega[r]\cap\Pi[n-2]$. Then $\langle \Omega[r], \Gamma[n-r-3]\rangle$ is an $(n-2)$-dimensional space. Since $\Omega[r]$ is contained in $\langle \Omega[r], \Gamma[n-r-3]\rangle$, the intersection of $\langle \Omega[r], \Gamma[n-r-3]\rangle$ and $\Pi[n-2]$ in an $(n-4)$-dimensional space. Let $\Pi^*[3]$ be a three-dimensional space skew to this $(n-4)$-dimensional space. Then $\Pi[n-2]$ and $\langle \Omega[r], \Gamma[n-r-3]\rangle$ intersect $\Pi^*[3]$ each in a line M_1 resp. M_2 and these two lines are disjoint. In $\Pi^*[3]$ it is possible to take a third line M_3 disjoint from M_1 and from M_2. Then the $(n-r-1)$-dimensional space $\langle M_3, \Gamma[n-r-3]\rangle$ is skew to $\Omega[r]$ and it intersects

$\Pi[n-2]$ in $\Gamma[n-r-3]$. Now let $\Lambda[d-r-3]$ be a $(d-r-3)$-dimensional subspace of $\Gamma[n-r-3]$ and let $\Lambda'[n-d+1]$ be an $(n-d+1)$-dimensional subspace of $\langle M_3, \Gamma[n-r-3]\rangle$ skew to $\Lambda[d-r-3]$. Then $\Lambda'[n-d+1]$ intersects $\Pi[n-2]$ in an $(n-d-1)$-dimensional space. Indeed, $\Lambda'[n-d+1] \cap \Pi[n-2]$ is contained in $\Gamma[n-r-3]$ so if its dimension would be greater than or equal to $n-d$, then it would intersect $\Lambda[d-r-3]$ in a point, a contradiction because we have chosen $\Lambda'[n-d+1]$ skew to $\Lambda[d-r-3]$. Now by [3] there exists a partial spread Σ_1 of lines of $\Lambda'[n-d+1]\backslash \mathrm{PG}(n,q)$. Let Σ_2 be the set of $(d-r-1)$-dimensional spaces spanned by $\Lambda[d-r-3]$ and a line of Σ_1. Then the elements of the partition Σ are the d-dimensional spaces spanned by an element of Σ_2 and $\Omega[r]$.
Hence the lines of $\mathcal{S}$ are all the lines that contain $q+1$ points of $\mathcal{S}$ or they are the lines not contained in a partition Σ of the points of $\mathcal{S}$, where each element of Σ is d-dimensional and contains $\Omega[r]$. This partition exists if and only if $(d-r)|(n-r)$, and always gives a geometry. □

2.2. *The case* $\alpha = q - \sqrt{q}$

Assuming q to be odd, there exists no non-trivial maximal arc in a Desarguesian plane [1], i.e. there cannot be $q-\sqrt{q}$-planes contained in $\mathrm{PG}(n,q)$. As in the previous section, we distinguish three types of planes that contain an antiflag of $\mathcal{S}$.

- Type I are the q-planes.
- Type II are the planes in which the restriction of $\mathcal{S}$ is the closure of a net. Note that such planes contain exactly $\sqrt{q}+1$ points that do not belong to $\mathcal{S}$.
- Type III are the planes in which all points are points of $\mathcal{S}$ and lines of $\mathcal{S}$ are the secant lines to a unital or the tangent lines to a Baer subplane.

Remark 13. Let $\mathcal{S}$ be a proper $(q-\sqrt{q}, q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, for q even, q a square, $q \neq 4$. Then every plane containing an antiflag of $\mathcal{S}$ is a plane of type I, II or III as defined above. This follows in the same way as in the case $\alpha = q-1$. Note that also in this case α-planes cannot exist. Indeed, $q-\sqrt{q}$ does not divide q, so a maximal arc of degree $q-\sqrt{q}$ does not exist. However, not everything what follows will be valid for q even, so we will mention explicitly q even or odd.

Lemma 14. *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *be a proper* $(q-\sqrt{q}, q)$*-geometry fully embedded in* $\mathrm{PG}(n,q)$*, with* q *a square,* $q \neq 4$*. Every line of* $\mathrm{PG}(n,q)$ *contains* 0, 1, $\sqrt{q}+1$ *or* $q+1$ *points of* $\mathrm{PG}(n,q)\backslash\mathcal{S}$.

Proof. One proves this lemma in the same way as Lemma 5. □

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper $(q-\sqrt{q}, q)$-geometry fully embedded in a subspace $\mathrm{PG}(m,q)$ of $\mathrm{PG}(n,q)$. Let $\Pi[n-m-1]$ be an $(n-m-1)$-dimensional subspace of $\mathrm{PG}(n,q)$ skew to $\mathrm{PG}(m,q)$. We use the following notation: $\Pi[n-m-1]\mathcal{S}$ is the cone with vertex $\Pi[n-m-1]$, projecting the $(q-\sqrt{q}, q)$-geometry $\mathcal{S}$. Now define

$\mathscr{S}^* = (\mathscr{P}^*, \mathscr{L}^*, \mathrm{I}^*)$ to be the following incidence structure: $\mathscr{P}^*$ is the set of points of the cone $\Pi[n-m-1]\mathscr{S}$ that are not contained in the vertex $\Pi[n-m-1]$, $\mathscr{L}^*$ is the set of lines that are contained in a plane $\langle x, L\rangle$, for any $x \in \Pi[n-m-1]$ and any $L \in \mathscr{L}$, and that do not contain x, I^* is the restriction of the incidence of $\mathrm{PG}(n,q)$ to $\mathscr{S}^*$. Then clearly also $\mathscr{S}^*$ is a proper $(q-\sqrt{q}, q)$-geometry fully embedded in $\mathrm{PG}(n,q)$. Such a $(q-\sqrt{q}, q)$-geometry $\mathscr{S}^*$ will be called *degenerate*. It suffices to classify all non-degenerate proper $(q-\sqrt{q}, q)$-geometries to obtain a full classification.

Lemma 15. *Let $\mathscr{S}$ be a proper non-degenerate $(q-\sqrt{q}, q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with q an odd square. If there is a line that contains $\sqrt{q}+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, then there is no line containing $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$.*

Proof. Let $M_{\sqrt{q}+1}$ be a line on which there are $\sqrt{q}+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. If $\mathrm{PG}(n,q)$ contains no line on which there are $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, then the lemma is proved. So we may assume that $\mathrm{PG}(n,q)$ contains such a line. From Lemma 14 it follows that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ form a $(0,1,\sqrt{q}+1,q+1)$-set. We will prove that $\mathscr{S}$ has to be degenerate. A plane through $M_{\sqrt{q}+1}$ that contains an antiflag of $\mathscr{S}$ is a plane of type II. Let ρ_1 and ρ_2 be two planes of type II through $M_{\sqrt{q}+1}$.

