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Total edge irregularity strength of complete graphs and complete bipartite graphs

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Dedicated to professor Jarík Nešetřil on the occasion of his sixtieth birthday

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ABSTRACT

A total edge irregular k-labelling ν of a graph G is a labelling of the vertices and edges of G with labels from the set $\{1, \dots, k\}$ in such a way that for any two different edges e and f their weights $\varphi(f)$ and $\varphi(e)$ are distinct. Here, the weight of an edge $g = uv$ is $\varphi(g) = \nu(g) + \nu(u) + \nu(v)$, i. e. the sum of the label of g and the labels of vertices u and v. The minimum k for which the graph G has an edge irregular total k-labelling is called *the total edge irregularity strength* of G.

We have determined the exact value of the total edge irregularity strength of complete graphs and complete bipartite graphs.

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1. Introduction

In [3], Chartrand et al. proposed the following problem:

Assign positive integer labels to the edges of a connected graph of order at least 3 in such a way that the graph becomes irregular, i.e. the weights (label sums of edges incident with the vertex) of vertices are distinct. What is the minimum value of the largest label over all such irregular assignments?

This parameter of a graph G is well known as the *irregularity strength* of the graph G, $s(G)$. Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [4,5,11]. For recent results see the papers by Amar and Togni [1], Jacobson and Lehel [8], Nierhoff [10], Frieze at al. [5]. Similarly Karoński, Łuczak, and Thomason [9] conjectured that the edges of every connected graph of order at least 3 can be assigned labels from $\{1, 2, 3\}$, such that for all pairs of adjacent vertices the sums of the labels of the incident edges are different. Motivated by these papers and by a book of Wallis [12], Bača et al. [2] started to investigate the total edge irregularity strength of a graph G, an invariant analogous to the irregularity strength for total labelling.

For a graph $G = (V, E)$ we define a labelling $\nu : V \cup E \to \{1, 2, \dots, k\}$ to be a total edge irregular k-labelling of the graph G if for every two different edges e and f of G one has $\varphi(e) \neq \varphi(f)$ where the weight of an edge $e = \{u, v\}$ in the labelling ν is $\varphi(e) = \nu(u) + \nu(v) + \nu(e)$. The minimum k for which the graph G has an edge irregular total k-labelling is called *the total edge irregularity strength* of G, $tes(G)$. Let us mention a result from [2] giving a lower bound on the total edge irregularity strength of a graph.

Firstly, let ν be an edge irregular total k-labelling of a graph G. Since $3 \leq \varphi(uv) = \nu(u) + \nu(uv) + \nu(v) \leq 3k$ for every edge $uv \in E$, we have $|E(G)| \leq 3k - 2$ which implies $tes(G) \geq \left\lceil \frac{|E(G)|+2}{3} \right\rceil$. Similarly, if $u \in V(G)$ is a vertex of maximum

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Table 1
The other residue classes.

n	t	$\|E(A, A)\|$	$\|E(A, B)\|$	$\|E(B, B)\|$	$\|E(A, C)\|$
$3l$	$l^2 + \frac{l\cdot(l-1)}{2} + 1$	$\frac{l\cdot(l-1)}{2}$	l^2	$\frac{l\cdot(l-1)}{2}$	l^2
$3l + 1$	$l^2 + \frac{l\cdot(l+1)}{2} + 1$	$\frac{l\cdot(l-1)}{2}$	$l \cdot (l + 1)$	$\frac{l\cdot(l+1)}{2}$	l^2

degree $\Delta = \Delta(G)$, then there is a range of $2k - 1$ possible weights $\nu(u) + 2 \leq \varphi(uv) \leq \nu(u) + 2k$ for the edges $uv \in E$ incident with u which implies $\text{tes}(G) \geq \lceil \frac{\Delta+1}{2} \rceil$. We obtain

$$\text{tes}(G) \geq \max \left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta + 1}{2} \right\rceil \right\}. \tag{1}$$

The authors of [2] present also a few families of graphs for which they found the exact value of the total edge irregularity strength. Recently Ivančo and Jendrol' [7] determined the total edge irregularity strength for any tree. They proved that for any tree T, $tes(T)$ is equal to its lower bound.

