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Categories enriched on two sides

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Abstract

We introduce morphisms $\mathscr{V} \to \mathscr{W}$ of bicategories, more general than the original ones of Bénabou. When $\mathscr{V} = \mathbf{1}$, such a morphism is a category enriched in the bicategory $\mathscr{W}$. Therefore, these morphisms can be regarded as categories enriched in bicategories "on two sides". There is a composition of such enriched categories, leading to a tricategory **Caten** of a simple kind whose objects are bicategories. It follows that a morphism from $\mathscr{V}$ to $\mathscr{W}$ in **Caten** induces a 2-functor $\mathscr{V}$-**Cat** $\to \mathscr{W}$-**Cat**, while an adjunction between $\mathscr{V}$ and $\mathscr{W}$ in **Caten** induces one between the 2-categories $\mathscr{V}$-**Cat** and $\mathscr{W}$-**Cat**. Left adjoints in **Caten** are necessarily homomorphisms in the sense of Bénabou, while right adjoints are not. Convolution appears as the internal hom for a monoidal structure on **Caten**. The 2-cells of **Caten** are functors; modules can also be defined, and we examine the structures associated with them. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

For any monoidal category $\mathscr{M} = (\mathscr{M}, \otimes, \mathrm{I})$ we have the notion of a *category enriched in* $\mathscr{M}$ (or an $\mathscr{M}$-*category*), along with the notions of $\mathscr{M}$-*functor* and $\mathscr{M}$-*natural*

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transformation. The totality of all these things constitutes a 2-category $\mathscr{M}$-**Cat**; see [2,13,18]. Appearing in [1] is the notion of what is now called a *monoidal functor* $\Phi : \mathscr{M} \to \mathscr{M}'$, consisting of a functor $\phi : \mathscr{M} \to \mathscr{M}'$, a morphism $\phi_0 : I' \to \phi I$, and a natural transformation ϕ_2 having components $\phi_{2;X,Y} : \phi X \otimes' \phi Y \to \phi(X \otimes Y)$, with these data satisfying three "coherence" axioms. A monoidal functor Φ induces a 2-functor $\Phi_* : \mathscr{M}$-**Cat** $\to \mathscr{M}'$-**Cat** which we may think of as a "change of base". Further introduced in [13] is the notion of a *monoidal natural transformation* $\alpha : \Phi \Rightarrow \Psi : \mathscr{M} \to \mathscr{M}'$ providing the 2-cells for a 2-category **MonCat**. The process sending $\mathscr{M}$ to $\mathscr{M}$-**Cat** and Φ to Φ_* extends to a 2-functor $(\ )_* :$ **MonCat** $\to$ 2-**Cat**.

The nature of adjunctions $\Psi \dashv \Phi : \mathscr{M} \to \mathscr{M}'$ in **MonCat** was determined in [17]. Indeed, the monoidal $\Psi = (\psi, \psi_0, \psi_2) : \mathscr{M} \to \mathscr{M}'$ admits a right adjoint in **MonCat** precisely when the functor $\psi : \mathscr{M} \to \mathscr{M}'$ admits a right adjoint in **Cat** and all the morphisms $\psi_0, \psi_{2;X,Y}$ are invertible. We note, without going into details here, that we can repeat the above with monoidal categories replaced by the more general *promonoidal categories* of [9].

Our primary concern in the present paper is with a different generalization. To give a bicategory $\mathscr{V}$ with a single object $*$ is equally to give the monoidal category $\mathscr{M} = \mathscr{V}(*, *)$; and such a $\mathscr{V}$ is called the *suspension* $\Sigma\mathscr{M}$ of $\mathscr{M}$ (although often one speaks loosely of "the bicategory $\mathscr{M}$", meaning the bicategory $\Sigma\mathscr{M}$). Around 1980 it was observed that certain important mathematical structures can be fruitfully described as *categories enriched in a bicategory* $\mathscr{V}$, or $\mathscr{V}$*-categories*. There is a 2-category $\mathscr{V}$-**Cat** of $\mathscr{V}$-categories, $\mathscr{V}$-functors, and $\mathscr{V}$-natural transformations, which reduces to the 2-category $\mathscr{M}$-**Cat** above when $\mathscr{V} = \Sigma\mathscr{M}$ has one object. (No real ambiguity arises in practice from the fact that $(\Sigma\mathscr{M})$-**Cat** is another name for $\mathscr{M}$-**Cat**.) Categories enriched in a bicategory were first treated in print in the articles [25,26] of Walters, who acknowledges earlier notes [4] on the subject by Renato Betti (see also [5]). A little later, more complete and systematic treatments of the 2-category $\mathscr{V}$-**Cat** were given in [24,6]. Familiarity with the basic results concerning $\mathscr{V}$-**Cat** contained in those papers is not a prerequisite for reading the present paper, since these results recur as special cases of our results below. Finally, we mention that Bénabou's fundamental paper [3] on bicategories already contains, under the name of *polyad*, the definition of a $\mathscr{V}$-category for a general bicategory $\mathscr{V}$—this, however, not being developed further except in the case $\mathscr{V} = \Sigma\mathscr{M}$.

The present investigation began as the study of "change of base" for categories enriched in bicategories. Given bicategories $\mathscr{V}$ and $\mathscr{W}$, we seek a notion of "morphism" $F : \mathscr{V} \to \mathscr{W}$ that will induce, in a well-behaved functorial way, a 2-functor $F_* : \mathscr{V}$-**Cat** $\to \mathscr{W}$-**Cat**. A first idea, since it reduces when $\mathscr{V}$ and $\mathscr{W}$ are suspensions of monoidal categories $\mathscr{M}$ and $\mathscr{N}$ to a monoidal functor $\Phi : \mathscr{M} \to \mathscr{N}$, is to take for F a *lax functor* $F : \mathscr{V} \to \mathscr{W}$ (that is, a *morphism of bicategories* in the terminology of [3]). Recall that such an F takes an object X of $\mathscr{V}$ to an object FX of $\mathscr{W}$, and comprises functors $F_{X,Y} : \mathscr{V}(X,Y) \to \mathscr{W}(FX, FY)$, along with arrows $F_{0;X} : 1_{FX} \to F1_X$ and arrows $F_{2;f,g} : Ff \otimes' Fg \to F(f \otimes g)$ natural in f and g and subject to coherence conditions: here $\otimes$ and $\otimes'$ denote horizontal composition in $\mathscr{V}$ and $\mathscr{W}$, respectively. Certainly such an

F does indeed give a 2-functor $F_* : \mathscr{V}\text{-}\mathbf{Cat} \to \mathscr{W}\text{-}\mathbf{Cat}$ with $1_* = 1$ and $(HF)_* = H_* F_*$, just as in the more classical special case where $\mathscr{V} = \Sigma\mathscr{M}$ and $\mathscr{W} = \Sigma\mathscr{N}$. However, the following consideration led us to look for "morphisms" $\mathscr{V} \to \mathscr{W}$ between bicategories that are more general than lax functors.

When the lax functor $F : \mathscr{V} \to \mathscr{W}$ is such that each functor $F_{X,Y} : \mathscr{V}(X,Y) \to \mathscr{W}(FX,FY)$ admits a right adjoint $R_{X,Y}$ in **Cat** and such that all the arrows $F_{0;X}, F_{2;f,g}$ are invertible, it turns out that the 2-functor $F_* : \mathscr{V}\text{-}\mathbf{Cat} \to \mathscr{W}\text{-}\mathbf{Cat}$ admits a right adjoint $F^* : \mathscr{W}\text{-}\mathbf{Cat} \to \mathscr{V}\text{-}\mathbf{Cat}$. Yet there is, in general, no lax functor $G : \mathscr{W} \to \mathscr{V}$ here for which $G_* \cong F^*$. There will, however, be such a G among the more general morphisms we shall now introduce. (*Note*. A lax functor $F : \mathscr{V} \to \mathscr{W}$ with all the $F_{0;X}$ and all the $F_{2;f,g}$ invertible was called by Bénabou in [3] a *homomorphism of bicategories*; we shall also call it a *pseudo-functor* from $\mathscr{V}$ to $\mathscr{W}$.)

We obtain a type of "morphism" $F : \mathscr{V} \to \mathscr{W}$, more general than a lax functor, as follows. Instead of the function $\text{ob}\,F : \text{ob}\,\mathscr{V} \to \text{ob}\,\mathscr{W}$ which forms part of a lax functor F, we take instead a *span*

$$\text{ob}\,\mathscr{V} \xleftarrow{(\ )_-} S \xrightarrow{(\ )_+} \text{ob}\,\mathscr{W};$$

and instead of the $F_{X,Y} : \mathscr{V}(X,Y) \to \mathscr{W}(FX,FY)$ we take functors $F_{s,t} : \mathscr{V}(s_-, t_-) \to \mathscr{W}(s_+, t_+)$, along with appropriate analogues of $F_{0;X}$ and $F_{2;f,g}$. With these new morphisms and the evident notion of 2-cell, we get a bicategory $\mathbb{B}$ whose objects are the bicategories (in some universe); and we further get, as desired, a 2-functor $(\ )_* : \mathbb{B} \to 2\text{-}\mathbf{Cat}$ sending $\mathscr{V}$ to $\mathscr{V}\text{-}\mathbf{Cat}$. In fact, we see at once that the 2-functor $(\ )_*$ is representable: writing $\mathbf{1}$ for the "unit" bicategory with one object, one arrow, and one 2-cell, we find that $\mathbb{B}(\mathbf{1}, \mathscr{V}) \cong \mathscr{V}\text{-}\mathbf{Cat}$ (at least as categories—for $\mathbb{B}$ as yet has no 3-cells). This suggests a totally new point of view: a morphism $F : \mathscr{V} \to \mathscr{W}$ in $\mathbb{B}$ may be thought of as a *category enriched in* $\mathscr{V}$ *on one side*, *and in* $\mathscr{W}$ *on the other*; or better, *a category enriched from* $\mathscr{V}$ *to* $\mathscr{W}$. To accommodate this point of view, we use instead of F a letter more traditionally used for a "category", such as $\mathscr{A} : \mathscr{V} \to \mathscr{W}$, with

$$\text{ob}\,\mathscr{V} \xleftarrow{(\ )_-} \text{ob}\,\mathscr{A} \xrightarrow{(\ )_+} \text{ob}\,\mathscr{W}$$

for the span above, and

$$\mathscr{A}(A,B) : \mathscr{V}(A_-, B_-) \to \mathscr{W}(A_+, B_+)$$

for the earlier $F_{s,t}$. Bicategories, unlike categories, are often named for their morphisms; we shall use **Caten** for the $\mathbb{B}$ above, since its morphisms are enriched categories.

We begin our formal treatment in the next section, defining **Caten** as a bicategory, giving examples of its morphisms, and discussing its basic properties. Then in Section 3 we add the 3-cells, exhibiting **Caten** as a tricategory of a very special kind, which is almost a "3-category". The reader, in fact, needs no prior knowledge of $\mathscr{V}$-**Cat**, since we re-find it below as the 2-category $\mathbf{Caten}(\mathbf{1}, \mathscr{V})$; and the "change of base"

2-functor $\mathscr{V}$-**Cat** $\to \mathscr{W}$-**Cat** arising from $\mathscr{A} : \mathscr{V} \to \mathscr{W}$ in **Caten** is nothing but the 2-functor **Caten**$(\mathbf{1}, \mathscr{V}) \to$ **Caten**$(\mathbf{1}, \mathscr{W})$ given by composition with $\mathscr{A}$. Section 4 exhibits a monoidal structure on **Caten** and describes the internal homs **Conv**$(\mathscr{V}, \mathscr{W})$ when $\mathscr{V}$ is locally small and $\mathscr{W}$ is locally cocomplete. Local cocompletion is studied in Section 5, and used in Section 6 to compare **Caten** with a generalization **PCaten** in which the morphisms $\mathscr{V} \to \mathscr{W}$ are now *procategories*. Finally, we turn in Section 7 to *modules* between categories enriched from $\mathscr{V}$ to $\mathscr{W}$.

Before going on, we make some comments about questions of *size*, such as the distinction between small and large sets, or small and large categories. For the purposes of this Introduction, one may be content to interpret such symbols as the **Cat**, $\mathscr{M}$-**Cat**, **MonCat**, 2-**Cat**, $\mathscr{V}$-**Cat**, and **Caten** above purely in a "metacategorical" sense: we are merely talking about certain kinds of structure, with no reference whatever to matters of size; and observing that, for instance, in this context **Cat** and **MonCat** are 2-categories, while 2-**Cat** is a 3-category that may be seen merely as a 2-category, whereupon $\mathscr{M} \mapsto \mathscr{M}$-**Cat** is a 2-functor **MonCat** $\to$ 2-**Cat**. When, however, we leave the mere naming of structures and embark upon concrete mathematical arguments, which are to be free of Russell-type paradoxes, we need a safer context, such as is provided by supposing that the morphisms of any category—or equally the 2-cells of any bicategory—form a *set*. And by a *set* here is understood an object of a chosen category **Set** of sets—meaning a 2-valued Boolean topos with natural-number-object—large enough for the purpose at hand: moreover, being "large enough" includes the existence of another category **set** of sets, called *the category of small sets*, which is a category-object in **Set** (also called a category *internal* to **Set**).

Now, by "a category $\mathscr{A}$" is meant a category-object in **Set**; it is *locally small* if each $\mathscr{A}(\mathrm{A}, \mathrm{B})$ is small, and is *small* if $\mathrm{ob}\,\mathscr{A}$ is small; in particular the category **set** is locally small. Similarly a bicategory—or in particular a 2-category—is one internal to **Set**, and it is *small* when its set of 2-cells is in **set**; while an $\mathscr{M}$-category or a $\mathscr{V}$-category $\mathscr{A}$ has $\mathrm{ob}\,\mathscr{A} \in$ **Set**, being *small* if $\mathrm{ob}\,\mathscr{A} \in$ **set**.

We write **Cat**, $\mathscr{M}$-**Cat**, 2-**Cat** for the 2-categories of categories, $\mathscr{M}$-categories, or 2-categories (these last really form a 3-category) in the sense above. But now the category **Set** is not itself an object of **Cat**, since ob(**Set**) is not a set. Yet nothing is lost by this, since the meaning of "set" can be flexible (if one admits the existence of arbitrarily large inaccessible cardinals). For **Set** is an object of the 2-category **CAT** of category-objects in a larger category **SET** of sets, containing **Set** as a category-object. Similarly, 2-**Cat** is an object of the appropriate 2-**CAT** (or 3-**CAT**), and so on. It suffices, of course, to discuss **Cat** and 2-**Cat**, since whatever is true of these (in the appropriate language) is also true of **CAT** and of 2-**CAT**.