Assume first that $\langle\rho_1,\rho_2\rangle$ contains two points $y_1, y_2 \in \mathrm{PG}(n,q)\backslash\mathscr{S}$, such that $\langle y_1, y_2\rangle$ is skew to $M_{\sqrt{q}+1}$. The line $\langle y_1, y_2\rangle$ contains exactly $\sqrt{q}+1$ points $y_1, y_2, \ldots, y_{\sqrt{q}+1}$ of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ (Lemma 14). From Lemma 16 it follows that each plane $\langle y_i, M_{\sqrt{q}+1}\rangle$ $(i=1,\ldots,\sqrt{q}+1)$ intersects $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in the points of a unital or a Baer subplane, or in $\sqrt{q}+1$ lines through a point. Hence $\langle\rho_1,\rho_2\rangle$ contains at least $(\sqrt{q}+1)q+\sqrt{q}+1=(q+1)(\sqrt{q}+1)$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Now let L be a line of $\mathscr{S}$ in ρ_1. Every plane through L contains at most $\sqrt{q}+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. So there are at most $(q+1)(\sqrt{q}+1)$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ contained in $\langle\rho_1,\rho_2\rangle$. From these two inequalities we get that $\langle\rho_1,\rho_2\rangle$ contains exactly $(q+1)(\sqrt{q}+1)$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ and hence that every plane $\langle y_i, M_{\sqrt{q}+1}\rangle$ intersects $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in a Baer subplane. This implies that no line contains $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. From Lemma 17 it now follows that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle\rho_1,\rho_2\rangle$ are the points of a 3-dimensional Baer subspace in $\langle\rho_1,\rho_2\rangle$. Assume next that all points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle\rho_1,\rho_2\rangle$ lie on the line $M_{\sqrt{q}+1}$. Then there is a plane in $\langle\rho_1,\rho_2\rangle$ that contains a line of $\mathscr{B}$, a point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ and a line of $\mathscr{S}$. This is a contradiction, as every plane containing an antiflag of $\mathscr{S}$ is of type I, II or III. Hence this case does not occur. Assume finally that the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle\rho_1,\rho_2\rangle$ are contained in a plane $\langle y, M_{\sqrt{q}+1}\rangle$ through $M_{\sqrt{q}+1}$ and a point y of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, $y \notin M_{\sqrt{q}+1}$. Then there cannot be a line of $\mathscr{B}$ or a line of $\mathscr{S}$ contained in $\langle y, M_{\sqrt{q}+1}\rangle$ (Lemma 16). Hence the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle\rho_1,\rho_2\rangle$ are the points of a $(1,\sqrt{q}+1,q+1)$-set contained in the plane $\langle y, M_{\sqrt{q+1}}\rangle$. From [12], Section 23.5, it follows that such a set is a unital, a Baer subplane, or a set of $\sqrt{q}+1$ concurrent lines. Hence the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle\rho_1,\rho_2\rangle$ are the points of a $(1,\sqrt{q}+1)$-set or a cone with base a $(1,\sqrt{q}+1)$-set.

We conclude that the 3-dimensional space $\langle\rho_1,\rho_2\rangle$ intersects $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in a 3-dimensional Baer subspace $\mathrm{PF}(3,\sqrt{q})$ or in a (cone on a) $(1,\sqrt{q}+1)$-set contained in a plane. If $n=3$, then since we assumed that there is a line containing $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, it follows that the points of $\mathrm{PG}(n,s)\backslash\mathscr{S}$ are the points of $\sqrt{q}+1$

concurrent lines contained in a plane. It easily follows that there cannot be a line of $\mathcal{B}$, as otherwise there would be a plane containing a line of $\mathcal{S}$, a line of $\mathcal{B}$ and a point of $\mathrm{PG}(n,q)\backslash\mathcal{S}$, a contradiction. It follows that $\mathcal{S}$ is degenerate.

Now assume that $n \neq 3$ and that in the m-dimensional subspace $\Gamma[m]$ of $\mathrm{PG}(n,q)$, the points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$ span $\mathrm{PG}(n,q)$ and are the points of a Baer subspace or of a cone with vertex a subspace and base a Baer subspace of dimension a least 3; or the points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$ are the points of a $(1, \sqrt{q}+1, q+1)$-set in a hyperplane $\Upsilon[m-1]$ of $\mathrm{PG}(n,q)$. In the last case the points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$ are the points of a (possibly degenerate) Hermitian variety or a cone with vertex a subspace and base a Baer subplane or a Baer subline. If $m=n$, then the points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$ cannot be contained in $\Upsilon[m-1]$, as in this case there cannot be lines of $\mathcal{B}$ and hence the number of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ cannot be a constant. So if $m=n$, then the lemma is proved. Assume now $m<n$. Let $\Lambda[m+1]$ be an $(m+1)$-dimensional subspace of $\mathrm{PG}(n,q)$ through $\Gamma[m]$ and a point $u \in \mathcal{P}, u \notin \Gamma[m]$. We will determine how $\Lambda[m+1]$ intersects $\mathrm{PG}(n,q)\backslash\mathcal{S}$. It follows immediately that $\Lambda[m+1]$ contains no lines of $\mathcal{B}$, for otherwise one can construct a plane containing a line of $\mathcal{B}$, a line of $\mathcal{S}$ and a point of $\mathrm{PG}(n,q)\backslash\mathcal{S}$, which would give a contradiction. Thus every plane of a $\Lambda[m+1]$ containing an antiflag of $\mathcal{S}$, has to contain a point of $\mathrm{PG}(n,q)\backslash\mathcal{S}$. It follows that either the points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$ in $\Lambda[m+1]$ are the points of a $(1, \sqrt{q}+1, q+1)$-set in a hyperplane of $\Lambda[m+1]$, or these points span $\mathrm{PG}(n,q)$. In the first case, from [12], Theorem 23.5.1, the result follows. So lets consider the second case. Let $y \in \Lambda[m+1]\backslash\Gamma[m]$ be a point of $\mathrm{PG}(n,q)\backslash\mathcal{S}$. If no line through y contains $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$, then no line of $\Lambda[m+1]$ can contain $q+1$ points of $\mathrm{PG}(n,q)/\mathcal{S}$. Indeed, from [12], Theorem 23.5.1, it follows that a plane through y and such a line would intersect $\mathrm{PG}(n,q)\backslash\mathcal{S}$ in $\sqrt{q}+1$ concurrent lines and hence contain a line through y on which there are $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$. Hence the points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$ in $\Lambda[m+1]$ are the points of a Baer subspace. If there is a line $\langle y,z\rangle$ through y containing $q+1$ points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$, then it follows that there is a point $\bar{y} \in \Lambda[m+1]\backslash\Gamma[m]$ such that the points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$ are the points of a cone with vertex $\bar{y}$ and base the points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$ in $\Gamma[m]$. So also in this case, the result follows. Continuing in this way, after a finite number of steps, the result of the theorem follows. □

Lemma 16. [1] *Let $\mathcal{K}$ be a $(0,1,\sqrt{q}+1)$-set with respect to lines in* $\mathrm{PG}(n,q)$, $n \geqslant 2$. *Then $\mathcal{K}$ intersects each plane of* $\mathrm{PG}(n,q)$ *in* 0 *points, a singleton, a unital,* $\sqrt{q}+1$ *collinear points or a Baer subplane.*

Proof. Let π be a plane of $\mathrm{PG}(n,q)$. If π contains no line on which there are 0 points of $\mathcal{K}$, then the points of $\mathcal{K}$ in π form a $(1,\sqrt{q}+1)$-set in π. Hence $\mathcal{K}$ intersects π in a unital or a Baer subplane (see [11, Theorem 12.17]). So we may assume that π contains a line L on which there are no points of $\mathcal{K}$. We denote by m_0 (resp. m_1 and $m_{\sqrt{q}+1}$) the number of lines of π that contain no (resp. 1 and $\sqrt{q}+1$) points of $\mathcal{K}$.