2. Main result

In this paper we deal with complete and complete bipartite graphs. We have determined the exact value of the total edge irregularity strength for graphs from these classes of graphs.

2.1. Complete graphs

The total edge irregularity strength is a monotone graph invariant, hence we have the following lemma.

Lemma 2.1. *Let $G = (V, E)$ be a graph and H its subgraph. Then $tes(H) \leq tes(G)$.*

Every graph G with vertex set $V(G)$ is a subgraph of the complete graph on the same vertex set $V(G)$. This gives an upper bound on the total edge irregularity strength of a graph G, $tes(G) \leq tes(K_{|V(G)|})$.

In what follows, we show that all complete graphs except for K_5 have total edge irregularity strength equal to the lower bound from (1). For the sake of completeness we show that the total edge irregularity strength of the K_5 is 5 (see also [2]) while its lower bound from (1) is 4.

Proposition 2.2. $tes(K_5) = 5$.

Proof. Let us assume for a contradiction that $tes(K_5) = 4$. Since the weight of an edge is the sum of three natural numbers its minimum value is 3. Using the same argument observe that its maximum value is 12. As K_5 has ten edges there have to be ten distinct weights. Therefore all the possible weights between 3 and 12 have to appear. To obtain weight 3 there must be two vertices labelled with the label 1 and to obtain weight 12 there must be two vertices labelled with the label 4. If the fifth vertex had a label ≤ 2 then there would be no possibility to obtain weight 11 but if it had a label ≥ 3 there would be no possibility to obtain weight 4. □

The first main result of this paper is the following:

Theorem 2.3. *Let $n \in \mathbb{N}$ and K_n be the complete graph on n vertices, $n \neq 5$. Then $tes(K_n) = \left\lceil \frac{n^2-n+4}{6} \right\rceil$.*

For sets of vertices X and Y of a graph G let us define $E(X, Y)$ to be the set of edges in G that have one end vertex in the set X and the other in the set Y. To prove Theorem 2.3 we need the following lemma:

Lemma 2.4. *Let n be an integer and let $t = \lceil \frac{n^2-n+4}{6} \rceil$. Let A, B, C be the sets of vertices of complete graph K_n with cardinalities $|A| = \lfloor \frac{n+1}{3} \rfloor$, $|B| = n - 2 \cdot \lfloor \frac{n+1}{3} \rfloor$ and $|C| = \lfloor \frac{n+1}{3} \rfloor$. Then the following hold:*

(i) $|E(A, A)| + |E(A, B)| = t - 1$
(ii) $|E(C, C)| + |E(B, C)| = t - 1$
(iii) $|E(A, C)| + |E(B, B)| \in \{t - 1, t\}$.

Proof. The proof is divided into three cases according to residue classes of n modulo 3.

Let $n = 3l - 1$, $l \in \mathbb{N}$. Then $t = 3 \cdot \frac{l\cdot(l-1)}{2} + 1$ and for the relations (i) and (ii) we have $|E(A, A)| = \frac{l\cdot(l-1)}{2}$ and $|E(A, B)| = l \cdot (l - 1)$ and so their sum is $3\frac{l\cdot(l-1)}{2} = t - 1$. For the relation (iii) we have $|E(B, B)| = \frac{l^2-3l}{2} + 1$ and $|E(A, C)| = l^2$ and hence their sum is $3\frac{l\cdot(l-1)}{2} + 1 = t$. Proofs for the other residue classes can be derived from Table 1. □

Now we are able to prove Theorem 2.3:

Proof. According to (1) it is enough to prove $tes(K_n) \leq \left\lceil \frac{n^2-n+4}{6} \right\rceil$. The main idea of the proof is to split the vertices of K_n into three mutually disjoint subsets (parts) A, B, and C with the cardinalities as mentioned in Lemma 2.4. Then we label the vertices from the set A by label 1, vertices from the set C by label $t = \left\lceil \frac{n^2-n+4}{6} \right\rceil$. To complete the labelling of vertices we label the vertices from the set B with labels $b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_m$, where $m = |B|$. We label the edges of the graph according to which of the six families they belong to:

(i) We label the edges from $E(A, A)$ with consecutive integers from 1 to $\binom{|A|}{2}$ to obtain the first $\binom{|A|}{2}$ weights, i.e. we create the weights from the integer interval $\left[3, \binom{|A|}{2} + 2\right]$.