So we continue to understand "category" and "bicategory" in the internal-to-**Set** sense above, writing "large category" or "large bicategory" for those internal to some larger **SET**; and we turn now to a precise definition of that version of the tricategory **Caten** which is based on **Set**: in the sense that its objects are the bicategories—meaning those internal-to-**Set** ones—and each morphism $\mathscr{A} : \mathscr{V} \to \mathscr{W}$ in **Caten** has $\mathrm{ob}\,\mathscr{A} \in$ **Set**.

2. The bicategory *Caten*

2.1. We suppose the reader to be familiar with the bicategory **Span** (= Span(**Set**)) whose objects are sets, whose hom-category **Span**(X, Y) is **Set**/(X × Y), and whose composition law is that formed in the obvious way using pullbacks (defined in **Set** by the usual canonical construction); see again [3]. Given a function $f : X \to Y$, we write $f_* : X \to Y$ and $f^* : Y \to X$ for the respective spans

$$X \xleftarrow{1_X} X \xrightarrow{f} Y, \quad Y \xleftarrow{f} X \xrightarrow{1_X} X.$$

There is an adjunction $f_* \dashv f^*$ in **Span**, and in fact [8] every left adjoint $\phi : X \to Y$ in **Span** is isomorphic to f_* for a unique $f : X \to Y$.

2.2. Let $\mathscr{V}$ and $\mathscr{W}$ be bicategories in which horizontal composition is denoted by $\otimes$ and $\otimes'$ respectively. A *category* $\mathscr{A}$ *enriched from* $\mathscr{V}$ *to* $\mathscr{W}$, or just a *category* $\mathscr{A} : \mathscr{V} \to \mathscr{W}$, is given by the following data:

(i) a set $\mathrm{ob}\,\mathscr{A}$ of *objects of* $\mathscr{A}$, provided with functions $(\,)_-, (\,)_+$ as in

$$\mathrm{ob}\,\mathscr{V} \xleftarrow{(\,)_-} \mathrm{ob}\,\mathscr{A} \xrightarrow{(\,)_+} \mathrm{ob}\,\mathscr{W}; \tag{2.1}$$

equivalently, we are given a span $(\mathrm{ob}\,\mathscr{A}, (\,)_-, (\,)_+) : \mathrm{ob}\,\mathscr{V} \to \mathrm{ob}\,\mathscr{W}$;

(ii) for each pair A, B of objects of $\mathscr{A}$, a functor

$$\mathscr{A}(A, B) : \mathscr{V}(A_-, B_-) \to \mathscr{W}(A_+, B_+); \tag{2.2}$$

(iii) for each object A of $\mathscr{A}$, a morphism (providing "identities")

$$\eta_A : 1_{A_+} \to \mathscr{A}(A, A)(1_{A_-}) \tag{2.3}$$

in $\mathscr{W}(A_+, A_+)$;

(iv) for each triple A, B, C of objects of $\mathscr{A}$, a natural transformation (providing "composition")

$$\begin{aligned} \mu^B_{A,C} : \otimes'(\mathscr{A}(B, C) \times \mathscr{A}(A, B)) \Rightarrow \mathscr{A}(A, C) \otimes : \\ \mathscr{V}(B_-, C_-) \times \mathscr{V}(A_-, B_-) \to \mathscr{W}(A_+, C_+), \end{aligned} \tag{2.4}$$

whose component at $(g, f) \in \mathscr{V}(B_-, C_-) \times \mathscr{V}(A_-, B_-)$ we may write as

$$\mu^B_{A,C}(g, f) : \mathscr{A}(B, C)(g) \otimes' \mathscr{A}(A, B)(f) \to \mathscr{A}(A, C)(g \otimes f). \tag{2.5}$$

These data are to satisfy the following left unit, right unit, and associativity axioms:

$$\begin{array}{ccc} \mathscr{A}(B, B)(1_{B_-}) \otimes' \mathscr{A}(A, B)(f) & \xrightarrow{\mu^B_{A,B}(1_{B_-}, f)} & \mathscr{A}(A, B)(1_{B_-} \otimes f) \\ {\scriptstyle \eta_B \otimes' 1} \big\uparrow & & \big\downarrow {\scriptstyle \mathscr{A}(A,B)(\ell)} \\ 1_{B_+} \otimes' \mathscr{A}(A, B)(f) & \xrightarrow[\ell']{} & \mathscr{A}(A, B)(f), \end{array} \tag{2.6}$$

$$\begin{array}{ccc}
\mathscr{A}(A,B)(f)\otimes'\mathscr{A}(A,A)(1_{A_-}) & \xrightarrow{\mu^{A}_{A,B}(f,1_{A_-})} & \mathscr{A}(A,B)(f\otimes 1_{A_-}) \\
\uparrow{\scriptstyle 1\otimes'\eta_A} & & \downarrow{\scriptstyle \mathscr{A}(A,B)(r)} \\
\mathscr{A}(A,B)(f)\otimes' 1_{A_+} & \xrightarrow[r']{} & \mathscr{A}(A,B)(f),
\end{array} \tag{2.7}$$

$$\begin{array}{ccc}
(\mathscr{A}(C,D)(h)\otimes'\mathscr{A}(B,C)(g))\otimes'\mathscr{A}(A,B)(f) & \xrightarrow{a'} & \mathscr{A}(C,D)(h)\otimes'(\mathscr{A}(B,C)(g)\otimes'\mathscr{A}(A,B)(f)) \\
\downarrow{\scriptstyle \mu^{C}_{B,D}(h,g)\otimes' 1} & & \downarrow{\scriptstyle 1\otimes'\mu^{B}_{A,C}(g,f)} \\
\mathscr{A}(B,D)(h\otimes g)\otimes'\mathscr{A}(A,B)(f) & & \mathscr{A}(C,D)(h)\otimes'\mathscr{A}(A,C)(g\otimes f) \\
\downarrow{\scriptstyle \mu^{B}_{A,D}(h\otimes g,f)} & & \downarrow{\scriptstyle \mu^{C}_{A,D}(h,g\otimes f)} \\
\mathscr{A}(A,D)((h\otimes g)\otimes f) & \xrightarrow[\mathscr{A}(A,D)(a)]{} & \mathscr{A}(A,D)(h\otimes(g\otimes f)),
\end{array} \tag{2.8}$$

wherein a, ℓ, r and a', ℓ', r' denote the associativity and unit constraints in $\mathscr{V}$ and $\mathscr{W}$, respectively.

2.3. Examples. (a) When $\mathscr{V}$ is the unit bicategory **1**, a category $\mathscr{A}$ from $\mathscr{V}$ to $\mathscr{W}$ is in effect just a $\mathscr{W}$-category in the sense of [24,6]: the function $()_+ : \mathrm{ob}\,\mathscr{A} \to \mathrm{ob}\,\mathscr{W}$ sends each $A \in \mathscr{A}$ to its underlying $\mathscr{W}$-value, and $\mathscr{A}(A,B) : \mathbf{1} \to \mathscr{W}(A_+, B_+)$ is the hom-arrow $\mathscr{A}(A,B) : A_+ \to B_+$ in $\mathscr{W}$, while μ and η provide the composition and its identities.

(b) Among the categories $\mathscr{A}$ enriched from $\mathscr{V}$ to $\mathscr{W}$ are those for which the span (2.1) is of the form

$$\mathrm{ob}\,\mathscr{V} \xleftarrow{1} \mathrm{ob}\,\mathscr{V} \xrightarrow{()_+} \mathrm{ob}\,\mathscr{W}, \tag{2.9}$$

so that in particular $\mathrm{ob}\,\mathscr{A} = \mathrm{ob}\,\mathscr{V}$. Such $\mathscr{A}$ are precisely the *lax functors* $F : \mathscr{V} \to \mathscr{W}$, where $FX = X_+$ and where $F_{XY} = \mathscr{A}(X,Y) : \mathscr{V}(X,Y) \to \mathscr{W}(FX,FY)$.

(c) We spoke in the Introduction of the case where a lax functor $F : \mathscr{V} \to \mathscr{W}$ has the $F_{0;X}$, $F_{2;f,g}$ invertible, while each $F_{XY} : \mathscr{V}(X,Y) \to \mathscr{W}(FX,FY)$ has a right adjoint $R_{XY} : \mathscr{W}(FX,FY) \to \mathscr{V}(X,Y)$ in **Cat**. Here we obtain as follows a category $\mathscr{B}$ enriched from $\mathscr{W}$ to $\mathscr{V}$. The objects of $\mathscr{B}$ are those of $\mathscr{V}$, and for the span (2.1) we take the span

$$\mathrm{ob}\,\mathscr{W} \xleftarrow{\mathrm{ob}\,F} \mathrm{ob}\,\mathscr{B} = \mathrm{ob}\,\mathscr{V} \xrightarrow{1} \mathrm{ob}\,\mathscr{V};$$

for the functor $\mathscr{B}(X,Y)$ we take R_{XY}; the unit $1_X \to R_{XX}(1_{FX})$ is the transpose of $F^{-1}_{0;X} : F_{XX}(1_X) \to 1_{FX}$ under the adjunction $F_{XX} \dashv R_{XX}$; and the composition

$\otimes(R_{YZ} \times R_{XY}) \to R_{XZ}\otimes'$ is the mate (see [19]) of $F_2^{-1}: F_{XZ}\otimes \to \otimes'(F_{YZ} \times F_{XY})$ under the adjunctions $F_{XY} \dashv R_{XY}$ and $F_{YZ} \times F_{XY} \dashv R_{YZ} \times R_{XY}$.

(d) When $\mathscr{V} = \Sigma\mathscr{M}$ and $\mathscr{W} = \Sigma\mathscr{N}$ for monoidal categories $\mathscr{M}$ and $\mathscr{N}$, to give a lax functor $F: \mathscr{V} \to \mathscr{W}$ is just to give a monoidal functor $\Phi: \mathscr{M} \to \mathscr{N}$.

(e) The general category $\mathscr{A}$ enriched from $\Sigma\mathscr{M}$ to $\Sigma\mathscr{N}$, however, does not reduce thus to a monoidal functor $\mathscr{M} \to \mathscr{N}$. It is given by a set $\mathrm{ob}\,\mathscr{A}$, along with functors $\mathscr{A}(A,B): \mathscr{M} \to \mathscr{N}$ for $A, B \in \mathrm{ob}\,\mathscr{A}$, morphisms $\eta_A: I' \to \mathscr{A}(A,A)(I)$, and morphisms $\mu^B_{A,C}(Y,X): \mathscr{A}(B,C)(Y) \otimes' \mathscr{A}(A,B)(X) \to \mathscr{A}(A,C)(Y \otimes X)$ for $X, Y \in \mathscr{M}$, satisfying the appropriate axioms.

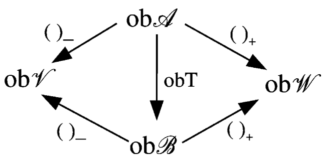
(f) As a particular example of (e), let $\mathscr{C}$ be an ordinary category provided with *actions*

$$\circ: \mathscr{C} \times \mathscr{M} \to \mathscr{C} \quad \text{and} \quad * : \mathscr{N} \times \mathscr{C} \to \mathscr{C}$$

of the monoidal categories $\mathscr{M}$ and $\mathscr{N}$, in the usual "to within isomorphism" sense; and let there further be coherent natural isomorphisms $P * (A \circ X) \cong (P * A) \circ X$, so that $\mathscr{C}$ is a "left $\mathscr{N}$-, right $\mathscr{M}$-bimodule". Finally, suppose that each $- * A: \mathscr{N} \to \mathscr{C}$ has a right adjoint $[A, -]: \mathscr{C} \to \mathscr{N}$. Then we get a category $\mathscr{A}$ enriched from $\Sigma\mathscr{M}$ to $\Sigma\mathscr{N}$, as in (e), by taking $\mathrm{ob}\,\mathscr{A} = \mathrm{ob}\,\mathscr{C}$ and $\mathscr{A}(A,B)(X) = [A, B \circ X]$.

2.4. Given bicategories $\mathscr{V}$ and $\mathscr{W}$ and categories $\mathscr{A}$ and $\mathscr{B}$ enriched from $\mathscr{V}$ to $\mathscr{W}$, a *functor* $T: \mathscr{A} \to \mathscr{B}$ *enriched from* $\mathscr{V}$ *to* $\mathscr{W}$, or simply a *functor* $T: \mathscr{A} \to \mathscr{B}$, is given by the following data:

(i) a morphism



(2.10)

of spans; that is, a function $\mathrm{ob}\,T: \mathrm{ob}\,\mathscr{A} \to \mathrm{ob}\,\mathscr{B}$, for whose value $(\mathrm{ob}\,T)(A)$ we in fact write TA, satisfying the conditions

$$(TA)_- = A_-, \quad (TA)_+ = A_+; \tag{2.11}$$

(ii) for each pair A, B of objects of $\mathscr{A}$, a natural transformation

$$T_{AB}: \mathscr{A}(A,B) \to \mathscr{B}(TA,TB): \mathscr{V}(A_-,B_-) \to \mathscr{W}(A_+,B_+), \tag{2.12}$$

whose component at $f \in \mathscr{V}(A_-,B_-)$ we may write as

$$T_{AB}(f): \mathscr{A}(A,B)(f) \to \mathscr{B}(TA,TB)(f). \tag{2.13}$$

These data are to satisfy the following two axioms, expressing the compatibility of the T_{AB} with the identities and composition. Firstly, we require commutativity of the following diagram in the category $\mathscr{W}(A_+, A_+)$:

$$\begin{array}{ccc} 1_{A_+} \xrightarrow{\eta_A} \mathscr{A}(A,A)(1_{A_-}) & \xrightarrow{T_{AA}(1_{A_-})} & \mathscr{B}(TA,TA)(1_{A_-}) \\ \Big\| & & \Big\| \\ 1_{(TA)_+} & \xrightarrow[\eta_{TA}]{} & \mathscr{B}(TA,TA)(1_{(TA)_-}). \end{array} \tag{2.14}$$

Secondly, we require commutativity of the following diagram of natural transformations (between functors from $\mathscr{V}(B_-, C_-) \times \mathscr{V}(A_-, B_-)$ to $\mathscr{W}(A_+, C_+)$):

$$\begin{array}{ccc} \otimes'(\mathscr{A}(B,C) \times \mathscr{A}(A,B)) & \xrightarrow{\mu^B_{A,C}} & \mathscr{A}(A,C)\otimes \\ {\scriptstyle \otimes'(T_{BC} \times T_{AB})}\Big\downarrow & & \Big\downarrow{\scriptstyle T_{AC}\otimes} \\ \otimes'(\mathscr{B}(TB,TC) \times \mathscr{B}(TA,TB)) & \xrightarrow[\mu^{TB}_{TA,TC}]{} & \mathscr{B}(TA,TC)\otimes, \end{array} \tag{2.15}$$

which may equally be written, in terms of the (g,f)-components for $g \in \mathscr{V}(B_-, C_-)$ and $f \in \mathscr{V}(A_-, B_-)$, as the commutativity of

$$\begin{array}{ccc} \mathscr{A}(B,C)(g) \otimes' \mathscr{A}(A,B)(f) & \xrightarrow{\mu^B_{A,C}(g,f)} & \mathscr{A}(A,C)(g \otimes f) \\ {\scriptstyle T_{BC}(g)\otimes' T_{AB}(f)}\Big\downarrow & & \Big\downarrow{\scriptstyle T_{AC}(g\otimes f)} \\ \mathscr{B}(TB,TC)(g) \otimes' \mathscr{B}(TA,TB)(f) & \xrightarrow[\mu^{TB}_{TA,TC}(g,f)]{} & \mathscr{B}(TA,TC)(g \otimes f). \end{array} \tag{2.16}$$

2.5. Examples. (a) When $\mathscr{V}$ here is the unit bicategory **1**, so that $\mathscr{A}$ and $\mathscr{B}$ are just $\mathscr{W}$-categories, a functor $T: \mathscr{A} \to \mathscr{B}$ is just a $\mathscr{W}$-functor in the sense of [24,6]; in particular, it is just an $\mathscr{N}$-functor [13] when $\mathscr{W} = \Sigma\mathscr{N}$.