[1] This Lemma has been proved by Ueberberg [15].

Then it follows that

$$m_0 + m_1 + m_{\sqrt{q}+1} = q^2 + q + 1,$$
$$m_1 + (\sqrt{q}+1)m_{\sqrt{q}+1} = (q+1)|\mathcal{K}|,$$
$$\sqrt{q}(\sqrt{q}+1)m_{\sqrt{q}+1} = |\mathcal{K}|(|\mathcal{K}|-1). \quad (3)$$

The first equation we obtain by counting all lines of π, the second by counting pairs (p, M), with $p \in \mathcal{K} \cap \pi$, M a line of π and $p \in M$, the third equation we get by counting triples (p, p', M), with $p, p' \in \mathcal{K} \cap \pi$, $p \neq p'$, M a line of π and $p, p' \in M$.

Let x be a point of $\mathcal{K} \cap \pi$. Counting the points of $\mathcal{K}$ on the lines through x in π, we get that $|\mathcal{K}| = a\sqrt{q}+1$, where a is the number of lines through x in π on which there are $\sqrt{q}+1$ points of $\mathcal{K}$. It is clear that $0 \leqslant a \leqslant q+1$. If $a = 0$, then $\mathcal{K}$ is a single point. Assume from now on that $a > 0$.

From (3) it follows that

$$m_{\sqrt{q}+1} = a^2 + \frac{a-a^2}{\sqrt{q}+1},$$
$$m_1 = aq\sqrt{q} + a\sqrt{q} + q + 1 - a^2\sqrt{q} - a,$$
$$m_0 = q^2 - aq\sqrt{q} - a\sqrt{q} + a^2\sqrt{q} + a - a^2 - \frac{a-a^2}{\sqrt{q}+1}.$$

Since m_0 and $m_{\sqrt{q}+1}$ have to be integers, we get that $\sqrt{q}+1 \mid a^2 - a$.

Also, it is clear that m_0 has to be greater than or equal to 0. From the above we get that $m_0(\sqrt{q}+1) = q(a-q)(a-\sqrt{q}-1)$, so either $a \leqslant 1+\sqrt{q}$, or $a \geqslant q$. Since by assumption $1 \leqslant a \leqslant q+1$, there are the following cases to consider:

(1) $a = q+1$. It is easy to check that in this case the condition $\sqrt{q}+1 \mid a^2 - a$ is not satisfied.
(2) $a = q$. Then $|\mathcal{K}| = q\sqrt{q}+1$ and from (3) it follows that $m_0 = 0$, giving a contradiction, as we assumed that π contains a line that contains no points of $\mathcal{K}$.
(3) $a = \sqrt{q}+1$. Then $|\mathcal{K}| = q+\sqrt{q}+1$. From (3) it follows that $m_0 = 0$, again giving a contradiction, since there is a line in π that does not contain points of $\mathcal{K}$.
(4) $a = 1$. Then the plane π contains $\sqrt{q}+1$ collinear points of $\mathcal{K}$.
(5) $2 \leqslant a \leqslant \sqrt{q}$. By definition, a is equal to the number of lines of π through a point x of $\mathcal{K}$ containing $\sqrt{q}+1$ points of $\mathcal{K}$. Now take a line through 2 points of $\mathcal{K}$ in π that does not contain x. This line contains at least 2 points of $\mathcal{K}$ and at most $\sqrt{q}$ such points, since at most $\sqrt{q}$ lines through x contain points of $\mathcal{K}$. But the existence of such a line contradicts our hypotheses. Hence this case does not occur.

This proves that every plane in $\mathrm{PG}(n,q)$ intersects $\mathcal{K}$ in 0 points, one point, $\sqrt{q}+1$ collinear points, a Baer subplane or a unital. □

Lemma 17. [2] *Let $\mathcal{K}$ be a $(0,1,\sqrt{q}+1)$-set with respect to lines in* $\mathrm{PG}(n,q)$, $n \geqslant 3$ and $q \neq 4$. *Assume that there is a line that contains $\sqrt{q}+1$ points of $\mathcal{K}$. Then $\mathcal{K}$ is a Baer subspace of* $\mathrm{PG}(n,q)$, *a Baer subspace of some subspace of* $\mathrm{PG}(n,q)$, $\sqrt{q}+1$ *collinear points, or a unital in a plane of* $\mathrm{PG}(n,q)$.

[2] This Lemma has been proved by Ueberberg [15].

Proof. From Lemma 16 it follows that every plane of $PG(n,q)$ intersects $\mathscr{K}$ in 0 points, a singleton, $\sqrt{q}+1$ collinear points, a Baer subplane or a unital.

We will prove that if there exists a plane π_U in $PG(n,q)$ that intersects $\mathscr{K}$ in a unital, then $\mathscr{K} \subset \pi_U$. So let π_U be a plane intersecting $\mathscr{K}$ in a unital and let p be a point of $\mathscr{K}$ that does not belong to π_U.

Suppose that $\langle p, \pi_U \rangle$ contains a line L that is exterior to $\mathscr{K}$. Let $L \cap \pi_U = \{x\}$. Let M be a line of π_U through x that intersects $\mathscr{K}$ in $\sqrt{q}+1$ points. The plane $\langle L, M \rangle$ contains $\sqrt{q}+1$ collinear points of $\mathscr{K}$ and a line exterior to $\mathscr{K}$. From Lemma 16 it follows that $\langle L, M \rangle$ intersects $\mathscr{K}$ in $\sqrt{q}+1$ collinear points. Hence $p \notin \langle L, M \rangle$. Through x there are q lines exterior to $\mathscr{K}$ that are contained in $\langle L, M \rangle$. We look at the planes through $\langle p, x \rangle$ in $\langle p, \pi_U \rangle$. There are $q - \sqrt{q}$ lines through x in π_U that contain $\sqrt{q}+1$ points of $\mathscr{K}$. There are q lines through x in $\langle L, M \rangle$ that are exterior to $\mathscr{K}$. Hence we can take a plane ρ through $\langle p, x \rangle$ that contains an exterior line to $\mathscr{K}$ and $\sqrt{q}+1$ collinear points of $\mathscr{K}$ on its intersection line with π_U. Hence ρ contains a line that is exterior to $\mathscr{K}$ and at least $\sqrt{q}+2$ points of $\mathscr{K}$. This is a contradiction, by Lemma 16. This proves that $\langle p, \pi_U \rangle$ cannot contain a line on which there are no points of $\mathscr{K}$. Hence $\mathscr{K} \cap \langle p, \pi_U \rangle$ is a $(1, \sqrt{q}+1)$-set. In particular it is also a $(1, \sqrt{q}+1, q+1)$-set that does not contain lines with $q+1$ points of $\mathscr{K}$. From [12, Theorem 23.5.1], it follows that $\mathscr{K} \cap \langle p, \pi_U \rangle$ is either singular, or it is a hermitian variety $H(3,q)$ (since we assumed q odd). However, this implies that $\mathscr{K} \cap \langle p, \pi_U \rangle$ contains lines. This gives us a contradiction, because $\mathscr{K}$ is a $(1, \sqrt{q}+1)$-set. We conclude that if $PG(n,q)$ contains a plane π_U that intersects $\mathscr{K}$ in a unital, then $\mathscr{K} \subset \pi_U$.