(ii) We label the edges from $E(C, C)$ with consecutive integers from t down to $t+1-\binom{|C|}{2}$ to obtain the last $\binom{|C|}{2}$ weights, i.e. the weights from the interval $\left[3t - \binom{|C|}{2} + 1, 3t\right]$.

(iii) We label the edges from $E(A, B)$ to obtain weights creating the interval $\left[\binom{|A|}{2} + 3, \binom{|A|}{2} + 2 + |A| \cdot |B|\right]$. Appropriate weights are created on the edges that have one end vertex labelled with the label from $b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_m$.

(iv) With the same technique we create the interval $\left[3t + 1 - \binom{|C|}{2} - |C| \cdot |B|, 3t - \binom{|C|}{2}\right]$ of different weights on the edges from the set $E(C, B)$.

(v) We label the edges from $E(B, B)$ to obtain different weights from the interval $\left[\binom{|A|}{2} + |A| \cdot |B| + 3, 3t - \binom{|C|}{2} - |C| \cdot |B|\right]$.

(vi) We label the edges from $E(A, C)$ to obtain weights from the interval $\left[\binom{|A|}{2} + |A| \cdot |B| + 3, 3t - \binom{|C|}{2} - |C| \cdot |B|\right]$ different from all existing weights.

The strategy of the labelling of edges is as follows. The edges from the family $E(A, A)$ are labelled successively from 1 to $\binom{|A|}{2}$. In a similar way we label edges from $E(C, C)$. To be able to realize the labelling in the families $E(A, B)$ and $E(C, B)$ we first show that a suitable choice of labels $b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_m$ with $m = |B|$ satisfying below derived estimations is possible. These estimations on b_i are derived from the requirements on the families (v) and (vi).

According to Lemma 2.4 it is possible to choose A, B, and C such that $|A| \cdot |B| + \binom{|A|}{2} = t - 1$ and also $|C| \cdot |B| + \binom{|C|}{2} = t - 1$. This ensures us that in the families (v) and (vi) it is sufficient to reach a common interval $[t + 2, 2t + 1]$. There also holds in the family (vi) that an arbitrary allowed label of the edge causes the weight from this interval and, conversely, for any weight from this interval it is possible to choose a label for an edge to reach this weight.

Now we estimate the bounds for the labels b_i of the vertices in the part B, i.e. the labels $b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_m$. To create the smallest weights from the interval $\left[\binom{|A|}{2} + 3, t + 1\right]$ we have

$$1 + b_1 + 1 \leq \binom{|A|}{2} + 3 \ \wedge \ 1 + b_1 + t \geq \binom{|A|}{2} + 2 + |A|.$$

This yields

$$|A| + \binom{|A|}{2} + 1 - t \leq b_1 \leq \binom{|A|}{2} + 1.$$

Next we have for $i \in [1, |B|]$

$$i|A| + \binom{|A|}{2} + 1 - t \leq b_i \leq \binom{|A|}{2} + 1 + (i-1) \cdot |A|.$$

Similarly, analysing the requirements for the family (iv) we obtain

$$t + b_m + t \geq 3t - \binom{|C|}{2} \ \wedge \ t + b_m + 1 \leq 3t - \binom{|C|}{2} + 1 - |C|.$$

This gives

$$t - \binom{|C|}{2} \leq b_m \leq 2t - \binom{|C|}{2} - |C|.$$

Next for $j \in [0, |B| - 1]$ we have

$$t - \binom{|C|}{2} - j \cdot |C| \leq b_{m-j} \leq 2t - \binom{|C|}{2} - (j+1) \cdot |C|.$$

Moreover, if the labels $b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_m$ satisfy these inequalities it is possible to label edges in families (iii) and (iv). The labelling in families (i) and (ii) is also possible because $\binom{|A|}{2} \leq t$ and $\binom{|C|}{2} \leq t$.

Next we show that it is possible to finish a required labelling in all families (i), (ii), (iii), (iv) and (vi). First we describe a choice of the labels $b_1 \le b_2 \le b_3 \le \cdots \le b_m$ to finish the labelling of the edges in the family (v).