(b) Consider the case when $\mathscr{A}$ and $\mathscr{B}$ both arise as in Example 2.3(b) from lax functors: say from the respective lax functors $F, G: \mathscr{V} \to \mathscr{W}$. Then we necessarily have $\mathrm{ob}\,\mathscr{A} = \mathrm{ob}\,\mathscr{B} = \mathrm{ob}\,\mathscr{V}$, and the function $\mathrm{ob}\,T$ of (2.10) must be the identity; so that (2.10) becomes the assertion that $FX = GX$ for all objects X of $\mathscr{V}$. Here, therefore, the natural transformations (2.12) have the form

$$T_{XY}: F_{XY} \to G_{XY}: \mathscr{V}(X,Y) \to \mathscr{W}(FX,FY) = \mathscr{W}(GX,GY), \tag{2.17}$$

with component at $f \in \mathscr{V}(X, Y)$ a 2-cell in $\mathscr{W}$ of the form

$$T_{XY}(f): F_{XY}(f) \to G_{XY}(f). \tag{2.18}$$

When we rewrite (2.18) as

$$\begin{array}{ccc} FX & \xrightarrow{\;1\;} & GX \\ {\scriptstyle F(f)}\downarrow & \overset{\Rightarrow}{\scriptstyle T_{XY}(f)} & \downarrow{\scriptstyle G(f)} \\ FY & \xrightarrow[\;1\;]{} & GY, \end{array} \tag{2.19}$$

and recall axioms (2.14) and (2.15), along with the naturality of (2.19) in f, we see that such a functor $T: \mathscr{A} \to \mathscr{B}$ is just what has been called an *optransformation* [3, p. 59], a *right lax transformation* [21, p. 222], or an *oplax natural transformation* [16, p. 189], with the extra property that each component $T_X: FX \to GX$ is an identity.

(c) When the $\mathscr{V}$ and $\mathscr{W}$ in (b) are of the forms $\Sigma\mathscr{M}$ and $\Sigma\mathscr{N}$ for monoidal $\mathscr{M}$ and $\mathscr{N}$ we observed in Example 2.3(d) that to give such lax functors F and G is just to give monoidal functors $\Phi, \Psi: \mathscr{M} \to \mathscr{N}$. In this case (2.17) reduces to a single natural transformation $T: \Phi \to \Psi: \mathscr{M} \to \mathscr{N}$, and axioms (2.14) and (2.16) are just the conditions for T to be a *monoidal* natural transformation in the sense of [13, p. 474].

2.6. We henceforth denote a category $\mathscr{A}$ enriched from $\mathscr{V}$ to $\mathscr{W}$ by using the arrow notation $\mathscr{A}: \mathscr{V} \to \mathscr{W}$, and look upon a functor $T: \mathscr{A} \to \mathscr{B}: \mathscr{V} \to \mathscr{W}$ as a 2-cell of the form

$$\mathscr{V} \underset{\mathscr{B}}{\overset{\mathscr{A}}{\rightrightarrows}} \mathscr{W}, \quad \Downarrow T \;; \tag{2.20}$$

we sometimes, as here, use a *double* arrow for such a T, to emphasize its "dimension"—but have no fixed rule about using double or single arrows. There is an evident "vertical" composite $ST: \mathscr{A} \to \mathscr{C}: \mathscr{V} \to \mathscr{W}$ of $T: \mathscr{A} \to \mathscr{B}: \mathscr{V} \to \mathscr{W}$ and $S: \mathscr{B} \to \mathscr{C}: \mathscr{V} \to \mathscr{W}$, as well as an evident identity $1_{\mathscr{A}}: \mathscr{A} \to \mathscr{A}: \mathscr{V} \to \mathscr{W}$, so that the categories enriched from $\mathscr{V}$ to $\mathscr{W}$ and the functors between these constitute a (large) category $Caten(\mathscr{V}, \mathscr{W})$.

Then, for bicategories $\mathscr{V}, \mathscr{W}, \mathscr{U}$, it is straightforward to define a "horizontal composition" functor

$$\circ = \circ^{\mathscr{W}}_{\mathscr{V}\mathscr{U}}: Caten(\mathscr{W}, \mathscr{U}) \times Caten(\mathscr{V}, \mathscr{W}) \to Caten(\mathscr{V}, \mathscr{U}). \tag{2.21}$$

We describe this first at the object level: categories $\mathscr{A}: \mathscr{V} \to \mathscr{W}$ and $\mathscr{C}: \mathscr{W} \to \mathscr{U}$ have a composite $\mathscr{C} \circ \mathscr{A}: \mathscr{V} \to \mathscr{U}$ where $ob(\mathscr{C} \circ \mathscr{A})$ is the span composite

$$ob\mathscr{V} \xleftarrow{(\,)_-} ob\mathscr{A} \xrightarrow{(\,)_+} ob\mathscr{W} \xleftarrow{(\,)_-} ob\mathscr{C} \xrightarrow{(\,)_+} ob\mathscr{U}, \tag{2.22}$$

so that

$$\mathrm{ob}(\mathscr{C}\circ\mathscr{A})=\{(\mathrm{C},\mathrm{A})\in\mathrm{ob}\,\mathscr{C}\times\mathrm{ob}\,\mathscr{A}\mid \mathrm{C}_-=\mathrm{A}_+\} \tag{2.23}$$

with

$$(\mathrm{C},\mathrm{A})_-=\mathrm{A}_- \quad\text{and}\quad (\mathrm{C},\mathrm{A})_+=\mathrm{C}_+, \tag{2.24}$$

and where

$$(\mathscr{C}\circ\mathscr{A})((\mathrm{C},\mathrm{A}),(\mathrm{C}',\mathrm{A}')):\ \mathscr{V}(\mathrm{A}_-,\mathrm{A}'_-)\to\mathscr{U}(\mathrm{C}_+,\mathrm{C}'_+)$$

is the composite

$$\mathscr{V}(\mathrm{A}_-,\mathrm{A}'_-)\xrightarrow{\mathscr{A}(\mathrm{A},\mathrm{A}')}\mathscr{W}(\mathrm{A}_+,\mathrm{A}'_+)=\mathscr{W}(\mathrm{C}_-,\mathrm{C}'_-)\xrightarrow{\mathscr{C}(\mathrm{C},\mathrm{C}')}\mathscr{U}(\mathrm{C}_+,\mathrm{C}'_+), \tag{2.25}$$

the identity

$$\eta_{(\mathrm{C},\mathrm{A})}:1_{(\mathrm{C},\mathrm{A})_+}\to\mathscr{C}(\mathrm{C},\mathrm{C})(\mathscr{A}(\mathrm{A},\mathrm{A})(1_{(\mathrm{C},\mathrm{A})_-}))$$

for $\mathscr{C}\circ\mathscr{A}$ being given by the composite

$$1_{\mathrm{C}_+}\xrightarrow{\eta_{\mathrm{C}}}\mathscr{C}(\mathrm{C},\mathrm{C})(1_{\mathrm{C}_-})=\mathscr{C}(\mathrm{C},\mathrm{C})(1_{\mathrm{A}_+})\xrightarrow{\mathscr{C}(\mathrm{C},\mathrm{C})(\eta_{\mathrm{A}})}\mathscr{C}(\mathrm{C},\mathrm{C})(\mathscr{A}(\mathrm{A},\mathrm{A})(1_{\mathrm{A}_-})), \tag{2.26}$$

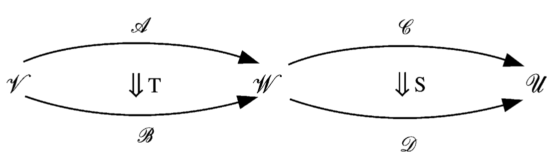
and the composition

$$\begin{aligned}\mu^{(\mathrm{C}',\mathrm{A}')}_{(\mathrm{C},\mathrm{A}),(\mathrm{C}'',\mathrm{A}'')}:{}&\otimes''(\mathscr{C}(\mathrm{C}',\mathrm{C}'')\mathscr{A}(\mathrm{A}',\mathrm{A}'')\times\mathscr{C}(\mathrm{C},\mathrm{C}')\mathscr{A}(\mathrm{A},\mathrm{A}'))\\ &\to\mathscr{C}(\mathrm{C},\mathrm{C}'')\mathscr{A}(\mathrm{A},\mathrm{A}'')\otimes\end{aligned}$$

for $\mathscr{C}\circ\mathscr{A}$ being given by the composite

$$\begin{aligned}&\otimes''(\mathscr{C}(\mathrm{C}',\mathrm{C}'')\times\mathscr{C}(\mathrm{C},\mathrm{C}'))(\mathscr{A}(\mathrm{A}',\mathrm{A}'')\times\mathscr{A}(\mathrm{A},\mathrm{A}'))\xrightarrow{\mu^{\mathrm{C}'}_{\mathrm{C},\mathrm{C}''}(\mathscr{A}(\mathrm{A}',\mathrm{A}'')\times\mathscr{A}(\mathrm{A},\mathrm{A}'))}\\ &\quad\mathscr{C}(\mathrm{C},\mathrm{C}'')\otimes'(\mathscr{A}(\mathrm{A}',\mathrm{A}'')\times\mathscr{A}(\mathrm{A},\mathrm{A}'))\xrightarrow{\mathscr{C}(\mathrm{C},\mathrm{C}'')\mu^{\mathrm{A}'}_{\mathrm{A},\mathrm{A}''}}\mathscr{C}(\mathrm{C},\mathrm{C}'')\mathscr{A}(\mathrm{A},\mathrm{A}'')\otimes;\end{aligned} \tag{2.27}$$

verification of axioms (2.6)–(2.8) is immediate. Next, we define the horizontal-composition functor $\circ$ on morphisms, its value $\mathrm{S}\circ\mathrm{T}:\mathscr{C}\circ\mathscr{A}\to\mathscr{D}\circ\mathscr{B}$ in the situation



being given on objects by

$$(\mathrm{S}\circ\mathrm{T})(\mathrm{C},\mathrm{A})=(\mathrm{SC},\mathrm{TA}), \tag{2.28}$$

while the "effect on homs"

$$(\mathrm{S}\circ\mathrm{T})_{(\mathrm{C},\mathrm{A}),(\mathrm{C}',\mathrm{A}')}:(\mathscr{C}\circ\mathscr{A})((\mathrm{C},\mathrm{A}),(\mathrm{C}',\mathrm{A}'))\to(\mathscr{D}\circ\mathscr{B})((\mathrm{SC},\mathrm{TA}),(\mathrm{SC}',\mathrm{TA}'))$$

is the natural transformation given by the horizontal composite

$$S_{CC'} \cdot T_{AA'}:\mathscr{C}(C,C')\mathscr{A}(A,A') \to \mathscr{D}(SC,SC')\mathscr{B}(TA,TA'); \tag{2.29}$$

that these data satisfy axioms (2.14)–(2.15) is immediate. Finally, it is clear from the definition of (vertical) composition in *Caten*$(\mathscr{V},\mathscr{W})$ that the operation $\circ^{\mathscr{W}}_{\mathscr{V}\mathscr{U}}$ of (2.21) is indeed a functor.

In the situation

$$\mathscr{V} \xrightarrow{\mathscr{A}} \mathscr{W} \xrightarrow{\mathscr{C}} \mathscr{U} \xrightarrow{\mathscr{E}} \mathscr{Z},$$

the only difference between $(\mathscr{E}\circ\mathscr{C})\circ\mathscr{A}$ and $\mathscr{E}\circ(\mathscr{C}\circ\mathscr{A})$ is that the objects of the first are triples $((E,C),A)$ with $E_- = C_+$ and $C_- = A_+$, while the objects of the second are triples $(E,(C,A))$ having the same properties. So we have an associativity isomorphism

$$a:(\mathscr{E}\circ\mathscr{C})\circ\mathscr{A} \to \mathscr{E}\circ(\mathscr{C}\circ\mathscr{A}) \tag{2.30}$$

which is clearly natural with respect to functors $T:\mathscr{A}\to\mathscr{B}$, $S:\mathscr{C}\to\mathscr{D}$, and $R:\mathscr{E}\to\mathscr{F}$. Moreover, the isomorphism (2.30) clearly satisfies Mac Lane's pentagonal coherence axiom.

Finally, there is an *identity category* $1_{\mathscr{V}}:\mathscr{V}\to\mathscr{V}$ for each bicategory $\mathscr{V}$, given by the identity span on ob $\mathscr{V}$ and the identity functors $\mathscr{V}(X,Y)\to\mathscr{V}(X,Y)$. The categories $\mathscr{A}\circ 1_{\mathscr{V}}$ and $1_{\mathscr{W}}\circ\mathscr{A}$ differ from $\mathscr{A}:\mathscr{V}\to\mathscr{W}$ only in the names of their objects, an object of $\mathscr{A}\circ 1_{\mathscr{V}}$, for instance, being a pair $(A,X)\in \mathrm{ob}\,\mathscr{A}\times\mathrm{ob}\,\mathscr{V}$ with $A_- = X$. So there are also natural isomorphisms

$$\ell:1_{\mathscr{W}}\circ\mathscr{A}\to\mathscr{A}, \quad r:\mathscr{A}\circ 1_{\mathscr{V}}\to\mathscr{A}, \tag{2.31}$$

which clearly satisfy the usual coherence axiom involving a, ℓ, and r.

Proposition 2.6. *The data above constitute a* (*large*) *bicategory Caten with bicategories as its objects and with the* (*large*) *hom-categories Caten*$(\mathscr{V},\mathscr{W})$. *There is an evident "forgetful" pseudofunctor* ob: *Caten* $\to$ **Span** *sending a bicategory* $\mathscr{V}$ *to its set* ob $\mathscr{V}$ *of objects and a category* $\mathscr{A}:\mathscr{V}\to\mathscr{W}$ *to the span* ob $\mathscr{A}$.