Now suppose that $PG(n,q)$ does not contain a plane that intersects $\mathscr{K}$ in a unital. From Lemma 16 it follows that every plane of $PG(n,q)$ intersects $\mathscr{K}$ in 0 points, one point, $\sqrt{q}+1$ collinear points or a Baer subplane. We will prove that $\mathscr{K}$ has to be a Baer subspace of $PG(n,q)$.

To prove that the points and lines of $\mathscr{K}$ are the points and lines of a projective geometry, we check whether the axioms of Dembowski hold (see [9]). From now on we call the lines that contain $\sqrt{q}+1$ points of $\mathscr{K}$ $\mathscr{K}$-lines.

1. Through every 2 points of $\mathscr{K}$ there has to be exactly one $\mathscr{K}$-line. This follows immediately from the fact that $\mathscr{K}$ is a $(0, 1, \sqrt{q}+1)$-set.
2. On every $\mathscr{K}$-line there have to be at least 3 points of $\mathscr{K}$. This is true, because every $\mathscr{K}$-line contains $\sqrt{q}+1 \geqslant 3$ points of $\mathscr{K}$.
3. Let L and M be two $\mathscr{K}$-lines that intersect in the point $p \in \mathscr{K}$. Let $q, r \in L$ and $s, t \in M$ be 4 different points of $\mathscr{K}$. Assume that $q \neq p \neq r$ and $s \neq p \neq t$. Then the lines $\langle q, s \rangle$ and $\langle r, t \rangle$ of $\mathscr{K}$ intersect in a point u of $\mathscr{K}$. Indeed, two intersecting $\mathscr{K}$-lines span a plane that intersects $\mathscr{K}$ in a Baer subplane. Since u lies on two $\mathscr{K}$-lines, which are Baer sublines, u clearly belongs to the Baer subplane and hence also to $\mathscr{K}$.

Hence the points and lines of $\mathscr{K}$ are the points and lines of a projective geometry. Since there are $\sqrt{q}+1$ points of $\mathscr{K}$ on a line of $\mathscr{K}$, we get that $\mathscr{K}$ is a Baer subspace of $PG(n,q)$, a Baer subspace of some subspace of $PG(n,q)$, or $\sqrt{q}+1$ collinear points. □

Theorem 18. *Let $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be a proper non-degenerate $(q-\sqrt{q},q)$-geometry fully embedded in* PG(n,q), *with q a square. Assume that* PG(n,q) *contains a line on which there are $\sqrt{q}+1$ points of* PG$(n,q)\backslash\mathcal{S}$. *Then $n=3$ or $n=4$ and there exists an n-dimensional Baer subspace* PG$(n,\sqrt{q})$ *of* PG(n,q) *such that $\mathcal{P}$ is the set of all points of* PG$(n,q)\backslash$PG$(n,\sqrt{q})$, *$\mathcal{L}$ is the set of all lines not intersecting* PG$(n,\sqrt{q})$ *and incidence is the one of* PG(n,q).

Proof. Let $\mathcal{S}$ be a proper $(q-\sqrt{q},q)$-geometry fully embedded in PG(n,q), with q a square. Assume that PG(n,q) contains a line on which there are $\sqrt{q}+1$ points of PG$(n,q)\backslash\mathcal{S}$. From Lemma 15, we know that no line of PG(n,q) contains $q+1$ points of PG$(n,q)\backslash\mathcal{S}$. From Lemma 14 we know that the points of PG$(n,q)\backslash\mathcal{S}$ form a $(0,1,\sqrt{q}+1,q+1)$-set of PG(n,q). Since no line of PG(n,q) contains $q+1$ points of PG$(n,q)\backslash\mathcal{S}$, the points of PG$(n,q)\backslash\mathcal{S}$ form a $(0,1,\sqrt{q}+1)$-set. From Lemma 17 it follows that the points of PG$(n,q)\backslash\mathcal{S}$ are the points of a Baer subspace or a unital in some plane.

If the points of PG$(n,q)\backslash\mathcal{S}$ are the points of a unital, then in the same way as in Lemma 15 one proves that $\mathcal{S}$ cannot be a proper $(q-\sqrt{q},q)$-geometry. Remark that in Lemma 15 we did assume that $H(d,q)$ contains a line, but we used this only to prove that the points of PG$(n,q)\backslash\mathcal{S}$ are the points of a hermitian variety. The proof of the second part of the theorem, namely assume that the points of PG$(n,q)\backslash\mathcal{S}$ are the points of a hermitian variety $H(d,q)$, then there is no $(q-\sqrt{q},q)$-geometry, is also valid for $H(d,q)$ being a unital.

So the points of PG$(n,q)\backslash\mathcal{S}$ are the points of a Baer subspace $B(d,\sqrt{q})$ of dimension d, for $0<d\leqslant n$. Let $\Upsilon[d]$ be the d-dimensional space containing $B(d,\sqrt{q})$. If $d=n$, then $\Upsilon[d]=$ PG(n,q). Let $\mathcal{B}$ be the set of lines containing $q+1$ points of $\mathcal{S}$ but not belonging to $\mathcal{S}$.

Suppose first that $d<n$. We will prove that every line of $\mathcal{B}$ is disjoint from $\Upsilon[d]$. Assume therefore that there is a line N of $\mathcal{B}$ that contains a point z of $\Upsilon[d]$. Let $M_{\sqrt{q}+1}$ be a line through z that intersects $B(d,\sqrt{q})$ in $\sqrt{q}+1$ points. The plane $\langle M_{\sqrt{q}+1},N\rangle$ contains points of PG$(n,q)\backslash\mathcal{S}$ and a line of $\mathcal{B}$. Hence this plane cannot contain lines of $\mathcal{S}$. Let L be a line of $\mathcal{S}$ through z. Then the plane $\langle L,M_{\sqrt{q}+1}\rangle$ contains an antiflag of $\mathcal{S}$ and points of PG$(n,q)\backslash\mathcal{S}$. So $\langle L,M_{\sqrt{q}+1}\rangle$ cannot contain elements of $\mathcal{B}$.

Now we will prove that $B(d,\sqrt{q})$ intersects the three-dimensional space $\langle L,N,M_{\sqrt{q}+1}\rangle$ in the $\sqrt{q}+1$ points of $B(d,\sqrt{q})$ on $M_{\sqrt{q}+1}$. Indeed, assume that there is a point $y\notin M_{\sqrt{q}+1}$ of $B(d,\sqrt{q})$ contained in $\langle L,N,M_{\sqrt{q}+1}\rangle$. Then $\langle L,N,M_{\sqrt{q}+1}\rangle\cap\Upsilon[d]$ contains a plane intersecting $B(d,\sqrt{q})$ in a Baer subplane. So the plane $\langle L,N\rangle$ contains at least one point of $B(d,\sqrt{q})$. Since $\langle L,N\rangle$ contains an antiflag of $\mathcal{S}$ and a line of $\mathcal{B}$, this gives a contradiction. Hence $\langle L,N,M_{\sqrt{q}+1}\rangle$ intersects $B(d,\sqrt{q})$ in the $\sqrt{q}+1$ points of $B(d,\sqrt{q})$ on $M_{\sqrt{q}+1}$.