Let us denote $n = 3l + \epsilon$ where $\epsilon \in \{-1, 0, 1\}$. Then

$$t = \left\lceil \frac{(3l+\epsilon)^2 - (3l+\epsilon) + 4}{6} \right\rceil$$
$$= \left\lceil \frac{3l^2 - l}{2} + l\epsilon + \frac{\epsilon^2 - \epsilon + 4}{6} \right\rceil = \frac{3l^2 - l}{2} + l\epsilon + \left\lceil \frac{\epsilon^2 - \epsilon + 4}{6} \right\rceil.$$

Hence

$$t = \frac{3l^2 - l}{2} + l\epsilon + 1.$$

With respect to Lemma 2.4 observe that $|A| = |C| = l$ and $|B| = l + \epsilon$. Then we revise the inequalities for the labels $b_1 \le b_2 \le b_3 \le \cdots \le b_{l+\epsilon}$ and obtain:

$$\binom{l}{2} + 1 + il - t \le b_i \le \binom{l}{2} + 1 + (i-1)l$$

and

$$t - \binom{l}{2} - (l + \epsilon - i)l \le b_i \le 2t - \binom{l}{2} - l(l + \epsilon - i + 1).$$

Observe that $\binom{l}{2} + 1 + il - t \le 0$ and $2t - \binom{l}{2} - l(l + \epsilon - i + 1) \ge t$ and hence it is sufficient to consider the next inequalities

$$t - \binom{l}{2} - (l + \epsilon - i)l \le b_i \le \binom{l}{2} + 1 + (i-1)l$$

or, equivalently,

$$li + 1 \le b_i \le \binom{l}{2} + 1 + li - l.$$

The received interval for b_i is nonempty for $l \ge 3$, because

$$\binom{l}{2} + 1 + li - l - (li + 1) = \binom{l}{2} - l \ge 0.$$

It is not hard to find a suitable labelling in the cases $n \le 7$ except when $n = 5$.

Now we show that there are $b_1 \le b_2 \le b_3 \le \cdots \le b_{l+\epsilon}$ fulfilling our inequalities such that we are able to label edges from the family (v). We have to show that the weights of the edges can reach different values from interval $[t+2, 2t+1]$ (the same as in the family (vi)). We need $\binom{l+\epsilon}{2}$ distinct weights. To see that this is possible we define bipartite graph, where in one part there are vertices denoted by the $t+2, t+3, \ldots, 2t+1$ and in the other part there are vertices corresponding to the edges with the end vertices in the part B. We denote them by e_{ij} where $1 \le i < j \le l + \epsilon$. There is an edge between e_{ij} and $t + 1 + k$ if and only if on the edge e_{ij} it is possible to reach the weight $t + 1 + k$ (i.e. $b_i + b_j + 1 \le t + 1 + k \le b_i + b_j + t$).

The degree of the vertex e_{ij}, $\deg(e_{ij})$, in our bipartite graph is the number of integers k that fulfill the condition

$$b_i + b_j - t \le k \le b_i + b_j - 1 \wedge 1 \le k \le t.$$

The degree of the vertex $e_{i,j}$ in our bipartite graph is at least

$$1 + \min\{t - 1, t - (b_i + b_j - t), (b_i + b_j - 1) - 1, (b_i + b_j - 1) - (b_i + b_j - t)\}.$$

Now we show that every vertex e_{ij} has at least $\binom{l+\epsilon}{2}$ neighbors and due to Hall's theorem [6] there exists a matching such that all the vertices e_{ij} are in this matching. But

$$\deg(e_{ij}) \ge \min\{t, 2t + 1 - (b_i + b_j), (b_i + b_j - 1)\}$$
$$\ge \min\{t, 2t + 1 - 2b_{l+\epsilon}, 2b_1 - 1\}.$$

We choose $b_{l+\epsilon}$ to be the minimum possible (from the derived estimations) and b_1 to be the maximum possible. Hence

$$\deg(e_{ij}) \ge \min\left\{t, 2t + 1 - 2(l \cdot (l + \epsilon) + 1), 2\left(\binom{l}{2} + 1\right)\right\} \ge l(l-1) + 1.$$

And $l \cdot (l-1) + 1 \ge \binom{l+\epsilon}{2}$ holds for all $\epsilon \in \{-1, 0, 1\}$ and all $l \ge 0$. $\square$

2.2. Complete bipartite graphs

Every bipartite graph G with partite sets of cardinalities m and n is a subgraph of the complete bipartite graph $K_{m,n}$. Hence form Lemma 2.1 we have an upper bound on the total edge irregularity strength of a bipartite graph G, $tes(G) \leq tes(K_{m,n})$. We show that all complete bipartite graphs have the total edge irregularity strength equal to the lower bound from (1). The case when $n = 1$ or $m = 1$ was discussed before, since $K_{1,m}$ and $K_{n,1}$ are trees, see [7].