In Section 3 we shall provide *Caten* with 3-cells, turning it from a bicategory (with the italic name *Caten*) to a tricategory with the bold-face name **Caten**.

2.7. We now examine the adjunctions in the bicategory *Caten*. First, consider a lax functor $F:\mathscr{V}\to\mathscr{W}$, giving as in Example 2.3(b) a category $\mathscr{A}:\mathscr{V}\to\mathscr{W}$, and suppose that

(i) the morphisms $F_{0;X}$ and $F_{2;f,g}$ are invertible (so that F is a pseudofunctor) and

(ii) each $F_{XY}:\mathscr{V}(X,Y)\to\mathscr{W}(FX,FY)$ has a right adjoint R_{XY} in **Cat**.

Then we obtain, as in Example 2.3(c), a category $\mathscr{B}:\mathscr{W}\to\mathscr{V}$. In fact, we shall now see that $\mathscr{B}$ is right adjoint to $\mathscr{A}$ in *Caten*. The object span of $\mathscr{B}\circ\mathscr{A}$ consists of the set $\{(X,X')\,|\,FX=FX'\}$ together with the two projections, and there is an evident functor $\eta : 1_{\mathscr{V}} \to \mathscr{B} \circ \mathscr{A}$ which is the diagonal on objects and for which the natural transformation

$$\eta_{XY} : 1_{\mathscr{V}}(X, Y) \to (\mathscr{B} \circ \mathscr{A})((X, X), (Y, Y)), \text{ or } \eta_{XY} : 1_{\mathscr{V}(X,Y)} \to \mathscr{B}(X, Y)\mathscr{A}(X, Y),$$

is just the unit $1 \to R_{XY}F_{XY}$ of the adjunction $F_{XY} \dashv R_{XY}$. Again, the object span of $\mathscr{A} \circ \mathscr{B}$ is in effect

$$\text{ob}\,\mathscr{W} \overset{\text{ob}\,F}{\longleftarrow} \text{ob}\,\mathscr{V} \overset{\text{ob}\,F}{\longrightarrow} \text{ob}\,\mathscr{W},$$

although an object of $\mathscr{A} \circ \mathscr{B}$ is more properly, by (2.23), a pair (X, X) with $X \in \text{ob}\,\mathscr{V}$. There is an evident functor $\varepsilon : \mathscr{A} \circ \mathscr{B} \to 1_{\mathscr{W}}$ which is ob F on objects and for which the natural transformation $\varepsilon_{(X,X),(Y,Y)} : (\mathscr{A} \circ \mathscr{B})((X, X), (Y, Y)) \to 1_{\mathscr{W}}(FX, FY)$, or $\varepsilon_{(X,X),(Y,Y)} : \mathscr{A}(X, Y)\mathscr{B}(X, Y) \to 1_{\mathscr{W}(FX,FY)}$, is just the counit $F_{XY}R_{XY} \to 1$ of the adjunction $F_{XY} \dashv R_{XY}$. Finally, the triangular equations for η and ε follow at once from those for the adjunction $F_{XY} \dashv R_{XY}$, confirming that we do indeed have an adjunction $\eta, \varepsilon : \mathscr{A} \dashv \mathscr{B} : \mathscr{W} \to \mathscr{V}$ in *Caten*.

In fact, the adjunctions above are, to within isomorphism, the only adjunctions in *Caten*. For, if $\eta, \varepsilon : \mathscr{A} \dashv \mathscr{B} : \mathscr{W} \to \mathscr{V}$ is an adjunction, application of the pseudofunctor ob : *Caten* $\to$ **Span** gives an adjunction $\text{ob}\,\mathscr{A} \dashv \text{ob}\,\mathscr{B}$ in **Span**. So, as we noted in Section 2.1, the span $\text{ob}\,\mathscr{V} \leftarrow \text{ob}\,\mathscr{A} \to \text{ob}\,\mathscr{W}$ may, after replacement by an isomorph, be supposed to be of the form

$$\text{ob}\,\mathscr{V} \overset{1}{\longleftarrow} \text{ob}\,\mathscr{V} \overset{f}{\longrightarrow} \text{ob}\,\mathscr{W},$$

so that, as in Example 2.3(b), $\mathscr{A}$ arises from a lax functor F with $\text{ob}\,F = f$; and the span $\text{ob}\,\mathscr{W} \leftarrow \text{ob}\,\mathscr{B} \to \text{ob}\,\mathscr{V}$ may, again after replacement by an isomorph, be supposed to be of the form

$$\text{ob}\,\mathscr{W} \overset{\text{ob}\,F}{\longleftarrow} \text{ob}\,\mathscr{V} \overset{1}{\longrightarrow} \text{ob}\,\mathscr{V},$$

so that the $\mathscr{B}(X, Y)$ have the form $R_{XY} : \mathscr{W}(FX, FY) \to \mathscr{V}(X, Y)$. Now η and ε provide us with adjunctions $F_{XY} \dashv R_{XY}$. Moreover, the composition μ for $\mathscr{B}$ reduces to morphisms $\otimes\,(R_{YZ} \times R_{XY}) \to R_{XZ}\otimes'$, whose mates under the adjunctions $F_{XZ} \dashv R_{XZ}$ and $F_{YZ} \times F_{XY} \dashv R_{YZ} \times R_{XY}$ are morphisms $\nu : F_{XZ} \otimes\ \to\ \otimes'\,(F_{YZ} \times F_{XY})$, which are easily shown to be inverses to the $\mu : \otimes'\,(F_{YZ} \times F_{XY}) \to F_{XZ}\otimes$, whose components are the $F_{2;f,g}$; the argument in the monoidal case of one-object bicategories is given in [17]. Finally, a similar argument shows the invertibility of $F_{0;X}$, which completes the proof.

Proposition 2.7. *A category* $\mathscr{A} : \mathscr{V} \to \mathscr{W}$ *has a right adjoint in Caten if and only if it arises from a pseudofunctor* F *and each of the functors* $F_{V\,V'} : \mathscr{V}(V, V') \to \mathscr{W}(FV, FV')$, *which we also write as* $\mathscr{A}(V, V') : \mathscr{V}(V, V') \to \mathscr{W}(\mathscr{A}(V), \mathscr{A}(V'))$, *has a right adjoint*.

2.8. We now exhibit a canonical decomposition of a general category $\mathscr{A} : \mathscr{V} \to \mathscr{W}$ in *Caten*. We have the function $()_- : \text{ob}\,\mathscr{A} \to \text{ob}\,\mathscr{V}$. Define a bicategory $\mathscr{Z}$ with $\text{ob}\,\mathscr{Z} = \text{ob}\,\mathscr{A}$ by setting $\mathscr{Z}(A, B) = \mathscr{V}(A_-, B_-)$ and by using the composition in $\mathscr{V}$ to define one in $\mathscr{Z}$, and similarly for identities. We have, of course, a lax functor

$\mathrm{L}:\mathscr{X}\to\mathscr{V}$ which is in fact a pseudofunctor, and more: $\mathrm{L}_{\mathrm{AB}}:\mathscr{X}(\mathrm{A},\mathrm{B})\to\mathscr{V}(\mathrm{LA},\mathrm{LB})=\mathscr{V}(\mathrm{A}_-,\mathrm{B}_-)$ is actually an equality of categories. As such, it has of course a right adjoint $\mathrm{R}_{\mathrm{AB}}:\mathscr{V}(\mathrm{LA},\mathrm{LB})\to\mathscr{X}(\mathrm{A},\mathrm{B})$, which is itself an equality. Let us write $\mathscr{L}:\mathscr{X}\to\mathscr{V}$ for the category determined by L, and $\mathscr{R}:\mathscr{V}\to\mathscr{X}$ for its right adjoint given by the R_{AB}. Now observe that there is a category $\mathscr{B}:\mathscr{X}\to\mathscr{W}$, whose object span $\mathrm{ob}\,\mathscr{X}\xleftarrow{()_-}\mathrm{ob}\,\mathscr{B}\xrightarrow{()_+}\mathrm{ob}\,\mathscr{W}$ is $\mathrm{ob}\,\mathscr{A}\xleftarrow{1}\mathrm{ob}\,\mathscr{A}\xrightarrow{()_+}\mathrm{ob}\,\mathscr{W}$, and whose effect-on-homs $\mathscr{B}(\mathrm{A},\mathrm{B}):\mathscr{X}(\mathrm{A},\mathrm{B})\to\mathscr{W}(\mathrm{A}_+,\mathrm{B}_+)$ is just $\mathscr{A}(\mathrm{A},\mathrm{B}):\mathscr{V}(\mathrm{A}_-,\mathrm{B}_-)\to\mathscr{W}(\mathrm{A}_+,\mathrm{B}_+)$; of course $\mathscr{B}$ too arises from (let us henceforth say that $\mathscr{B}$ *is*) a lax functor $\mathscr{X}\to\mathscr{W}$. Moreover, the composite

$$\mathscr{V}\xrightarrow{\mathscr{R}}\mathscr{X}\xrightarrow{\mathscr{B}}\mathscr{W} \tag{2.32}$$

is isomorphic to $\mathscr{A}:\mathscr{V}\to\mathscr{W}$; one could say that it *is* $\mathscr{A}$, except that, by our definition in Section 2.6 of $\mathrm{ob}(\mathscr{B}\circ\mathscr{R})$, the latter is not $\mathrm{ob}\,\mathscr{A}$ but the diagonal $\{(\mathrm{A},\mathrm{A})\mid \mathrm{A}\in\mathrm{ob}\,\mathscr{A}\}$.

Proposition 2.8. *Every category $\mathscr{A}:\mathscr{V}\to\mathscr{W}$ admits an isomorphism $\mathscr{A}\cong\mathscr{B}\circ\mathscr{R}$ where $\mathscr{R}:\mathscr{V}\to\mathscr{X}$ is a right-adjoint category whose $\mathscr{R}(\mathrm{V},\mathrm{V}')$ are equivalences, and where $\mathscr{B}:\mathscr{X}\to\mathscr{W}$ is a lax functor. Furthermore, this gives a factorization system on Caten in the sense* (*see* [7] *for example*) *appropriate to bicategories.*

2.9. We have a principle of *duality*, in that there is an involutory automorphism of bicategories

$$()^{\circ}:\mathit{Caten}\to\mathit{Caten} \tag{2.33}$$

given as follows. First, for a bicategory $\mathscr{V}$, we set

$$\mathscr{V}^{\circ}=\mathscr{V}^{\mathrm{op}} \tag{2.34}$$

in the usual sense, whereby $\mathscr{V}^{\mathrm{op}}(\mathrm{X},\mathrm{Y})=\mathscr{V}(\mathrm{Y},\mathrm{X})$ and the composition

$$\otimes^{\mathrm{op}}:\mathscr{V}^{\mathrm{op}}(\mathrm{Y},\mathrm{Z})\times\mathscr{V}^{\mathrm{op}}(\mathrm{X},\mathrm{Y})\to\mathscr{V}^{\mathrm{op}}(\mathrm{X},\mathrm{Z})$$

is the composite

$$\mathscr{V}(\mathrm{Z},\mathrm{Y})\times\mathscr{V}(\mathrm{Y},\mathrm{X})\cong\mathscr{V}(\mathrm{Y},\mathrm{X})\times\mathscr{V}(\mathrm{Z},\mathrm{Y})\xrightarrow{\otimes}\mathscr{V}(\mathrm{Z},\mathrm{X}). \tag{2.35}$$

(Note, in particular, that $(\Sigma\mathscr{M})^{\mathrm{op}}=\Sigma(\mathscr{M}^{\mathrm{rev}})$ for a monoidal category $\mathscr{M}$; here $\mathscr{M}^{\mathrm{rev}}$ is $\mathscr{M}$ as a category but with $\mathrm{A}\otimes^{\mathrm{rev}}\mathrm{B}=\mathrm{B}\otimes\mathrm{A}$.) Now for a category $\mathscr{A}:\mathscr{V}\to\mathscr{W}$ in *Caten*, we define $\mathscr{A}^{\circ}:\mathscr{V}^{\circ}\to\mathscr{W}^{\circ}$ by setting $\mathrm{ob}(\mathscr{A}^{\circ})=\mathrm{ob}\,\mathscr{A}$ as spans and with $\mathscr{A}^{\circ}(\mathrm{A},\mathrm{B}):\mathscr{V}^{\mathrm{op}}(\mathrm{A}_-,\mathrm{B}_-)\to\mathscr{W}^{\mathrm{op}}(\mathrm{A}_+,\mathrm{B}_+)$ equal to

$$\mathscr{A}(\mathrm{B},\mathrm{A}):\mathscr{V}(\mathrm{B}_-,\mathrm{A}_-)\to\mathscr{W}(\mathrm{B}_+,\mathrm{A}_+). \tag{2.36}$$

Similarly, for a functor $\mathrm{T}:\mathscr{A}\to\mathscr{B}:\mathscr{V}\to\mathscr{W}$, we set $\mathrm{ob}(\mathrm{T}^{\circ})=\mathrm{ob}\,\mathrm{T}$ and take

$$(\mathrm{T}^{\circ})_{\mathrm{AB}}:\mathscr{A}^{\circ}(\mathrm{A},\mathrm{B})\to\mathscr{B}^{\circ}(\mathrm{TA},\mathrm{TB})$$

to be

$$\mathrm{T}_{\mathrm{BA}}:\mathscr{A}(\mathrm{B},\mathrm{A})\to\mathscr{B}(\mathrm{TB},\mathrm{TA}). \tag{2.37}$$

Note that $\mathscr{A}^{\mathrm{op}}$ would not be an appropriate name for $\mathscr{A}^{\circ}$ because, when we have a $\mathscr{W}$-category $\mathscr{A}:\mathbf{1}\to\mathscr{W}$, the usual meaning of $\mathscr{A}^{\mathrm{op}}$ is the composite

$$\mathbf{1}\xrightarrow{\mathscr{A}^{\circ}}\mathscr{W}^{\mathrm{op}}\xrightarrow{\mathrm{H}}\mathscr{W} \tag{2.38}$$

for a suitable isomorphism H of bicategories (often of the form ΣD for a monoidal isomorphism $\mathrm{D}:\mathscr{M}^{\mathrm{rev}}\to\mathscr{M}$). Similarly, tensor products of $\mathscr{W}$-categories arise from a homomorphism $\mathscr{W}\times\mathscr{W}\to\mathscr{W}$.

3. The tricategory Caten

The very name “functor” for the 2-cells of the bicategory *Caten* naturally leads to the expectation that there should be 3-cells called “natural transformations”. We now introduce these, which provide the 3-cells turning the bicategory *Caten* into a tricategory **Caten**.