Let w_1 be a point of N, $w_1\neq z$. The plane $\langle L,N\rangle$ contains an antiflag of $\mathcal{S}$. Hence we can take a line L_{w_1} of $\mathcal{S}$ through w_1 that intersects L in a point w_2. Let ρ be a plane through L_{w_1} and a point of $\mathcal{S}$ on $M_{\sqrt{q}+1}$. Then ρ contains an antiflag of $\mathcal{S}$ and no points of PG$(n,q)\backslash\mathcal{S}$. Hence it is of type III. Let N_{w_2} be a line of $\mathcal{B}$ through w_2 in ρ. Denote the points of PG$(n,q)\backslash\mathcal{S}$ on $M_{\sqrt{q}+1}$ by $x_1,\ldots,x_{\sqrt{q}+1}$. Then the plane $\langle N_{w_2},x_1\rangle$ contains at least one point of PG$(n,q)\backslash\mathcal{S}$ and a line of $\mathcal{B}$. This implies that it cannot contain a line of $\mathcal{S}$. So we can choose a line N' of $\mathcal{B}$ in $\langle N_{w_2}, x_1\rangle$ that is disjoint from L_{w_1} and $M_{\sqrt{q}+1}$. The plane $\langle N', x_2\rangle$ then contains no lines of $\mathcal{S}$. The plane $\langle L_{w_1}, x_1\rangle$ cannot contain elements of $\mathcal{B}$. These two planes are contained in a three-dimensional space. So they intersect in a line. Clearly this line contains no point of $\mathrm{PG}(n,q)\backslash\mathcal{S}$. Hence it has to be a line that belongs to $\mathcal{L}\cap\mathcal{B}$, giving a contradiction. This proves that there cannot be a line of $\mathcal{B}$ through a point of $\Upsilon[d]$.

Let u_1 and u_2 be points of $\mathcal{S}$, $u_1 \in \Upsilon[d]$ and $u_2 \notin \Upsilon[d]$. Then through u_1 there are more lines that do not intersect $B(d,\sqrt{q})$ than through u_2. Hence there must be a line of $\mathcal{B}$ through u_1. However, in the previous paragraph we proved that there cannot be a line of $\mathcal{B}$ that contains a point of $\Upsilon[d]$. So we have found a contradiction. We conclude that for $d < n$ there does not exist a $(q-\sqrt{q}, q)$-geometry that has point set $\mathrm{PG}(n,q)\backslash B(d,\sqrt{q})$.

Suppose next that $d = n$. From the above it follows that $\mathrm{PG}(n,q)$ contains no lines of $\mathcal{B}$. Hence every plane containing an antiflag of $\mathcal{S}$ is of type I or of type II, which implies that every plane containing an antiflag of $\mathcal{S}$ contains at least one point of $\mathrm{PG}(n,q)\backslash\mathcal{S}$. For $n \geqslant 5$, it is clear that $\mathrm{PG}(n,q)$ contains planes skew to $\mathrm{PG}(n,\sqrt{q})$. Hence for $n \geqslant 5$, we have found a contradiction. For $n \leqslant 4$, there are no planes skew to $\mathrm{PG}(n,\sqrt{q})$ contained in $\mathrm{PG}(n,q)$. Since $\mathcal{B} = \emptyset$, $\mathcal{L}$ is the set of all lines of $\mathrm{PG}(n,q)$ not containing a point of $\mathrm{PG}(n,\sqrt{q})$. One easily sees that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, with $\mathcal{P}$ the set of points of $\mathrm{PG}(n,q)\backslash\mathrm{PG}(n,\sqrt{q})$, $\mathcal{L}$ the set of lines of $\mathrm{PG}(n,q)$ not intersecting $\mathrm{PG}(n,\sqrt{q})$, is indeed a $(q-\sqrt{q}, q)$-geometry for $n = 3$ or 4. □

Theorem 19. *Let $\mathcal{S}$ be a proper $(q-\sqrt{q}, q)$-geometry fully embedded in* $\mathrm{PG}(n,q)$, *with q a square, $q \neq 4$. Assume that no line of* $\mathrm{PG}(n,q)$ *contains $\sqrt{q}+1$ points of* $\mathrm{PG}(n,q)\backslash\mathcal{S}$. *Assume also that $|\mathrm{PG}(n,q)\backslash\mathcal{S}| \neq \emptyset$. Then $\mathcal{S}$ is a degenerate $(q-\sqrt{q}, q)$-geometry.*

Proof. Let $\mathcal{S}$ be a proper $(q-\sqrt{q}, q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with q a square. Assume that no line of $\mathrm{PG}(n,q)$ contains $\sqrt{q}+1$ points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$. From Lemma 14 it follows that the points of $\mathrm{PG}(n,q)\backslash\mathcal{S}$ are the points of a $(0, 1, q+1)$-set, or in other words they are the points of an m-dimensional subspace $\Pi[m]$ of $\mathrm{PG}(n,q), 0 \leqslant m \leqslant n-2$. Let $\mathcal{B}$ be the set of all lines of $\mathrm{PG}(n,q)$ that contain $q+1$ points of $\mathcal{S}$, but that do not belong to $\mathcal{S}$. It is clear that $\mathcal{B} \neq \emptyset$, as otherwise $\mathcal{S}$ would be a $(q, q+1)$-geometry, a contradiction. Let $N \in \mathcal{B}$. Then the $(m+2)$-dimensional space $\langle N, \Pi[m]\rangle$ does not contain a line of $\mathcal{S}$. Indeed, if there would be a line L of $\mathcal{S}$ in $\langle N, \Pi[m]\rangle$, L intersecting N, then $\langle L, N\rangle$ would be a plane containing an antiflag of $\mathcal{S}$, a line of $\mathcal{B}$ and a point of $\Pi[m]$, a contradiction as such a plane cannot exist. This implies that $\langle N, \Pi[m]\rangle$ contains no lines of $\mathcal{S}$, as otherwise for such a line L' of $\mathcal{S}$, $i(z, L') = 0$, for a point $z \in N$, clearly a contradiction. Hence the $(m+2)$-dimensional spaces through $\Pi[m]$ in $\mathrm{PG}(n,q)$ either contain no lines of $\mathcal{S}$, or they contain no lines of $\mathcal{B}$.

Let $\Omega[n-m-1]$ be a subspace of $\mathrm{PG}(n,q)$ skew to $\Pi[m]$. Then each $(m+2)$-dimensional subspace of $\mathrm{PG}(n,q)$ that contains $\Pi[m]$, intersects $\Omega[n-m-1]$ in a line M. If $M \in \mathcal{B}$, then $\langle M, \Pi[m]\rangle$ contains no lines of $\mathcal{S}$. If $M \in \mathcal{L}$, then $\langle M, \Pi[m]\rangle$ contains no lines of $\mathcal{B}$. We will prove now that $\mathcal{S}$ intersects $\Omega[n-m-1]$ in a