Theorem 2.5. *Let $n, m \geq 2$ and $K_{m,n}$ be the complete bipartite graph with partite sets of cardinalities m and n. Then $tes(K_{m,n}) = \lceil \frac{m \cdot n+2}{3} \rceil$.*

For purposes of the proof we need the following lemma:

Lemma 2.6. *Let n, m be integers and let $t = \lceil \frac{m \cdot n+2}{3} \rceil$. Let A_1, B_1, C_1 be mutually disjoint sets of vertices in one part of $K_{n,m}$ and A_2, B_2, C_2 be mutually disjoint sets of vertices in the other part of $K_{n,m}$ with the cardinalities $|A_1| = \lfloor \frac{n+1}{3} \rfloor$, $|B_1| = n - 2 \cdot \lfloor \frac{n+1}{3} \rfloor$, $|C_1| = \lfloor \frac{n+1}{3} \rfloor$ and $|A_2| = \lfloor \frac{m+1}{3} \rfloor$, $|B_2| = m - 2 \cdot \lfloor \frac{m+1}{3} \rfloor$, $|C_2| = \lfloor \frac{m+1}{3} \rfloor$. Then the following hold:*

(i) $|E(A_1, A_2)| + |E(A_1, B_2)| + |E(A_2, B_1)| = t - 1$
(ii) $|E(C_1, C_2)| + |E(C_1, B_2)| + |E(C_2, B_1)| = t - 1$
(iii) $|E(B_1, B_2)| + |E(A_1, C_2)| + |E(A_2, C_1)| \in \{t - 2, t - 1, t\}$.

Proof. Let us denote $n = 3k + \epsilon, \epsilon \in \{-1, 0, 1\}$ and $m = 3l + \gamma, \gamma \in \{-1, 0, 1\}$ then

$$t = \left\lceil \frac{m \cdot n + 2}{3} \right\rceil = \left\lceil \frac{(3k + \epsilon) \cdot (3l + \gamma) + 2}{3} \right\rceil$$

$$t = \left\lceil \frac{9kl + 3k\gamma + 3l\epsilon + \gamma\epsilon + 2}{3} \right\rceil = 3kl + k\gamma + l\epsilon + \left\lceil \frac{\epsilon\gamma + 2}{3} \right\rceil$$
$$= 3kl + k\gamma + l\epsilon + 1.$$

The sets $A_1, \ldots, C_2$ have the following cardinalities:
$|A_1| = |C_1| = \lfloor \frac{n+1}{3} \rfloor = k$
$|B_1| = n - 2 \cdot \lfloor \frac{n+1}{3} \rfloor = k + \epsilon$
$|A_2| = |C_2| = \lfloor \frac{m+1}{3} \rfloor = l$
$|B_2| = m - 2 \cdot \lfloor \frac{m+1}{3} \rfloor = l + \gamma$.
We now compute the number of edges in the case (i) (the same for (ii))

$$lk + l(k + \epsilon) + k(l + \gamma) = 3kl + k\gamma + l\epsilon = t - 1.$$

For the number of edges in the case (iii) we have

$$(k + \epsilon)(l + \gamma) + kl + kl = 3kl + k\gamma + l\epsilon + \epsilon\gamma \in \{t - 2, t - 1, t\}. \quad \square$$

Now we are able to prove the second main result, Theorem 2.5.