3.1. Given bicategories $\mathscr{V}$ and $\mathscr{W}$, categories $\mathscr{A},\mathscr{B}:\mathscr{V}\to\mathscr{W}$, and functors $\mathrm{T},\mathrm{S}:\mathscr{A}\to\mathscr{B}$, we now define the notion of a *natural transformation* $\alpha:\mathrm{T}\to\mathrm{S}$, which we may also write as

$$\alpha:\mathrm{T}\to\mathrm{S}:\mathscr{A}\to\mathscr{B}:\mathscr{V}\to\mathscr{W}$$

to present the information succinctly. There is no real need to speak of such a natural transformation as “enriched from $\mathscr{V}$ to $\mathscr{W}$”: since the categories $\mathscr{A}$ and $\mathscr{B}$ are so enriched, the functors T and S are necessarily so, as is the “natural transformation” α. Such an α is a function assigning to each pair A, B of objects of $\mathscr{A}$ a natural transformation (in the usual sense)

$$\alpha_{\mathrm{AB}}:\mathscr{A}(\mathrm{A},\mathrm{B})\to\mathscr{B}(\mathrm{TA},\mathrm{SB}):\mathscr{V}(\mathrm{A}_-,\mathrm{B}_-)\to\mathscr{W}(\mathrm{A}_+,\mathrm{B}_+), \tag{3.1}$$

whose component at $\mathrm{f}\in\mathscr{V}(\mathrm{A}_-,\mathrm{B}_-)$ we may write as

$$\alpha_{\mathrm{AB}}(\mathrm{f}):\mathscr{A}(\mathrm{A},\mathrm{B})(\mathrm{f})\to\mathscr{B}(\mathrm{TA},\mathrm{SB})(\mathrm{f}), \tag{3.2}$$

subject to the condition that, for all $\mathrm{f}\in\mathscr{V}(\mathrm{A}_-,\mathrm{B}_-)$ and $\mathrm{g}\in\mathscr{V}(\mathrm{B}_-,\mathrm{C}_-)$, we have commutativity in the diagram

$$\begin{array}{ccc}
\mathscr{A}(\mathrm{B},\mathrm{C})(\mathrm{g})\otimes'\mathscr{A}(\mathrm{A},\mathrm{B})(\mathrm{f}) & \xrightarrow{\alpha_{\mathrm{BC}}(\mathrm{g})\otimes'\mathrm{T}_{\mathrm{AB}}(\mathrm{f})} & \mathscr{B}(\mathrm{TB},\mathrm{SC})(\mathrm{g})\otimes'\mathscr{B}(\mathrm{TA},\mathrm{TB})(\mathrm{f}) \\
{\scriptstyle \mathrm{S}_{\mathrm{BC}}(\mathrm{g})\otimes'\alpha_{\mathrm{AB}}(\mathrm{f})}\downarrow & & \downarrow{\scriptstyle \mu^{\mathrm{TB}}_{\mathrm{TA,SC}}(\mathrm{g},\mathrm{f})} \\
\mathscr{B}(\mathrm{SB},\mathrm{SC})(\mathrm{g})\otimes'\mathscr{B}(\mathrm{TA},\mathrm{SB})(\mathrm{f}) & \xrightarrow[\mu^{\mathrm{SB}}_{\mathrm{TA,SC}}(\mathrm{g},\mathrm{f})]{} & \mathscr{B}(\mathrm{TA},\mathrm{SC})(\mathrm{g}\otimes\mathrm{f})
\end{array} \tag{3.3}$$

of the category $\mathscr{W}(\mathrm{A}_+,\mathrm{C}_+)$.

The (classical) natural transformations α_{AB} above (which themselves have the components $\alpha_{AB}(f)$) might be called the *two-sided components* of the natural transformation $\alpha : T \to S : \mathscr{A} \to \mathscr{B} : \mathscr{V} \to \mathscr{W}$; alongside these, it is useful to introduce what we might call the *one-sided components*, or simply the *components*, of such a natural transformation α, which provide an alternative way of describing α. For each $A \in \mathrm{ob}\,\mathscr{A}$, the (one-sided) *component* of α is the morphism

$$\alpha_A : 1_{A_+} \to \mathscr{B}(TA, SA)(1_{A_-})$$

of $\mathscr{W}(A_+, C_+)$ given by the composite

$$1_{A_+} \xrightarrow{\eta_A} \mathscr{A}(A,A)(1_{A_-}) \xrightarrow{\alpha_{AA}(1_{A_-})} \mathscr{B}(TA,SA)(1_{A_-}). \tag{3.4}$$

Using (2.14), (2.6), and (2.7) as well as (3.3), we observe that these components make commutative the diagram

$$\begin{array}{ccccc}
\mathscr{A}(A,B)(f) \xrightarrow{\ell'^{-1}} 1_{B_+} \otimes' \mathscr{A}(A,B)(f) & & \xrightarrow{\alpha_B \otimes' T_{AB}(f)} & & \mathscr{B}(TB,SB)(1_{B_-}) \otimes' \mathscr{B}(TA,TB)(f) \\
\downarrow{\scriptstyle r'^{-1}} & & & & \downarrow{\scriptstyle \mu^{TB}_{TA,SB}(1_{B_-},f)} \\
\mathscr{A}(A,B)(f) \otimes' 1_{A_+} & & & & \mathscr{B}(TA,SB)(1_{B_-} \otimes f) \\
\downarrow{\scriptstyle S_{AB}(f) \otimes' \alpha_A} & & & & \downarrow{\scriptstyle \mathscr{B}(TA,SB)(\ell)} \\
\mathscr{B}(SA,SB)(f) \otimes' \mathscr{B}(TA,SA)(1_{A_-}) & \xrightarrow[\mu^{SA}_{TA,SB}(f,1_{A_-})]{} & \mathscr{B}(TA,SB)(f \otimes 1_{A_-}) & \xrightarrow[\mathscr{B}(TA,SB)(r)]{} & \mathscr{B}(TA,SB)(f),
\end{array} \tag{3.5}$$

each leg being the morphism $\alpha_{AB}(f)$.

Conversely, given a family of morphisms $\alpha_A : 1_{A_+} \to \mathscr{B}(TA,SA)(1_{A_-})$ in $\mathscr{W}$ making (3.5) commutative, upon defining $\alpha_{AB}(f)$ to be the diagonal of the square (3.5), we easily see that each α_{AB} is natural and (using (2.8) and (2.16)) that each leg of (3.3) is equal to the composite

$$\mathscr{A}(B,C)(g) \otimes' \mathscr{A}(A,B)(f) \xrightarrow{\mu^B_{AC}} \mathscr{A}(A,C)(g \otimes f) \xrightarrow{\alpha_{AC}(g \otimes f)} \mathscr{B}(TA,SC)(g \otimes f); \tag{3.6}$$

moreover, the composite (3.4) gives back α_A, as we see using (2.14) and (2.7). Thus, a natural transformation $\alpha : T \to S : \mathscr{A} \to \mathscr{B}$ may equally be defined as a family of (one-sided) components α_A satisfying (3.5).

It is, of course, the one-sided components α_A that correspond to the familiar $\alpha_A : TA \to SA$ for a classical natural transformation, or to the somewhat less familiar $\alpha_A : I \to \mathscr{B}(TA,SA)$ when $T, S : \mathscr{A} \to \mathscr{B}$ are $\mathscr{M}$-functors for a monoidal category $\mathscr{M}$; while in the classical case $\alpha_{AB} : \mathscr{A}(A,B) \to \mathscr{B}(TA,SA)$ takes $f \in \mathscr{A}(A,B)$ to the common value of $S(f)\alpha_A$ and $\alpha_B T(f)$. In the present generality, although we find it convenient to refer both to the α_{AB} and to the α_A, it is the former that we use in our basic definition: essentially because the α_{AB} are simply described as classical natural transformations, while it would require a lengthy diversion to establish the existence and properties of certain "underlying ordinary categories" $\mathscr{B}_{A_-A_+}(TA, SA)$ containing as morphisms the $\alpha_A : 1_{A_+} \to \mathscr{B}(TA, SA)(1_{A_-})$. (These "underlying ordinary categories" are in fact studied in Example 6.10 (d) below.)

3.2. We now describe a category **Caten**$(\mathscr{V}, \mathscr{W})(\mathscr{A}, \mathscr{B})$ whose objects are the functors $T : \mathscr{A} \to \mathscr{B}$ and whose arrows $\alpha : T \to S$ are the natural transformations. The composite $\beta \cdot \alpha : T \to R$ of $\alpha : T \to S$ and $\beta : S \to R$ is defined by taking the (one-sided) component $(\beta \cdot \alpha)_A$ to be the composite

$$\begin{aligned} 1_{A_+} &\xrightarrow{\imath'^{-1}} 1_{A_+} \otimes' 1_{A_+} \xrightarrow{\quad \beta_A \otimes' \alpha_A \quad} \mathscr{B}(SA, RA)(1_{A_-}) \otimes' \mathscr{B}(TA, SA)(1_{A_-}) \\ &\xrightarrow{\mu^{SA}_{TA,RA}(1_{A_-}, 1_{A_-})} \mathscr{B}(TA, RA)(1_{A_-} \otimes 1_{A_-}) \xrightarrow{\mathscr{B}(TA,RA)(\imath)} \mathscr{B}(TA, RA)(1_{A_-}); \end{aligned} \tag{3.7}$$

given the coherence of a, ℓ, $\imath$ and a', ℓ', $\imath'$, the associativity of this composition follows at once from (2.8). Again, we obtain a natural transformation $1_T : T \to T : \mathscr{A} \to \mathscr{B}$ on taking $(1_T)_A : 1_{A_+} \to \mathscr{B}(TA, TA)(1_{A_-})$ to be η_{TA}; for when we set $S = T$ and $\alpha_A = \eta_{TA}$ in (3.5), it follows directly from (2.6) and (2.7) that each leg equals $T_{AB}(f)$. That 1_T is the identity for the composition above also follows at once from (2.6) and (2.7). Note that the two-sided component $(1_T)_{AB}$ of 1_T is $T_{AB} : \mathscr{A}(A, B) \to \mathscr{B}(TA, TB)$.

3.3. We now go on to show that we have a (large) 2-category **Caten**$(\mathscr{V}, \mathscr{W})$ whose underlying category is *Caten*$(\mathscr{V}, \mathscr{W})$ and whose (large) hom-categories are none other than the **Caten**$(\mathscr{V}, \mathscr{W})(\mathscr{A}, \mathscr{B})$ of 3.2. We must extend the vertical composition

$$Caten(\mathscr{V}, \mathscr{W})(\mathscr{B}, \mathscr{C}) \times Caten(\mathscr{V}, \mathscr{W})(\mathscr{A}, \mathscr{B}) \to Caten(\mathscr{V}, \mathscr{W})(\mathscr{A}, \mathscr{C})$$

of the bicategory *Caten* by defining it on natural transformations

$$\alpha : T \to S : \mathscr{A} \to \mathscr{B}, \quad \gamma : P \to Q : \mathscr{B} \to \mathscr{C}$$

in such a way as to obtain a functor

$$\mathbf{Caten}(\mathscr{V}, \mathscr{W})(\mathscr{B}, \mathscr{C}) \times \mathbf{Caten}(\mathscr{V}, \mathscr{W})(\mathscr{A}, \mathscr{B}) \to \mathbf{Caten}(\mathscr{V}, \mathscr{W})(\mathscr{A}, \mathscr{C}). \tag{3.8}$$

To this end, we define the composite

$$\gamma\alpha : PT \to QS : \mathscr{A} \to \mathscr{C}$$

by taking for its two-sided components $(\gamma\alpha)_{AB}$ the composite natural transformations

$$\mathscr{A}(A, B) \overset{\alpha_{AB}}{\Rightarrow} \mathscr{B}(TA, SB) \overset{\gamma_{TA,SB}}{\Rightarrow} \mathscr{C}(PTA, QSB). \tag{3.9}$$

The reader will easily verify the commutativity of the diagram (3.3) for $\gamma\alpha$, using the commutativity of the diagram (3.3) for α and that the diagram (3.3) for γ not only commutes but has the γ-version of (3.6) as its diagonal.

The proof that (3.8) is indeed a functor is complicated by the fact that we found it convenient to use one-sided components in the definition (3.7) of *vertical composition* in the 2-category **Caten**$(\mathscr{V}, \mathscr{W})$, but to use two-sided components in the definition (3.9) of *horizontal composition*. The following is a fairly short proof using the *partial functors* of (3.8).

First, note from (3.9) and two applications of (3.4) that the one-sided component $(\gamma\alpha)_A$ is the composite

$$1_{A_+} \xrightarrow{\alpha_A} \mathscr{B}(TA, SA)(1_{A_-}) \xrightarrow{\gamma_{TA,SA}(1_{A_-})} \mathscr{C}(PTA, QSA)(1_{A_-}). \tag{3.10}$$

Let us write $P\alpha : PT \to PS$ for $1_P\alpha$ and $\gamma T : PT \to QT$ for $\gamma 1_T$. Since $(1_P)_{TA,SA}$, as we saw in Section 3.2, is $P_{TA,SA}$, (3.10) gives:

$$(P\alpha)_A \text{ is } 1_{A_+} \xrightarrow{\alpha_A} \mathscr{B}(TA, SA)(1_{A_-}) \xrightarrow{P_{TA,SA}(1_{A_-})} \mathscr{C}(PTA, PSA)(1_{A_-}). \tag{3.11}$$

Again, since $(1_T)_A = \eta_{TA}$, (3.10) and (3.4) give:

$$(\gamma T)_A \text{ is } \gamma_{TA} : 1_{A_+} \to \mathscr{C}(PTA, QTA)(1_{A_-}). \tag{3.12}$$

In particular, either of (3.11) or (3.12) gives

$$1_P 1_T \; (= P1_T = 1_P T) = 1_{PT}. \tag{3.13}$$

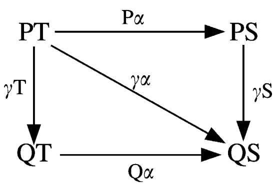
Now, we verify the functoriality of

$$(T \mapsto PT, \alpha \mapsto P\alpha) : \mathbf{Caten}(\mathscr{V}, \mathscr{W})(\mathscr{A}, \mathscr{B}) \to \mathbf{Caten}(\mathscr{V}, \mathscr{W})(\mathscr{A}, \mathscr{C}).$$

In fact, it preserves identities by (3.13), and is easily seen to preserve composition by (3.11), (3.7) and diagram (2.16) for P. Next, the functoriality of

$$(P \mapsto PT, \gamma \mapsto \gamma T) : \mathbf{Caten}(\mathscr{V}, \mathscr{W})(\mathscr{B}, \mathscr{C}) \to \mathbf{Caten}(\mathscr{V}, \mathscr{W})(\mathscr{A}, \mathscr{C})$$

is immediate from (3.12), (3.13), and (3.7). It now remains to show that these are indeed the partial functors of (3.8), in the sense that each triangle in



(3.14)

commutes. If we use the top leg of (3.5) to express $\gamma_{TA,SA}$ in terms of γ_{SA}, and so to express $(\gamma\alpha)_A$ in terms of α_A and γ_{SA} using (3.10), we find that this is precisely the composite $((\gamma S) \cdot (P\alpha))_A$ given by (3.11), (3.12), and (3.7). Similarly, if we use instead the bottom leg of (3.5) to express $\gamma_{TA,SA}$ in terms of γ_{TA}, we find that

$$(\gamma\alpha)_A = ((Q\alpha) \cdot (\alpha T))_A.$$

To complete the proof that $\mathbf{Caten}(\mathscr{V}, \mathscr{W})$ is a 2-category, it remains only to verify the associativity and the unit laws for the horizontal composition. In fact, this associativity is immediate from (3.9), as is the fact that the natural transformations $1_{1_{\mathscr{A}}} : 1_{\mathscr{A}} \to 1_{\mathscr{A}}$ act as horizontal identities.