$(q-\sqrt{q},q)$-geometry $\mathcal{S}'$. It is clear that every line of $\mathcal{S}$ in $\Omega[n-m-1]$ contains $q+1$ points of $\mathcal{S}$ and that for every antiflag (p,L) of $\mathcal{S}$ in $\Omega[n-m-1]$, we have that $i(p,L)=q-\sqrt{q}$ or q. So we only need to prove that the number of lines of $\mathcal{S}$ in $\Omega[n-m-1]$ through a point of $\mathcal{S}$ in $\Omega[n-m-1]$ is a constant. Let u be a point of $\Omega[n-m-1]$. If L_1 is a line of $\mathcal{S}$ through u in $\Omega[n-m-1]$, then $\langle L_1,\Pi[m]\rangle$ contains $1+(q-1)(q^{m+1}-1)/(q-1)=q^{m+1}$ lines of $\mathcal{S}$ through u. If L_2 is a line of $\mathcal{S}$ through u in $\Omega[n-m-1]$, $L_2\neq L_1$, then $\langle L_1,\Pi[m]\rangle$ and $\langle L_2,\Pi[m]\rangle$ intersect in the $(m+1)$-dimensional space $\langle u,\Pi[m]\rangle$, that clearly contains no line of $\mathcal{S}$. Hence each line of $\mathcal{S}$ through u belongs to exactly one $(m+2)$-dimensional space $\langle L,\Pi[m]\rangle$ through $\langle u,\Pi[m]\rangle$. It follows that $t+1=q^{m+1}(t_u+1)$, where t_u+1 is the number of lines of $\mathcal{S}$ through u contained in $\Omega[n-m-1]$. Since $t+1$ is a constant, it follows that also t_u+1 is a constant, independent of the choice of the point $u\in\Omega[n-m-1]$. This proves that $\mathcal{S}$ intersects $\Omega[n-m-1]$ in a $(q-\sqrt{q},q)$-geometry $\mathcal{S}'$. Hence $\mathrm{PG}(n,q)$ contains a cone $\Pi[m]\mathcal{S}'$, projecting a $(q-\sqrt{q},q)$-geometry $\mathcal{S}'$ fully embedded in an $(n-m-1)$-dimensional subspace $\Omega[n-m-1]$ of $\mathrm{PG}(n,q)$ skew to $\Pi[m]$. By definition $\mathcal{S}$ is a degenerate $(q-\sqrt{q},q)$-geometry. □

Theorem 20. *There exists no proper $(q-\sqrt{q},q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, for q a square, $q\neq 4$ and $n>3$ with point set $\mathcal{P}=\mathrm{PG}(n,q)$.*

Proof. Let $\mathcal{S}$ be a proper $(q-\sqrt{q},q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with q a square. Assume that $n\neq 3$. Let $\mathcal{P}$ be the set of all points of $\mathrm{PG}(n,q)$. Since all points of $\mathrm{PG}(n,q)$ are points of $\mathcal{S}$, every plane containing an antiflag of $\mathcal{S}$ is of type III. Let $\mathcal{B}$ be the set of lines containing $q+1$ points of $\mathcal{S}$ but not belonging to $\mathcal{L}$.

Let p be an arbitrary point of $\mathrm{PG}(n,q)$. Let $\Pi[n-1]$ be a hyperplane of $\mathrm{PG}(n,q)$ that does not contain p. Then the elements of $\mathcal{B}$ through p intersect $\Pi[n-1]$ in a set of points, which we denote by $\mathcal{K}$. Since every plane through p contains 1, $\sqrt{q}+1$ or $q+1$ elements of $\mathcal{B}$ through p, the set $\mathcal{K}$ is a $(1,\sqrt{q}+1,q+1)$-set of $\Pi[n-1]$. If there is a line of $\Pi[n-1]$ that contains $\sqrt{q}+1$ points of $\mathcal{K}$, then [12, Theorems 23.5.1 and 23.5.119] the points of $\mathcal{K}$ are the points of either a hermitian variety, a Baer subplane, a cone with base either a hermitian variety or a Baer subplane, or $\sqrt{q}+1$ hyperplanes intersecting in an $(n-2)$-dimensional space. If no line of $\Pi[n-1]$ intersects $\mathcal{K}$ in $\sqrt{q}+1$ points, then the points of $\mathcal{K}$ are the points of a hyperplane of $\Pi[n-1]$.

Assume that there is a plane π in $\Pi[n-1]$ for which the points of $\mathcal{K}\cap\pi$ are the points of $\sqrt{q}+1$ concurrent lines. We denote these lines by $L_1,\ldots,L_{\sqrt{q}+1}$. Let w be the intersection point of $L_1,\ldots,L_{\sqrt{q}+1}$. We look at the three-dimensional space spanned by p and π. The plane $\langle p,L_i\rangle$, for $i\in\{1,\ldots,\sqrt{q}+1\}$, contains $q+1$ elements of $\mathcal{B}$ through p. Hence it cannot contain a line of $\mathcal{S}$, since, if there was a line L of $\mathcal{S}$ in this plane, then there would be 0 lines through p intersecting L, a contradiction. Hence the planes $\langle p,L_i\rangle$, for $i=1,\ldots,\sqrt{q}+1$ contain no lines of $\mathcal{S}$. Suppose that there is a line L' of $\mathcal{S}$ that is not contained in one of the planes $\langle p,L_i\rangle$, for $i=1,\ldots,\sqrt{q}+1$, and that is skew to the line $\langle p,w\rangle$. Let ρ be a plane through $\langle p,w\rangle$ different from $\langle p,L_i\rangle$, for all $i\in\{1,\ldots,\sqrt{q}+1\}$. Let $L'\cap\rho$ be the point x. The plane ρ contains an antiflag of $\mathcal{S}$, hence it is of type III. Through x we can take a line N of $\mathcal{B}$ such that $N\subset\rho$.

The lines N and $\langle p,w\rangle$ intersect in a point. We denote this point by y. Remark that y can be the point w. Now we look at the plane $\langle L',y\rangle$. It contains the line $L'\in\mathscr{L}$ and at least $\sqrt{q}+2$ elements of $\mathscr{B}$ through y, namely the intersection lines of $\langle L',y\rangle$ with $\langle p,L_i\rangle$, for $i=1,\ldots,\sqrt{q}+1$ and the line N. This is a contradiction, since $\alpha=q-\sqrt{q}$ and $\beta=q$. We conclude that the only lines of $\mathscr{S}$ in $\langle p,\pi\rangle$ are contained in the $q-\sqrt{q}$ planes through $\langle p,w\rangle$ and a line through w in π different from $L_1,\ldots,L_{\sqrt{q}+1}$. Now let M_1 and M_2 be two lines of $\mathscr{S}$ in $\langle p,\pi\rangle$ that contain p, such that the planes $\langle p,w,M_1\rangle$ and $\langle p,w,M_2\rangle$ are distinct. Then $\langle M_1,M_2\rangle$ is a plane containing an antiflag of $\mathscr{S}$. All the lines of $\mathscr{S}$ in this plane contain the point p. This is a contradiction because of $\alpha=q-\sqrt{q}$ and $\beta=q$. It follows that no plane intersects $\mathscr{K}$ in $\sqrt{q}+1$ concurrent lines.

Hence we have shown that the points of $\mathscr{K}$ have to be the points of either a Baer subplane or a unital in some plane of $\Pi[n-1]$, or a hyperplane of $\Pi[n-1]$. Indeed, a non-singular hermitian variety with $n>3$ contains planes that intersect it in $\sqrt{q}+1$ concurrent lines. Moreover every singular hermitian variety and every cone with base a Baer subplane contains such planes.

Assume first that the points of $\mathscr{K}$ are the points of a unital or a Baer subplane of $\Pi[n-1]$. It follows immediately that $n=3$, since $\Pi[n-1]$ cannot contain lines that are skew to $\mathscr{K}$. This is a contadiction, since we assumed $n>3$.