Proof. According to (1) it is enough to prove $tes(K_{m,n}) \geq \lceil \frac{m \cdot n+2}{3} \rceil$. The main idea of the proof is analogous to the proof of Theorem 2.3, i.e. we split the set of vertices into six mutually disjoint subsets (parts) A_1, B_1, C_1 and A_2, B_2, C_2 with cardinalities as in Lemma 2.6. We label the vertices from the parts A_1 and A_2 by label 1, vertices from the parts C_1 and C_2 by label $t = \lceil \frac{m \cdot n+2}{3} \rceil$. The vertices from the parts B_1 and B_2 are labelled with labels $b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_z$, where $z = |B_1|$ and $b'_1 \leq b'_2 \leq b'_3 \leq \cdots \leq b'_x$, where $x = |B_2|$. We label the edges of $K_{m,n}$ according to which of the following families they belong to.

(i) We label the edges from $E(A_1, A_2)$ with consecutive integers from 1 to $|A_1| \cdot |A_2|$ to obtain the first $|A_1| \cdot |A_2|$ weights, i.e. weights from the integer interval $[3, |A_1| \cdot |A_2| + 2]$.
(ii) We label the edges from $E(C_1, C_2)$ with consecutive integers from t down to $t + 1 - |C_1| \cdot |C_2|$ to obtain the last $|C_1| \cdot |C_2|$ weights, i.e. weights from the interval $[3t - |C_1| \cdot |C_2| + 1, 3t]$.
(iii) (a) We label the edges from $E(A_1, B_2)$ to obtain weights creating the interval $[|A_1| \cdot |A_2| + 3, |A_1| \cdot |A_2| + 2 + |A_1| \cdot |B_2|]$. Appropriate weights are created on the edges that have one end vertex labelled with the label from $b'_1 \leq b'_2 \leq b'_3 \leq \cdots \leq b'_x$.
(b) We label the edges from $E(A_2, B_1)$ to obtain weights creating the interval $[|A_1| \cdot |A_2| + 3 + |A_1| \cdot |B_2|, |A_1| \cdot |A_2| + 2 + |A_1| \cdot |B_2| + |A_2| \cdot |B_1|]$. Appropriate weights are created on the edges that have one end vertex labelled with the label from $b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_z$.
(iv) With the same technique we create the interval $[3t - |C_1| \cdot |C_2| + 1 - |B_1| \cdot |C_2| - |B_2| \cdot |C_1|, 3t - |C_1| \cdot |C_2|]$ of weights on the edges from $E(C_1, B_2)$ and $E(C_2, B_1)$.

(v) We label the edges from $E(B_1, B_2)$ to obtain different weights from the interval $[|A_1| \cdot |A_2| + |A_1| \cdot |B_2| + |A_2| \cdot |B_1| + 3, 3t - |C_1| \cdot |C_2| - |B_1| \cdot |C_2| - |B_2| \cdot |C_1|]$.
(vi) We label the edges from $E(A_1, C_2)$ and from the set $E(A_2, C_1)$ to obtain weights from the interval $[|A_1| \cdot |A_2| + |A_1| \cdot |B_2| + |A_2| \cdot |B_1| + 3, 3t - |C_1| \cdot |C_2| - |B_1| \cdot |C_2| - |B_2| \cdot |C_1|]$ different from all existing weights.

According to Lemma 2.6 it is possible to choose A_1, B_1, C_1 and A_2, B_2, C_2 such that $|A_1| \cdot |B_2| + |A_2| \cdot |B_1| + |A_1| \cdot |A_2| = t - 1$ and also $|C_1| \cdot |B_2| + |C_2| \cdot |B_1| + |C_1| \cdot |C_2| = t - 1$. This ensures us that for the edges from the families (v) and (vi) it is sufficient to create weights from the interval $[t + 2, 2t + 1]$, and for the edges from the family (vi) an arbitrary label of the edge causes the weight from this interval and conversely for any weight from this interval it is possible to choose a label for an edge to reach this weight.

Now we estimate the bounds for the labels of the vertices in the part B_1, i.e. the labels $b_1 \le b_2 \le b_3 \le \cdots \le b_z$ and the labels of the vertices in B_2, i.e. the labels $b'_1 \le b'_2 \le b'_3 \le \cdots \le b'_x$. To create the smallest weights from the interval $[|A_1| \cdot |A_2| + 3, t + 1]$ observe that

$$1 + b'_1 + 1 \le |A_1| \cdot |A_2| + 3 \wedge 1 + b'_1 + t \ge |A_1| \cdot |A_2| + 2 + |A_1|.$$