3.4. We now extend the functor (2.21) to a 2-functor

$$\circ = \circ^{\mathscr{W}}_{\mathscr{V}\mathscr{U}} : \mathbf{Caten}(\mathscr{W}, \mathscr{U}) \times \mathbf{Caten}(\mathscr{V}, \mathscr{W}) \to \mathbf{Caten}(\mathscr{V}, \mathscr{W}). \tag{3.15}$$

Given natural transformations

$$\alpha : \mathrm{T} \to \mathrm{P} : \mathscr{A} \to \mathscr{B} : \mathscr{V} \to \mathscr{W} \quad \text{and} \quad \beta : \mathrm{S} \to \mathrm{Q} : \mathscr{C} \to \mathscr{D} : \mathscr{W} \to \mathscr{U},$$

we define the natural transformation

$$\beta \circ \alpha : \mathrm{S} \circ \mathrm{T} \to \mathrm{Q} \circ \mathrm{P} : \mathscr{C} \circ \mathscr{A} \to \mathscr{D} \circ \mathscr{B} : \mathscr{V} \to \mathscr{U}$$

by taking the (classical) natural transformation

$$(\beta \circ \alpha)_{(\mathrm{C},\mathrm{A}),(\mathrm{D},\mathrm{B})} : (\mathscr{C} \circ \mathscr{A})((\mathrm{C},\mathrm{A}),(\mathrm{D},\mathrm{B})) \to (\mathscr{D} \circ \mathscr{B})((\mathrm{SC},\mathrm{TA}),(\mathrm{QD},\mathrm{PB}))$$

to be the horizontal composite

$$\beta_{\mathrm{CD}}\alpha_{\mathrm{AB}} : \mathscr{C}(\mathrm{C},\mathrm{D})\mathscr{A}(\mathrm{A},\mathrm{B}) \to \mathscr{D}(\mathrm{SC},\mathrm{QD})\mathscr{B}(\mathrm{TA},\mathrm{PB}) \tag{3.16}$$

of (classical) natural transformations. The commutativity of (3.3) for $\beta\circ\alpha$ follows easily from its commutativity for β and for α, using (3.16) along with (2.27) and (2.29), so that $\beta \circ \alpha$ is indeed a natural transformation $\mathrm{S} \circ \mathrm{T} \to \mathrm{Q} \circ \mathrm{P}$. To complete the verification that we now have a 2-functor (3.15), it remains to show that $\circ$ preserves both horizontal and vertical composites of natural transformations, as well as the horizontal and vertical identities. For the horizontal identities and composites, this is immediate from (3.16) and (3.9). In order to deal with vertical identities and composites, it is convenient to transform (3.16) using (3.4), to obtain the one-sided components of $\beta \circ \alpha$; in the light of (2.26), we easily obtain

$$\begin{aligned} &1_{\mathrm{C}_+} \xrightarrow{\gamma_\mathrm{A}} \mathscr{D}(\mathrm{SC},\mathrm{QC})(1_{\mathrm{C}_-}) \\ &\quad = \mathscr{D}(\mathrm{SC},\mathrm{QC})(1_{\mathrm{A}_+}) \xrightarrow{\mathscr{D}(\mathrm{SC},\mathrm{QC})(\alpha_\mathrm{A})} \mathscr{D}(\mathrm{SC},\mathrm{QC})(\mathscr{B}(\mathrm{TA},\mathrm{PA})(1_{\mathrm{A}_-})) \end{aligned} \tag{3.17}$$

as the value of

$$1_{(\mathrm{C},\mathrm{A})_+} \xrightarrow{(\beta\circ\alpha)_{(\mathrm{C},\mathrm{A})}} (\mathscr{D} \circ \mathscr{B})((\mathrm{SC},\mathrm{TA}),(\mathrm{QD},\mathrm{PB}))(1_{(\mathrm{C},\mathrm{A})_-}). \tag{3.18}$$

Now, using (2.26) and the result $(1_\mathrm{T})_\mathrm{A} = \eta_{\mathrm{TA}}$ from Section 3.2, it is immediate from (3.17) that $\circ$ preserves vertical identities; while, using (2.27) and (3.7), it is immediate from (3.17) that $\circ$ preserves vertical composition.

3.5. To conclude the proof that **Caten** is a (large) tricategory (of an especially well-behaved kind), it remains only to verify that the isomorphisms $a : (\mathscr{E} \circ \mathscr{C}) \circ \mathscr{A} \to \mathscr{E} \circ (\mathscr{C} \circ \mathscr{A})$, $\ell : 1_{\mathscr{W}} \circ \mathscr{A} \to \mathscr{A}$, and $r : \mathscr{A} \circ 1_{\mathscr{V}} \to \mathscr{A}$ of (2.30) and (2.31) are not only natural but 2-natural. This is immediate since, as we saw in Section 2.6, these correspond to a trivial re-naming of the objects of these categories.

3.6. More needs to be said about the well-behaved kind of tricategory exemplified by **Caten**. The structure is what one obtains by taking the "local definition" of bicategory as given in [3, pp. 1–6] and replacing the hom *categories* by hom *2-categories*, the composition *functors* by composition *2-functors*, and the unit and associativity *natural isomorphisms* by unit and associativity *2-natural isomorphisms*; let us call such a structure a *bi-2-category*. (In fact, every such tricategory is equivalent to a 3-category

[14, Corollary 8.4].) Similarly, we can rewrite, at this higher level, the notions of lax functor (= morphism of bicategories), of pseudofunctor (= homomorphism), of lax natural transformation (= transformation), of pseudonatural transformation, and of modification, while retaining the same terminology. Thus we may speak of pseudonatural transformations between pseudofunctors from one bi-2-category to another.

4. A monoidal structure on Caten, and convolution

4.1. Bicategories are algebraic structures and therefore there is a cartesian product $\mathscr{U} \times \mathscr{V}$ of two bicategories $\mathscr{U}, \mathscr{V}$. This is the product, in the usual categorical sense, in the category of bicategories and strict structure-preserving morphisms. It is *not* the bicategorical product in the bicategory *Caten*: the categories $Caten(\mathscr{W}, \mathscr{U} \times \mathscr{V})$ and $Caten(\mathscr{W}, \mathscr{U}) \times Caten(\mathscr{W}, \mathscr{V})$ are generally not equivalent. However, the cartesian product of bicategories is the object-function of a pseudofunctor

$$- \times -: Caten \times Caten \to Caten \tag{4.1}$$

making *Caten* into a monoidal bicategory (see [14;12;20, Appendix A]). The definition of the pseudofunctor on arrows and 2-cells is the straightforward pointwise one; and it can be extended in the same pointwise way to 3-cells: for $\alpha : \mathrm{T} \to \mathrm{S} : \mathscr{A} \to \mathscr{C} : \mathscr{V} \to \mathscr{V}'$ and $\beta : \mathrm{P} \to \mathrm{Q} : \mathscr{B} \to \mathscr{D} : \mathscr{W} \to \mathscr{W}'$ we have

$$\alpha \times \beta : \mathrm{T} \times \mathrm{P} \to \mathrm{S} \times \mathrm{Q} : \mathscr{A} \times \mathscr{B} \to \mathscr{C} \times \mathscr{D} : \mathscr{V} \times \mathscr{W} \to \mathscr{V}' \times \mathscr{W}',$$

where

$$(\alpha \times \beta)_{(\mathrm{A},\mathrm{B}),(\mathrm{A}',\mathrm{B}')} = \alpha_{\mathrm{AA}'} \times \beta_{\mathrm{BB}'} : \mathscr{A}(\mathrm{A}, \mathrm{A}') \times \mathscr{B}(\mathrm{B}, \mathrm{B}') \\ \to \mathscr{C}(\mathrm{TA}, \mathrm{SA}') \times \mathscr{D}(\mathrm{PB}, \mathrm{QB}').$$

Now the value of (4.1) on the hom-categories extends to a 2-functor

$$- \times -: \mathbf{Caten}(\mathscr{V}, \mathscr{V}') \times \mathbf{Caten}(\mathscr{W}, \mathscr{W}') \to \mathbf{Caten}(\mathscr{V} \times \mathscr{W}, \mathscr{V}' \times \mathscr{W}'), \tag{4.2}$$

and the coherent constraints of the pseudofunctor become 2-natural. The associativity and unit constraints

$$(\mathscr{U} \times \mathscr{V}) \times \mathscr{W} \cong \mathscr{U} \times (\mathscr{V} \times \mathscr{W}), \quad 1 \times \mathscr{V} \cong \mathscr{V}, \ \ \mathscr{V} \times 1 \cong \mathscr{V} \tag{4.3}$$

are the evident ones, and clearly satisfy the appropriate coherence conditions. Thus, **Caten** is a monoidal tricategory of a particularly simple kind: for example, (4.1) extends to a pseudofunctor **Caten** × **Caten** → **Caten** in the sense of Section 3.6. In view of the evident isomorphism

$$\mathscr{U} \times \mathscr{V} \cong \mathscr{V} \times \mathscr{U}, \tag{4.4}$$

the monoidal structure is symmetric.

4.2. Before discussing the extent to which this monoidal structure on **Caten** is *closed*, we need to introduce some further notions related to size. Recall from [6] that a bicategory $\mathscr{W}$ is said to be *locally cocomplete* when each hom-category $\mathscr{W}(W, W')$ is cocomplete (that is, admits small colimits) and each functor $\mathscr{W}(f, g)$ preserves small colimits. (In view of our terminology in Section 3.6, a locally cocomplete bicategory could be called a “bi-cocomplete-category”; however we shall retain the established term.) A bicategory $\mathscr{V}$ is *locally small* when each hom-category $\mathscr{V}(V, V')$ is a small category (at least to within equivalence).

4.3. Given bicategories $\mathscr{V}$ and $\mathscr{W}$, can we find a bicategory $\mathbf{Conv}(\mathscr{V}, \mathscr{W})$ such that to give a category $\mathscr{A} : \mathscr{U} \times \mathscr{V} \to \mathscr{W}$ is equally to give a category $\bar{\mathscr{A}} : \mathscr{U} \to \mathbf{Conv}(\mathscr{V}, \mathscr{W})$? The name $\mathbf{Conv}(\mathscr{V}, \mathscr{W})$ was chosen for this “internal hom” because we shall see that its horizontal composition is given by a *convolution formula* closely related to [11].

Since the object span for $\mathscr{A} : \mathscr{U} \times \mathscr{V} \to \mathscr{W}$ has the form

$$\mathrm{ob}\,\mathscr{U} \times \mathrm{ob}\,\mathscr{V} \xleftarrow{((\,)_-, (\,)_0)} \mathrm{ob}\,\mathscr{A} \xrightarrow{(\,)_+} \mathrm{ob}\,\mathscr{W} \tag{4.5}$$

and since to give such a span is equally to give a span

$$\mathrm{ob}\,\mathscr{U} \xleftarrow{(\,)_-} \mathrm{ob}\,\mathscr{A} \xrightarrow{((\,)_0, (\,)_+)} \mathrm{ob}\,\mathscr{V} \times \mathrm{ob}\,\mathscr{W}, \tag{4.6}$$

we are led to take

$$\mathrm{ob}\,\mathbf{Conv}(\mathscr{V}, \mathscr{W}) = \mathrm{ob}\,\mathscr{V} \times \mathrm{ob}\,\mathscr{W} \tag{4.7}$$

and

$$\mathrm{ob}\,\bar{\mathscr{A}} = \mathrm{ob}\,\mathscr{A} \tag{4.8}$$

with (4.6) providing the object span for $\bar{\mathscr{A}}$. Next, to give functors

$$\mathscr{A}(A, B) : \mathscr{U}(A_-, B_-) \times \mathscr{V}(A_0, B_0) \to \mathscr{W}(A_+, B_+) \tag{4.9}$$

describing the effect-on-homs of $\mathscr{A}$ is equally to give functors

$$\bar{\mathscr{A}}(A, B) : \mathscr{U}(A_-, B_-) \to [\mathscr{V}(A_0, B_0), \mathscr{W}(A_+, B_+)], \tag{4.10}$$

where square brackets denote the functor category; here (4.9) and (4.10) are connected by

$$\mathscr{A}(A, B)(f, g) = \bar{\mathscr{A}}(A, B)(f)(g), \tag{4.11}$$

along with a similar equation for morphisms $\alpha : f \to f'$ in $\mathscr{U}(A_-, B_-)$ and $\beta : g \to g'$ in $\mathscr{V}(A_0, B_0)$. Accordingly, we are led to take

$$\mathbf{Conv}(\mathscr{V}, \mathscr{W})((V, W), (V', W')) = [\mathscr{V}(V, V'), \mathscr{W}(W, W')], \tag{4.12}$$

with $\mathscr{A}(A, B)$ and $\bar{\mathscr{A}}(A, B)$ related as in (4.11).

To give the identities for $\mathscr{A}$ is, by (2.3), to give for each A in $\mathrm{ob}\,\mathscr{A}$ a morphism

$$\eta_A : 1_{A_+} \to \mathscr{A}(A, A)(1_{A_-}, 1_{A_0}) = \bar{\mathscr{A}}(A, A)(1_{A_-})(1_{A_0}). \tag{4.13}$$

We want this to be the same thing as giving for each A a natural transformation

$$\bar{\eta}_A : 1_{(A_0,A_+)} \to \bar{\mathscr{A}}(A,A)(1_{A_-}); \tag{4.14}$$

and we can achieve this when the category $\mathscr{V}(A_0,A_0)$ is locally small and $\mathscr{W}$ is locally cocomplete by taking for the identity $1_{(V,W)}$ in the bicategory $\mathbf{Conv}(\mathscr{V},\mathscr{W})$ the functor $\mathscr{V}(V,V) \to \mathscr{W}(W,W)$ given by

$$1_{(V,W)}(f) = \mathscr{V}(V,V)(1_V,f) \bullet 1_W, \tag{4.15}$$

where for a set Λ and a morphism $w : W \to W'$ in $\mathscr{W}$, the morphism $\Lambda \bullet w$ is the coproduct in $\mathscr{W}(W,W')$ of Λ copies of w. For then to give an $\bar{\eta}_A$ as in (4.14) is equally to give a natural transformation

$$\mathscr{V}(A_0,A_0)(1_{A_0},-) \to \mathscr{W}(A_+,A_+)(1_{A_+}, \bar{\mathscr{A}}(A,A)(1_{A_-})(-))$$

and hence by Yoneda to give a morphism η_A as in (4.13).