Assume that the points of $\mathscr{K}$ are the points of a hyperplane of $\Pi[n-1]$. Then clearly $t+1=q^{n-1}$. We denote the hyperplane containing the points of $\mathscr{K}$ by $\mathrm{PG}(n-2,q)$. Then the $(n-1)$-dimensional space $\langle p,\mathrm{PG}(n-2,q)\rangle$ contains no lines of $\mathscr{S}$. Indeed, if there would be a line L_p of $\mathscr{S}$ contained in this subspace, then there would be 0 lines through p intersecting L_p, a contradiction since $\mathscr{S}$ is a $(q-\sqrt{q},q)$-geometry. Let u be an arbitrary point of $\langle p,\mathrm{PG}(n-2,q)\rangle$. Then the $t+1=q^{n-1}$ lines of $\mathscr{S}$ through u are clearly all lines through u not in $\langle p,\mathrm{PG}(n-2,q)\rangle$. It follows that the lines of $\mathscr{S}$ in $\mathrm{PG}(n,q)$ are all lines not in $\langle p,\mathrm{PG}(n-2,q)\rangle$. However, this implies that for a point u' of $\mathrm{PG}(n,q)$ not in $\langle p,\mathrm{PG}(n-2,q)\rangle$ there are $q+1$ lines of $\mathscr{S}$ intersecting each line of $\mathscr{S}$ not through u'. This is a contradiction since $\mathscr{S}$ is a $(q-\sqrt{q},q)$-geometry. Hence the points of $\mathscr{K}$ cannot be the points of a hyperplane of $\Pi[n-1]$.

This proves that for $n\neq 3$ there cannot be a $(q-\sqrt{q},q)$-geometry contained in $\mathrm{PG}(n,q)$, q a square, with point set all the points of $\mathrm{PG}(n,q)$. □

Remark 21. The previous theorem does not classify proper $(q-\sqrt{q},q)$-geometries fully embedded in $\mathrm{PG}(3,q)$, for q a square and point set $\mathscr{P}=\mathrm{PG}(3,q)$. If there exists such a $(q-\sqrt{q},q)$-geometry $\mathscr{S}$, then one of the following would hold:

(1) $t+1=q\sqrt{q}+1$, the lines of $\mathscr{S}$ through a point p of $\mathrm{PG}(3,q)$ intersect each plane not through p in the points of a unital and the lines in each plane of $\mathrm{PG}(3,q)$ are the lines that intersect a unital in that plane in $\sqrt{q}+1$ points;
(2) $t+1=q+\sqrt{q}+1$, the lines of $\mathscr{S}$ through a point p of $\mathrm{PG}(3,q)$ intersect each plane not through p in the points of a Baer subplane and the lines in each plane of $\mathrm{PG}(3,q)$ are the lines that are tangent to a Baer subplane in that plane.

It is not known to us whether such a $(q-\sqrt{q},q)$-geometry exists.

2.3. The case $q-1 \neq \alpha \neq q-\sqrt{q}$

Let $\mathscr{S}$ be a proper (α, q)-geometry fully embedded in $\mathrm{PG}(n,q)$, q odd and $\alpha > 1$, for which $q-1 \neq \alpha \neq q-\sqrt{q}$. Then every plane that contains an antiflag of $\mathscr{S}$ is either a q-plane, or it intersects $\mathscr{S}$ in the closure of a net. Hence every plane containing an antiflag of $\mathscr{S}$ contains one point or $q+1-\alpha$ collinear points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$.

Lemma 22. *Let $\mathscr{S}$ be a proper (α, q)-geometry fully embedded in* $\mathrm{PG}(n,q)$, *q odd and $\alpha > 1$, for which $q-1 \neq \alpha \neq q-\sqrt{q}$. Then every line of* $\mathrm{PG}(n,q)$ *contains* 0,1, *$q+1-\alpha$ or $q+1$ points of* $\mathrm{PG}(n,q)\backslash\mathscr{S}$.

Proof. One proves this lemma in exactly the same way as Lemma 5. □

Theorem 23. *There exists no proper (α, q)-geometry fully embedded in* $\mathrm{PG}(n,q)$, *q odd and $\alpha > 1$, for which $q-1 \neq \alpha \neq q-\sqrt{q}$.*

Proof. Every plane containing an antiflag of $\mathscr{S}$ is a q-plane (plane of type 1) or a plane intersecting $\mathscr{S}$ in the closure of a net (plane of type II). It follows that every plane of $\mathrm{PG}(n,q)$ contains at least one point of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ and that every line containing $q+1$ points of $\mathscr{S}$ is a line of $\mathscr{S}$. Since $\mathscr{S}$ is proper, there is a plane ρ of type II. Let M be the line of ρ that contains $q+1-\alpha$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. Let $u \notin \rho$ be a point of $\mathscr{S}$. Then all points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle u,\rho\rangle$ are coplanar. Indeed, since $\alpha \neq q-\sqrt{q}$, every plane that contains no line of $\mathscr{S}$ intersects $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in the $q+1$ points on $q+1-\alpha$ concurrent lines. If the points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ in $\langle u,\rho\rangle$ would not be coplanar, then there would be exactly $q+1-\alpha$ planes through M containing no line of $\mathscr{S}$ (this follows from Lemma 22). Hence $\langle u,\rho\rangle$ would contain at least $(q-1)(q+1-\alpha)^2+2(q+1-\alpha)$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$. This is a contradiction, since $\langle u,\rho\rangle$ contains lines of $\mathscr{S}$ and every plane through such a line contains either 1 or $q+1-\alpha$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ and thus there are at most $(q+1)(q+1-\alpha)$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ contained in $\langle u,\rho\rangle$. Now let $\Pi[m], m \geqslant 3$ be an m-dimensional subspace of $\mathrm{PG}(n,q)$ through $\langle u,\rho\rangle$, such that all points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ are contained in a hyperplane of $\Pi[m]$. Let u' be a point of $\mathscr{S}, u' \notin \Pi[m]$. Then it follows that all points of $\langle u',\Pi[m]\rangle$ are contained in a hyperplane of $\langle u',\Pi[m]\rangle$, as otherwise one can find a 3-dimensional subspace of $\langle u',\Pi[m]\rangle$ that contains two skew lines on which there are $q+1-\alpha$ points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$, as well as lines of $\mathscr{S}$, which is a contradiction with the first part of the proof. After a finite number of steps, one gets that all points of $\mathrm{PG}(n,q)\backslash\mathscr{S}$ are contained in an $(n-1)$-dimensional subspace of $\mathrm{PG}(n,q)$, a contradiction since then the number $t+1$ of lines of $\mathscr{S}$ through a point of $\mathscr{S}$ cannot be a constant. This proves that $\mathscr{S}$ cannot exist. □

3. Conclusion

In this section we summarise the results of this paper and [5].

Let $\mathscr{S}=(\mathscr{P},\mathscr{L},\mathrm{I})$ be a proper (α,β)-geometry fully embedded in $\mathrm{PG}(n,q)$, q odd and $\alpha > 1$. In case $\alpha = q-\sqrt{q}$ we assume that $\mathscr{S}$ is non-degenerate. Assume that $\mathrm{PG}(n,q)$

contains at least one α-plane or one β-plane. Then $\mathcal{S}$ is one of the following.

(1) $\mathcal{S}$ is a $(q, q+1)$-geometry, with points the points of $\mathrm{PG}(n,q)\backslash\mathrm{PG}(m,q)$, for some $0 \leqslant m < n-2$, and lines those lines of $\mathrm{PG}(n,q)$ that are disjoint from $\mathrm{PG}(m,q)$.