We obtain

$$|A_1| + |A_1| \cdot |A_2| + 1 - t \le b'_1 \le |A_1| \cdot |A_2| + 1.$$

Next for $i \in [1, |B_2|]$ we have

$$i|A_1| + |A_1| \cdot |A_2| + 1 - t \le b'_i \le |A_1| \cdot |A_2| + 1 + (i - 1) \cdot |A_1|.$$

Similarly to create the smallest weights from the interval $[|A_1||A_2| + |A_1||B_2| + 3, t + 1]$ observe that

$$1 + b_1 + 1 \le |A_1||A_2| + |A_1||B_2| + 3 \wedge 1 + b_1 + t \ge |A_1||A_2| + |A_1||B_2| + 2 + |A_2|.$$

We obtain

$$|A_2| + |A_1| \cdot |A_2| + |A_1| \cdot |B_2| + 1 - t \le b_1 \le |A_1| \cdot |A_2| + |A_1| \cdot |B_2| + 1.$$

Next for $i \in [1, |B_1|]$ we have

$$i|A_2| + |A_1||A_2| + |A_1||B_2| + 1 - t \le b_i \le |A_1||A_2| + |A_1||B_2| + 1 + (i - 1)|A_2|.$$

Similarly, analysing the family (iv), we obtain

$$t + b_z + t \ge 3t - |C_1| \cdot |C_2| \wedge t + b_z + 1 \le 3t - |C_1| \cdot |C_2| + 1 - |C_2|.$$

We obtain

$$t - |C_1| \cdot |C_2| \le b_z \le 2t - |C_1| \cdot |C_2| - |C_2|.$$

Next for $j \in [0, |B_1| - 1]$ we have

$$t - |C_1| \cdot |C_2| - j \cdot |C_2| \le b_{z-j} \le 2t - |C_1| \cdot |C_2| - (j + 1) \cdot |C_2|$$

and also for $b'_1 \le b'_2 \le b'_3 \le \cdots \le b'_x$ we have

$$t + b'_x + t \ge 3t - |C_1||C_2| - |B_1||C_2| \wedge t + b'_x + 1 \le 3t - |C_1||C_2| - |B_1||C_2| + 1 - |C_1|.$$

We obtain

$$t - |C_1| \cdot |C_2| - |B_1| \cdot |C_2| \le b'_x \le 2t - |C_1| \cdot |C_2| - |B_1| \cdot |C_2| - |C_1|.$$

Next for $j \in [0, |B_2| - 1]$ we have

$$t - |C_1||C_2| - |B_1||C_2| - j|C_1| \le b'_{x-j} \le 2t - |C_1||C_2| - |B_1||C_2| - (j + 1)|C_1|.$$

Moreover if the labels $b_1 \le b_2 \le b_3 \le \cdots \le b_z$ and $b'_1 \le b'_2 \le b'_3 \le \cdots \le b'_x$ fulfill these inequalities it is possible to complete the labelling for edges from the families (iii), (iv) and (v). The labellings in the families (i) and (ii) are possible because $|A_1||A_2| \le t$ and $|C_1||C_2| \le t$.

We have seen that it is possible to finish the labelling of edges in all families (i), (ii), (iiia), (iiib), (iv) and (vi). To finish the labelling of the edges in the family (v) it is sufficient to describe how to choose the labels $b_1 \le b_2 \le b_3 \le \cdots \le b_z$ and $b'_1 \le b'_2 \le b'_3 \le \cdots \le b'_x$.

Let us denote $n = 3k + \epsilon$ where $\epsilon \in \{-1, 0, 1\}$ and $m = 3l + \gamma$ where $\gamma \in \{-1, 0, 1\}$. Then

$$t = 3kl + k\gamma + l\epsilon + 1.$$

With respect to Lemma 2.6 we can observe that if $|A_1| = |C_1| = k$, $|B_1| = k + \epsilon$, $|A_2| = |C_2| = l$ and $|B_2| = l + \gamma$. Then we can specify the inequalities for the labels $b_1 \le b_2 \le b_3 \le \cdots \le b_{k+\epsilon}$. We obtain

$$il + kl + k(l + \gamma) + 1 - t \le b_i \le kl + k(l + \gamma) + 1 + (i - 1)l$$

and

$$t - kl - (k + \epsilon - i) \cdot l \leq b_i \leq 2t - kl - (k + \epsilon - i + 1) \cdot l.$$