Finally, we have the composition law for $\mathscr{A}$, given by components

$$\mu^B_{A,C}((f,h),(g,k)) : \mathscr{A}(B,C)(f,h) \otimes \mathscr{A}(A,B)(g,k) \to \mathscr{A}(A,C)(f \otimes g, h \otimes k) \tag{4.16}$$

natural in $f \in \mathscr{U}(B_-,C_-)$, $h \in \mathscr{V}(B_0,C_0)$, $g \in \mathscr{U}(A_-,B_-)$ and $k \in \mathscr{V}(A_0,B_0)$; here we have abandonned our notational distinctions between the horizontal compositions in the three bicategories $\mathscr{U},\mathscr{V},\mathscr{W}$, denoting each by an unadorned $\otimes$. We want the giving of such a natural μ to be equivalent to the giving of components

$$\bar{\mu}^B_{A,C}(f,g) : \bar{\mathscr{A}}(B,C)(f) \bar{\otimes} \bar{\mathscr{A}}(A,B)(g) \to \bar{\mathscr{A}}(A,C)(f \otimes g) \tag{4.17}$$

in the functor category $[\mathscr{V}(A_0,C_0),\mathscr{W}(A_+,C_+)]$, natural in f and g, where $\bar{\otimes}$ denotes the (yet to be defined) horizontal composition in $\mathbf{Conv}(\mathscr{V},\mathscr{W})$. We can achieve this when $\mathscr{V}$ is locally small and $\mathscr{W}$ is locally cocomplete by defining the composition in $\mathbf{Conv}(\mathscr{V},\mathscr{W})$ as follows. The functor

$$\begin{aligned}\bar{\otimes} : [\mathscr{V}(V',V''),\mathscr{W}(W',W'')] \times [\mathscr{V}(V,V'),\mathscr{W}(W,W')] \\ \to [\mathscr{V}(V,V''),\mathscr{W}(W,W'')]\end{aligned} \tag{4.18}$$

is described on objects by the convolution formula

$$P \bar{\otimes} Q = \int^{h \in \mathscr{V}(V',V''),\ k \in \mathscr{V}(V,V')} \mathscr{V}(V,V'')(h \otimes k, -) \bullet (P(h) \otimes Q(k)); \tag{4.19}$$

that such a formula does describe a functor is classical—for example, see [18, Section 3.3]. Now to give (4.17), natural in f and g, is to give components

$$\mathscr{V}(A_0,C_0)(h \otimes k, -) \bullet (\mathscr{A}(B,C)(f,h) \otimes \mathscr{A}(A,B)(g,k)) \to \mathscr{A}(A,C)(f \otimes g, -)$$

natural in f, g, h, and k: which is equivalent by Yoneda to the giving of (4.16).

Notice that formula (4.19) says that $P \bar{\otimes} Q$ is the (pointwise) left Kan extension of the composite

$$\mathscr{V}(V',V'') \times \mathscr{V}(V,V') \xrightarrow{P \times Q} \mathscr{W}(W',W'') \times \mathscr{W}(W,W') \xrightarrow{\otimes'} \mathscr{W}(W,W'')$$

along the functor $\otimes : \mathscr{V}(V',V'') \times \mathscr{V}(V,V') \to \mathscr{V}(V,V'')$. Similarly, (4.15) says that $1_{(V,W)} : \mathscr{V}(V,V) \to \mathscr{W}(W,W)$ is the left Kan extension of $1_W : 1 \to \mathscr{W}(W,W)$ along $1_V : 1 \to \mathscr{V}(V,V)$. Note, too, that $\mathbf{Conv}(\mathscr{V},\mathscr{W})$ is, like $\mathscr{V}$ and $\mathscr{W}$, an honest bicategory—one internal to **Set**.

Proposition 4.3. *Consider bicategories $\mathscr{V}$ and $\mathscr{W}$ with $\mathscr{V}$ locally small and $\mathscr{W}$ locally cocomplete. There is a locally cocomplete bicategory $\mathbf{Conv}(\mathscr{V},\mathscr{W})$ defined by* (4.7), (4.12), (4.15) *and* (4.18), *and having certain canonical associativity and unit constraints described below. There is a family of isomorphisms*

$$\mathbf{Caten}(\mathscr{U} \times \mathscr{V}, \mathscr{W}) \cong \mathbf{Caten}(\mathscr{U}, \mathbf{Conv}(\mathscr{V},\mathscr{W}))$$

of 2-*categories, pseudonatural in $\mathscr{U} \in \mathbf{Caten}$, given on objects by* (4.8), (4.11) *and the bijections $\eta \leftrightarrow \bar{\eta}$, $\mu \leftrightarrow \bar{\mu}$ described above. In particular, taking $\mathscr{U} = 1$ gives a canonical bijection between categories enriched from $\mathscr{V}$ to $\mathscr{W}$ and categories enriched in* $\mathbf{Conv}(\mathscr{V},\mathscr{W})$.

Proof. We begin with a "several-object" version of the calculations of [9, pp. 19–20]; a more detailed account in the case where $\mathscr{V}$ and $\mathscr{W}$ are suspensions of monoidal categories appears in [15]. We first need to produce the coherent associativity constraints for the composition (4.18). The fact that colimits commute with colimits and are preserved by $-\otimes w$, along with the Yoneda isomorphism and the definition (4.19), give us a series of isomorphisms

$$\begin{aligned}
(P\bar{\otimes}Q)\bar{\otimes}R &= \int^{m,n} \mathscr{V}(V,V''')(m \otimes n, -) \bullet ((P\bar{\otimes}Q)(m) \otimes R(n)) \\
&\cong \int^{m,n} \mathscr{V}(V,V''')(m \otimes n, -) \bullet \\
&\qquad \left(\int^{h,k} \mathscr{V}(V',V''')(h \otimes k, m) \bullet (P(h) \otimes Q(k)) \otimes R(n) \right) \\
&\cong \int^{m,n,h,k} (\mathscr{V}(V,V''')(m \otimes n, -) \times \mathscr{V}(V',V''')(h \otimes k, m)) \bullet \\
&\qquad ((P(h) \otimes Q(k)) \otimes R(n)) \\
&\cong \int^{n,h,k} \mathscr{V}(V,V''')((h \otimes k) \otimes n, -) \bullet ((P(h) \otimes Q(k)) \otimes R(n)). \quad (4.20)
\end{aligned}$$

In the same way, we have

$$P\bar{\otimes}(Q\bar{\otimes}R) \cong \int^{n,h,k} \mathscr{V}(V,V''')(h \otimes (k \otimes n), -) \bullet (P(h) \otimes (Q(k) \otimes R(n))). \quad (4.21)$$

By (4.20) and (4.21), the associativity constraints for $\mathscr{V}$ and $\mathscr{W}$ give associativity constraint for $\mathbf{Conv}(\mathscr{V},\mathscr{W})$; moreover, the coherence pentagon for the latter constraint follows from the corresponding pentagons for the former ones. Similarly for the unit constraints; using (4.15), we have for $P \in [\mathscr{V}(V, V'), \mathscr{W}(W, W')]$ the isomorphisms

$$\begin{aligned} P \otimes 1_{(V,W)} &= \int^{h,k} \mathscr{V}(V, V')(h \otimes k, -) \bullet (P(h) \otimes 1_{(V,W)}(k)) \\ &= \int^{h,k} \mathscr{V}(V, V')(h \otimes k, -) \bullet (P(h) \otimes (\mathscr{V}(V, V)(1_V, k) \bullet 1_W)) \\ &\cong \int^{h,k} (\mathscr{V}(V, V')(h \otimes k, -) \times \mathscr{V}(V, V)(1_V, k)) \bullet (P(h) \otimes 1_W) \\ &\cong \int^{h,k} \mathscr{V}(V, V')(h \otimes 1_V, -) \bullet (P(h) \otimes 1_W), \end{aligned}$$

so that the right-unit constraints for $\mathscr{V}$ and $\mathscr{W}$ give the desired right-unit constraint

$$P \otimes 1_{(V,W)} \cong P$$

for **Conv**$(\mathscr{V}, \mathscr{W})$. Similarly, the coherence triangle relating the unit and associativity constraints follows from those for $\mathscr{V}$ and $\mathscr{W}$. Thus **Conv**$(\mathscr{V}, \mathscr{W})$ is a bicategory, which by (4.12) and (4.19) is clearly locally cocomplete.

We need to show that the data for $\mathscr{A} : \mathscr{U} \times \mathscr{V} \to \mathscr{W}$ satisfy axioms (2.6)–(2.8) if and only if those for $\bar{\mathscr{A}} : \mathscr{U} \to$ **Conv**$(\mathscr{V}, \mathscr{W})$ do so. This follows easily when, for instance, we extend the discussion in Section 4.3 of the relationship between $\mu^B_{A,C}$ and $\bar{\mu}^B_{A,C}$ to establish a bijection between natural transformations

$$(\bar{\mathscr{A}}(C, D)(f) \bar{\otimes} \bar{\mathscr{A}}(B, C)(g)) \bar{\otimes} \bar{\mathscr{A}}(A, B)(h) \to \bar{\mathscr{A}}(A, D)((f \otimes g) \otimes h)$$

and natural transformations

$$\begin{aligned} &(\mathscr{A}(C, D)(f, u) \otimes \mathscr{A}(B, C)(g, v)) \otimes \mathscr{A}(A, B)(h, w) \\ &\quad \to \mathscr{A}(A, D)((f \otimes g) \otimes h, (u \otimes v) \otimes w). \end{aligned}$$

So we do indeed have the object bijection $\mathscr{A} \leftrightarrow \bar{\mathscr{A}}$ for a pseudonatural isomorphism of 2-categories

$$\rho_{\mathscr{U},\mathscr{V},\mathscr{W}} : \mathbf{Caten}(\mathscr{U} \times \mathscr{V}, \mathscr{W}) \cong \mathbf{Caten}(\mathscr{U}, \mathbf{Conv}(\mathscr{V}, \mathscr{W})). \tag{4.22}$$

To save space, we leave it to the reader to complete the description of the isomorphism (4.22), showing that to give a functor $T : \mathscr{A} \to \mathscr{B} : \mathscr{U} \times \mathscr{V} \to \mathscr{W}$ is equally to give a functor $\bar{T} : \bar{\mathscr{A}} \to \bar{\mathscr{B}} : \mathscr{U} \to$ **Conv**$(\mathscr{V}, \mathscr{W})$, and similarly for natural transformations, with these bijections respecting all types of composition: the calculations, although a little long, are straightforward, and the reader will see that they basically depend on the *compactness* (sometimes called the *autonomy*) of the monoidal bicategory **Span** and the *cartesian closedness* of **Cat**. Finally, the reader will easily verify the pseudonaturality in $\mathscr{U}$ of the isomorphism (4.22). □

From general principles applied to the pseudonatural isomorphism (4.22), we see that **Conv** can be made the object-function of a pseudofunctor into **Caten** from the full subtricategory of $\mathbf{Caten}^{op} \times \mathbf{Caten}$ consisting of the pairs $(\mathscr{V}, \mathscr{W})$ of bicategories satisfying the conditions of Proposition 4.3; this construction is the essentially unique one forcing pseudonaturality of the isomorphisms (4.22). Again from the same kind of general principles, the pseudonaturality in the locally small $\mathscr{U}$ and $\mathscr{V}$, and in the locally cocomplete $\mathscr{W}$, implies a biequivalence of the bicategories $\mathbf{Conv}(\mathscr{U}\times\mathscr{V},\mathscr{W})$ and $\mathbf{Conv}(\mathscr{U},\mathbf{Conv}(\mathscr{V},\mathscr{W}))$. In fact, however, we have a stronger result: a direct calculation, which we leave to the reader, gives a pseudonatural *isomorphism*

$$\mathbf{Conv}(\mathscr{U}\times\mathscr{V},\mathscr{W})\cong\mathbf{Conv}(\mathscr{U},\mathbf{Conv}(\mathscr{V},\mathscr{W})) \tag{4.23}$$

of bicategories.

4.4. Finally, we note the special case given by $\mathbf{Conv}(\Sigma\mathscr{M},\Sigma\mathscr{N})$, where $\mathscr{M}=(\mathscr{M},\circ,\mathrm{J})$ is a small monoidal category and $\mathscr{N}=(\mathscr{N},\circ,\mathrm{I})$ is a cocomplete one for which $\mathrm{N}\otimes-$ and $-\otimes\mathrm{N}$ preserve small colimits. It is immediate that

$$\mathbf{Conv}(\Sigma\mathscr{M},\Sigma\mathscr{N})=\Sigma[\mathscr{M},\mathscr{N}], \tag{4.24}$$

where $[\mathscr{M},\mathscr{N}]$ is the functor category provided with Day's "convolution monoidal structure" $([\mathscr{M},\mathscr{N}],*,\mathrm{K})$ as in [9]. Thus $\mathrm{K}=\mathscr{M}(\mathrm{J},-)\bullet\mathrm{I}$, while

$$\mathrm{P}*\mathrm{Q}=\int^{\mathrm{h,k}}\mathscr{M}(\mathrm{h}\otimes\mathrm{k},-)\bullet(\mathrm{P(h)}\otimes\mathrm{Q(k)}). \tag{4.25}$$

5. Local cocompletion of bicategories

5.1. We say that a category $\mathscr{A}:\mathscr{V}\to\mathscr{W}$ is a *local left adjoint* when the functor $\mathscr{A}(\mathrm{A,B}):\mathscr{V}(\mathrm{A}_-,\mathrm{B}_-)\to\mathscr{W}(\mathrm{A}_+,\mathrm{B}_+)$ has a right adjoint for all objects A, B of $\mathscr{A}$. This is the case in particular when $\mathscr{A}$ is a left adjoint in *Caten* (see Proposition 2.7). We write $\mathbf{Lla}(\mathscr{V},\mathscr{W})$ for the full sub-2-category of $\mathbf{Caten}(\mathscr{V},\mathscr{W})$ consisting of the local left adjoint categories $\mathscr{A}:\mathscr{V}\to\mathscr{W}$.