(2) $\mathcal{S}$ is a $(q, q+1)$-geometry, with points the points of $\mathrm{PG}(n,q)\backslash\mathrm{PG}(m,q)$, with $0 \leqslant m \leqslant n-3$. Moreover there exists a partition of the points of $\mathcal{S}$ in m'-dimensional subspaces of $\mathrm{PG}(n,q)$ that pairwise intersect in $\mathrm{PG}(m,q)$, $m+2 \leqslant m' \leqslant n-2$, such that the lines of $\mathcal{S}$ are the lines that intersect $q+1$ of these m'-dimensional spaces in a point. A necessary and sufficient condition for this partition and the $(q, q+1)$-geometry to exist is that $(m'-m)|(n-m')$.

(3) $\mathcal{S}$ is a $(q-1, q)$-geometry, with points the points of $\mathrm{PG}(n,q)\backslash\mathrm{PG}(n-2,q)$, and lines those that do not contain a point of $\mathrm{PG}(n,q)\backslash\mathcal{S}$ and that do not belong to a partition Σ of the points of $\mathrm{PG}(n,q)\backslash\mathrm{PG}(n-2,q)$ in r-dimensional spaces meeting $\mathrm{PG}(n-2,q)$ in subspaces of dimension $r-2$, with $1 \leqslant r \leqslant n-2$. Further, such a partition exists for every $1 \leqslant r \leqslant n-2$, and gives a $(q-1, q)$-geometry.

(4) $\mathcal{S}$ is a $(q-1, q)$-geometry with points the points of $\mathrm{PG}(n,q)$ not contained in one of the two subspaces $\mathrm{PG}(n-2,q)$ and $\mathrm{PG}(r,q)$ of $\mathrm{PG}(n,q)$, for $1 \leqslant r \leqslant n-2$, for which $\mathrm{PG}(r,q) \cap \mathrm{PG}(n-2,q)$ is an $(r-2)$-dimensional space. The lines of $\mathcal{S}$ are either all lines of $\mathrm{PG}(n,q)$ that contain $q+1$ points of $\mathcal{S}$, or they are the lines not contained in a partition of the points of $\mathcal{S}$ in d-dimensional spaces pairwise intersecting in $\mathrm{PG}(r,q)$. A necessary and sufficient condition for such a partition to exist is that $(d-r)|(n-r)$ and that $n-2 \geqslant d \geqslant r+2$. Further, if $(d-r)|(n-r)$ and $n-2 \geqslant d \geqslant r+2$, then this partition gives a $(q-1, q)$-geometry.

(5) $\mathcal{S}$ is a non-degenerate $(q-\sqrt{q}, q)$-geometry with points those of $\mathrm{PG}(n,q)$ that do not belong to a Baer subspace $\mathrm{PG}(n,\sqrt{q})$ of $\mathrm{PG}(n,q)$ and lines the lines not intersecting $\mathrm{PG}(n,\sqrt{q})$. In this case $n=3$ or 4.

(6) $\mathcal{S}$ is a non-degenerate $(q-\sqrt{q}, q)$-geometry in $\mathrm{PG}(3,q)$, with points all points of $\mathrm{PG}(3,q)$, such that if p is a point of $\mathcal{S}$ and π is a plane of $\mathrm{PG}(3,q)$ not containing p, the lines of $\mathcal{S}$ through p intersect π in the points of a unital, and such that in every plane of $\mathrm{PG}(3,q)$ the lines of $\mathcal{S}$ are the lines that intersect a unital in this plane in $\sqrt{q}+1$ points. It is not known to us whether such a $(q-\sqrt{q}, q)$-geometry exists.

(7) $\mathcal{S}$ is a non-degenerate $(q-\sqrt{q}, q)$-geometry in $\mathrm{PG}(3,q)$, with points all points of $\mathrm{PG}(3,q)$, such that if p is a point of $\mathcal{S}$ and π is a plane of $\mathrm{PG}(3,q)$ not containing p, the lines of $\mathcal{S}$ through p intersect π in the points of a Baer subplane of π, and such that in every plane of $\mathrm{PG}(3,q)$ the lines of $\mathcal{S}$ are the lines that are tangent to a Baer subplane in this plane. It is not known to us whether such a $(q-\sqrt{q}, q)$-geometry exists.

If q is even, then we obtained the following results.

Theorem 24. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper $(q, q+1)$-geometry fully embedded in* $\mathrm{PG}(n,q)$, *q even. Assume that every plane of* $\mathrm{PG}(n,q)$ *that contains an antiflag of $\mathcal{S}$*

is a q-plane or a $(q+1)$*-plane. Then* $\mathcal{P}$ *is the set of points of* $\mathrm{PG}(n,q)\backslash\mathrm{PG}(m,q)$, *for some* $0 \leqslant m < n-2$ *and* $\mathcal{L}$ *is the set of the lines of* $\mathrm{PG}(n,q)$ *that are disjoint of* $\mathrm{PG}(m,q)$.

Theorem 25. *Let* $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ *be a proper* $(q-1,q)$*-geometry fully embedded in* $\mathrm{PG}(n,q)$, *q even*, $q \neq 2$. *Assume that all the planes containing an antiflag of* $\mathcal{S}$ *are q-planes or planes in which there is exactly one point p and exactly one line L not belonging to* $\mathcal{S}$, *such that p is not incident with L. Then the points of* $\mathrm{PG}(n,q)\backslash\mathcal{S}$ *are the points of a subspace* $\mathrm{PG}(n-2,q)$. *The lines of* $\mathcal{S}$ *are the lines that do not contain a point of* $\mathrm{PG}(n,q)\backslash\mathcal{S}$ *and that do not belong to a partition* Σ *of the points of* $\mathrm{PG}(n,q)\backslash\mathrm{PG}(n-2,q)$ *in r-dimensional spaces meeting* $\mathrm{PG}(n-2,q)$ *in subspaces of dimension* $r-2$, *with* $1 \leqslant r \leqslant n-2$. *Further, such a partition exists for every* $1 \leqslant r \leqslant n-2$, *and gives a* $(q-1,q)$*-geometry.*

Theorem 26. *Let* $\mathcal{S}$ *be a proper* $(q-1,q)$*-geometry fully embedded in* $\mathrm{PG}(n,q)$, *q even*, $q \neq 2$. *Assume that there is a plane in which the points not in* $\mathcal{S}$ *are two different points, while the lines of* $\mathcal{S}$ *are all the lines not containing one of these points. Then the points of* $\mathrm{PG}(n,q)\backslash\mathcal{S}$ *are the points of two subspaces* $\mathrm{PG}(n-2,q)$ *and* $\mathrm{PG}(r,q)$ *of* $\mathrm{PG}(n,q)$, *for* $0 \leqslant r \leqslant n-2$, *with* $\mathrm{PG}(r,q)\cap\mathrm{PG}(n-2,q)$ *an* $(r-2)$*-dimensional space. The lines of* $\mathcal{S}$ *are either all lines of* $\mathrm{PG}(n,q)$ *that contain* $q+1$ *points of* $\mathcal{S}$, *or they are the lines not contained in a partition of the points of* $\mathcal{S}$ *in d-dimensional spaces pairwise intersecting in* $\mathrm{PG}(r,q)$. *A necessary and sufficient condition for such a partition to exist is that* $(d-r)|(n-r)$ *and that* $n-2 \geqslant d \geqslant r+2$. *Further, if* $(d-r)|(n-r)$ *and* $n-2 \geqslant d \geqslant r+2$, *then this partition gives a* $(q-1,q)$*-geometry.*

Acknowledgements

I would like to thank Frank De Clerck and Nicholas Hamilton for their comments and advise while I was writing this paper and for the hours they spent reading a first version of this paper.

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