Observe that $il + kl + k(l + \gamma) + 1 - t \leq 0$ and $2t - kl - (k + \epsilon - i + 1) \cdot l \geq t$ and hence the next inequalities hold

$$t - kl - (k + \epsilon - i) \cdot l \leq b_i \leq kl + k(l + \gamma) + 1 + (i - 1)l$$
$$kl + k\gamma + li + 1 \leq b_i \leq 2kl + k\gamma + 1 + li - l.$$

For $k \geq 1$ and $l \geq 1$ it is possible to choose b_i from the prescribed interval which is nonempty. For $k \geq 1$ and $l \geq 1$ we have

$$2kl + k\gamma + 1 + li - l - (kl + k\gamma + li + 1) = l \cdot (k - 1) \geq 0.$$

Analogously the inequalities for the labels $b'_1 \leq b'_2 \leq b'_3 \leq \cdots \leq b'_{l+\gamma}$ can be estimated as follows:

$$ik + kl + 1 - t \leq b'_i \leq kl + 1 + (i - 1) \cdot k$$

and

$$t - kl - (k + \epsilon)l - (l + \gamma - i)k \leq b'_i \leq 2t - kl - (k + \epsilon)l - (l + \gamma - i + 1)k.$$

Observe that $ik + kl + 1 - t \leq 0$ and $2t - kl - (k + \epsilon)l - (l + \gamma - i + 1)k \geq t$ and hence the next inequalities hold

$$t - kl - (k + \epsilon)l - (l + \gamma - i)k \leq b'_i \leq kl + 1 + (i - 1)k$$
$$ki + 1 \leq b'_i \leq kl + 1 + ki - k.$$

The prescribed interval for b_i is nonempty for $k \geq 1$ and $l \geq 1$

$$kl + 1 + ik - k - ki + 1 = k(l - 1) \geq 0.$$

Now we show that with $b_1 \leq b_2 \leq b_3 \leq \cdots \leq b_{k+\epsilon}$ and $b'_1 \leq b'_2 \leq b'_3 \leq \cdots \leq b'_{l+\gamma}$ obtained in this way we are able to finish the labelling in the family (v). We have to prove that the weights of the edges can reach different values from the interval $[t + 2, 2t + 1]$. We need $(l + \gamma) \cdot (k + \epsilon)$ distinct weights. To see this we define a bipartite graph, where in one partition there are vertices denoted by $t + 2, t + 3, \ldots, 2t + 1$ and in the other partition there are vertices corresponding to the edges with one end vertex in the part B_1 and the second one in part B_2. We denote them by e_{ij} where $1 \leq i \leq k + \epsilon$ and $1 \leq j \leq l + \gamma$. There is an edge between $e_{ij} \in B_1$ and $t + 1 + k \in B_2$ if and only if there is a labelling of an edge $e_{ij} \in E(G)$ such that its weight is $t + 1 + k$ (i.e. $b_i + b'_j + 1 \leq t + 1 + k \leq b_i + b'_j + t$).

The degree of the vertex e_{ij} in our bipartite graph is the number of integers k that fulfill the condition

$$b_i + b'_j - t \leq k \leq b_i + b'_j - 1 \wedge 1 \leq k \leq t.$$

The degree of the vertex e_{ij} in our bipartite graph is at least

$$1 + \min\{t - 1, t - (b_i + b'_j - t), (b_i + b'_j - 1) - 1, (b_i + b'_j - 1) - (b_i + b'_j - t)\}.$$

Now we show that every vertex e_{ij} has at least $(k + \epsilon) \cdot (l + \gamma)$ neighbors and due to Hall's theorem there exists a matching such that all the vertices e_{ij} are in this matching. However, the degree

$$\deg(e_{ij}) \geq \min\{t, 2t + 1 - (b_i + b'_j), (b_i + b'_j - 1)\}$$
$$\geq \min\{t, 2t + 1 - b_{k+\epsilon} - b_{l+\gamma}, b_1 + b'_1 - 1\} \geq \min\{t, t - k\gamma, t - l\epsilon\}.$$

But this minimum is at least $(k + \varepsilon)(l + \gamma)$ for all $k, l \geq 1$. □

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