5.2. Suppose $\mathscr{V}$ and $\mathscr{W}$ are locally cocomplete. We say that a category $\mathscr{A}:\mathscr{V}\to\mathscr{W}$ is *locally cocontinuous* when each of the functors $\mathscr{A}(\mathrm{A,B}):\mathscr{V}(\mathrm{A}_-,\mathrm{B}_-)\to\mathscr{W}(\mathrm{A}_+,\mathrm{B}_+)$ preserves small colimits. This is the case in particular when $\mathscr{A}$ is a local left adjoint. If the homs of $\mathscr{V}$ are presheaf categories—that is, of the form $[\mathscr{K}^{\mathrm{op}},\mathbf{set}]$ for some small category $\mathscr{K}$ (see below)—then every locally cocontinuous $\mathscr{A}:\mathscr{V}\to\mathscr{W}$ is a local left adjoint.

5.3. Recall that **set** denotes the category of small sets. For a small category $\mathscr{K}$, let $\mathrm{P}\mathscr{K}$ denote the presheaf category $[\mathscr{K}^{\mathrm{op}},\mathbf{set}]$, with $\mathrm{Y}=\mathrm{Y}_{\mathscr{K}}:\mathscr{K}\to\mathrm{P}\mathscr{K}$ for the Yoneda embedding. Suspending the cartesian monoidal category **set** gives the locally cocomplete bicategory $\Sigma\mathbf{set}$, and for each locally small bicategory $\mathscr{V}$ we define a new locally-cocomplete bicategory $\mathscr{P}\mathscr{V}$ by setting

$$\mathscr{P}\mathscr{V}=\mathbf{Conv}(\mathscr{V}^{\mathrm{co}},\Sigma\mathbf{set}), \tag{5.1}$$

where $\mathscr{V}^{co}$, as usual, is the dual of $\mathscr{V}$ obtained by reversing 2-cells, so that $\mathscr{V}^{co}(V,V') = \mathscr{V}(V,V')^{op}$. Since the bicategory $\Sigma\mathbf{set}$ has only one object, we may identify ob($\mathscr{P}\mathscr{V}$) with ob $\mathscr{V}$; and then (4.12) gives

$$(\mathscr{P}\mathscr{V})(V,V') = [\mathscr{V}(V,V')^{op}, \mathbf{set}] = P(\mathscr{V}(V,V')), \tag{5.2}$$

which may also be written for brevity as $P\mathscr{V}(V,V')$. By (4.15), the identity 1_V of V in $\mathscr{P}\mathscr{V}$, which we shall write as $\bar{1}_V$ to distinguish it from the identity 1_V of V in $\mathscr{V}$, is given by

$$\bar{1}_V = \mathscr{V}(V,V)(-,1_V) = Y_{\mathscr{V}(V,V)}(1_V). \tag{5.3}$$

Finally, by (4.19), we not only have commutativity to within isomorphism in

$$\begin{array}{ccc} P\mathscr{V}(V',V'') \times P\mathscr{V}(V,V') & \xrightarrow{\bar{\otimes}} & P\mathscr{V}(V,V') \\ {\scriptstyle Y\times Y}\uparrow & \cong & \uparrow{\scriptstyle Y} \\ \mathscr{V}(V',V'') \times \mathscr{V}(V,V') & \xrightarrow[\otimes]{} & \mathscr{V}(V,V'), \end{array} \tag{5.4}$$

but in fact—see Section 3 of [15]—the functor $\bar{\otimes}$ here is the unique extension of $Y\otimes$ that is separately cocontinuous in each variable (or equivalently, separately left adjoint in each variable).

We reiterate that $\mathscr{P}\mathscr{V}$ is defined only for a locally small $\mathscr{V}$. Observe that there is then a category

$$\mathscr{Y} = \mathscr{Y}_{\mathscr{V}} : \mathscr{V} \to \mathscr{P}\mathscr{V} \tag{5.5}$$

which is, in fact, a pseudofunctor: it is the identity on objects, and its effect-on-homs

$$\mathscr{Y}(V,V') : \mathscr{V}(V,V') \to (\mathscr{P}\mathscr{V})(V,V')$$

is just the Yoneda embedding

$$Y_{\mathscr{V}(V,V')} : \mathscr{V}(V,V') \to P\mathscr{V}(V,V'), \tag{5.6}$$

whereupon (5.3) and (5.4) complete its structure as a pseudofunctor. This category $\mathscr{Y}_{\mathscr{V}}$ has the following universal property; note that, by (5.2), the locally left-adjoint categories $\mathscr{P}\mathscr{V} \to \mathscr{W}$ coincide with the locally cocontinuous ones, for a locally cocomplete $\mathscr{W}$.

Proposition 5.3. *When $\mathscr{V}$ is locally small, the bicategory $\mathscr{P}\mathscr{V}$ is defined, and the functor*

$$- \circ \mathscr{Y}_{\mathscr{V}} : \mathbf{Lla}(\mathscr{P}\mathscr{V}, \mathscr{W}) \to \mathbf{Caten}(\mathscr{V}, \mathscr{W}) \tag{5.7}$$

is an equivalence of 2-categories for each locally-cocomplete bicategory $\mathscr{W}$.

Proof. Abbreviate $\mathscr{Y}_{\mathscr{V}}$ to $\mathscr{Y}$. For any category $\mathscr{B}:\mathscr{P}\mathscr{V}\to\mathscr{W}$, the composite $\mathscr{A}=\mathscr{B}\circ\mathscr{Y}:\mathscr{V}\to\mathscr{W}$ has $\mathrm{ob}\,\mathscr{A}=\mathrm{ob}\,\mathscr{B}$ by (2.23)—an isomorphism that we may take to be an equality—while $\mathscr{A}(A,B)$ is by (2.25) the composite

$$\mathscr{V}(A_-,B_-)\xrightarrow{Y}P\mathscr{V}(A_-,B_-)\xrightarrow{\mathscr{B}(A,B)}\mathscr{W}(A_+,B_+). \tag{5.8}$$

In terms of the identity η_A for $\mathscr{B}$, that for $\mathscr{A}$, using equality (5.4), is the composite

$$1_{A_+}\xrightarrow{\eta_A}\mathscr{B}(A,A)(\bar{1}_{A_-})=\mathscr{B}(A,A)Y(1_{A_-})=\mathscr{A}(A,B)(1_{A_-}); \tag{5.9}$$

and in terms of the composition $\mu^B_{A,C}$ for $\mathscr{B}$, that for $\mathscr{A}$, using isomorphism (5.4), is the composite

$$\begin{aligned}\mathscr{A}(B,C)(g)\otimes\mathscr{A}(A,B)(f)=\mathscr{B}(B,C)(Yg)\otimes\mathscr{B}(A,B)(Yf)&\xrightarrow{\mu^B_{A,C}(Yg,Yf)}\\ \mathscr{B}(A,C)(Yg\bar{\otimes}Yf)&\cong\mathscr{B}(A,C)(Y(g\otimes f))\\ &=\mathscr{A}(A,B)(g\otimes f).\end{aligned} \tag{5.10}$$

Let us now show that the 2-functor (5.7) is essentially surjective on objects. Given a category $\mathscr{A}^*:\mathscr{V}\to\mathscr{W}$, we construct as follows a locally-left-adjoint category $\mathscr{B}:\mathscr{P}\mathscr{V}\to\mathscr{W}$ with $\mathscr{A}=\mathscr{B}\circ\mathscr{Y}$ isomorphic to $\mathscr{A}^*$. We take $\mathrm{ob}\,\mathscr{B}$ to be $\mathrm{ob}\,\mathscr{A}^*$, and take $\mathscr{B}(A,B):P\mathscr{V}(A_-,B_-)\to\mathscr{W}(A_+,B_+)$ to be the left-adjoint functor—unique to within isomorphism—whose restriction $\mathscr{B}(A,B)Y$ as in (5.8) is isomorphic to $\mathscr{A}^*(A,B)$. Now, (5.9) forces the value of the unit η_A for $\mathscr{B}$, and (5.10) forces the value of $\mu^B_{A,C}$ on the representables (Yf,Yg); but this suffices to determine $\mu^B_{A,C}$ completely, by Im and Kelly [15, Proposition 3.1], since each leg of

$$\begin{array}{ccc} P\mathscr{V}(B_-,C_-)\times P\mathscr{V}(A_-,B_-) & \xrightarrow{\mathscr{B}(B,C)\times\mathscr{B}(A,B)} & \mathscr{W}(B_+,C_+)\times\mathscr{W}(A_+,B_+)\\ \bar{\otimes}\Big\downarrow & \Downarrow\mu^B_{A,C} & \Big\downarrow\otimes\\ P\mathscr{V}(A_-,C_-) & \xrightarrow[\mathscr{B}(A,C)]{} & \mathscr{W}(A_+,B_+)\end{array}$$

is cocontinuous in each variable. That $\mathscr{B}$ satisfies axioms (2.6)–(2.8) now follows from the principles developed in [15].

It remains to show that the 2-functor (5.7) is fully faithful. A functor $T:\mathscr{B}\to\mathscr{B}':\mathscr{P}\mathscr{V}\to\mathscr{W}$ gives us $S=T\circ\mathscr{Y}:\mathscr{A}\to\mathscr{A}'$, where $\mathscr{A}=\mathscr{B}\circ\mathscr{Y}$ and $\mathscr{A}'=\mathscr{B}'\circ\mathscr{Y}$. Clearly, the spans $\mathrm{ob}\,S$ and $\mathrm{ob}\,T$ coincide, while S_{AB} is the restriction $T_{AB}Y$ of the natural transformation T_{AB} along $Y:\mathscr{V}(A_-,B_-)\to P\mathscr{V}(A_-,B_-)$. When $\mathscr{B}(A,B)$ and $\mathscr{B}'(A,B)$ are left adjoints, there is a unique T_{AB} with $T_{AB}Y$ equal to a given S_{AB}; and the T_{AB} satisfy the functorial axioms when the S_{AB} do so. Thus (5.7) is fully faithful at the level of 1-cells; and a similar argument shows it to be fully faithful at the level of 2-cells, a natural transformation $\alpha:T\to R:\mathscr{B}\to\mathscr{B}'$ being uniquely recoverable from the restriction of the natural transformation $\alpha_{AB}:\mathscr{B}(A,B)\to\mathscr{B}'(TA,RB)$ along $Y:\mathscr{V}(A_-,B_-)\to P\mathscr{V}(A_-,B_-)$. □

6. Procategories

We shall describe an extension of **Caten** to an autonomous (also called "compact" or "rigid") monoidal tricategory **PCaten** whose arrows are called two-sided enriched *procategories*.

6.1. We remind the reader of the bicategory **Mod** of modules (also called "profunctors", "distributors" or "bimodules"). The objects are (ordinary) categories (in our usual internal-to-**Set** sense, and so not necessarily small). The arrows $M : \mathscr{A} \to \mathscr{B}$ are functors $M : \mathscr{B}^{op} \times \mathscr{A} \to \mathbf{Set}$. Natural transformations $\theta : M \Rightarrow N : \mathscr{B}^{op} \times \mathscr{A} \to \mathbf{Set}$ provide the 2-cells $\theta : M \Rightarrow N : \mathscr{A} \to \mathscr{B}$ of **Mod**; they are called *module morphisms*. Vertical composition of 2-cells is vertical composition of natural transformations. The horizontal composite $NM : \mathscr{A} \to \mathscr{C}$ of $M : \mathscr{A} \to \mathscr{B}$ and $N : \mathscr{B} \to \mathscr{C}$ is given by the coend formula

$$(NM)(C, A) = \int^{B} N(C, B) \times M(B, A), \tag{6.1}$$

and this is clearly functorial in M and N. Each functor $F : \mathscr{A} \to \mathscr{B}$ can be identified with the module $F : \mathscr{A} \to \mathscr{B}$ having $F(B, A) = \mathscr{B}(B, FA)$, and this gives an inclusion $\mathbf{Cat} \to \mathbf{Mod}$. If idempotents split in $\mathscr{B}$ then the modules $M : \mathscr{A} \to \mathscr{B}$ with right adjoints in **Mod** are those isomorphic to arrows in **Cat**—that is, to functors.

It is useful to have at hand the following observation, whose proof (involving two applications of the Yoneda isomorphism) the reader will easily supply.

Lemma 6.1. *Let* $M : \mathscr{A} \to \mathscr{B}$ *and* $N : \mathscr{C} \to \mathscr{D}$ *be modules, and let* $T : \mathscr{A} \to \mathscr{C}$ *and* $S : \mathscr{B} \to \mathscr{D}$ *be functors, identified with modules as above. Then to give a module-morphism* $\theta : SM \to NT$ *is equally to give a family of functions* $\theta_{BA} : M(B, A) \to N(SB, TA)$, *natural in* B *and* A.

The cartesian product $\mathscr{A} \times \mathscr{B}$ of categories defines an autonomous monoidal structure on the bicategory **Mod**. The *dual* of $\mathscr{A}$ as an object of **Mod** is its usual dual $\mathscr{A}^{op}$ as a category, in view of the canonical isomorphism of categories

$$\mathbf{Mod}(\mathscr{C} \times \mathscr{A}, \mathscr{D}) \cong \mathbf{Mod}(\mathscr{C}, \mathscr{A}^{op} \times \mathscr{D}). \tag{6.2}$$

6.2. Suppose $\mathscr{V}$ and $\mathscr{W}$ are bicategories. A *procategory* $\mathscr{A} : \mathscr{V} \to \mathscr{W}$ is defined in the same way as a category from $\mathscr{V}$ to $\mathscr{W}$ except that in (2.2) we take a *module* $\mathscr{A}(A, B) : \mathscr{V}(A_-, B_-) \to \mathscr{W}(A_+, B_+)$ rather than a functor, with the consequent changes in the data (2.3) and (2.4). Thus, the unit (2.3) is now to be a module-morphism $\eta_A : 1_{A_+} \to \mathscr{A}(A, A)1_{A_-}$, where the functors $1_{A_+} : 1 \to \mathscr{W}(A_+, A_+)$ and $1_{A_-} : 1 \to \mathscr{V}(A_-, A_-)$ are identified with the corresponding modules: so that, by Lemma 6.1, to give η_A is equally to give an element

$$\eta_A \in \mathscr{A}(A, A)(1_{A_+}, 1_{A_-}). \tag{6.3}$$

Again, since (2.4) is now to be a module-morphism $\mu^{B}_{A,C} : \otimes'(\mathscr{A}(B,C)\times\mathscr{A}(A,B)) \Rightarrow \mathscr{A}(A,C)\otimes$, where $\otimes'$ and $\otimes$ are functors, it becomes by Lemma 6.1 a family of functions

$$\mu^{B}_{AC}(v,u;g,f) : \mathscr{A}(B,C)(v,g)\times\mathscr{A}(A,B)(u,f)\to\mathscr{A}(A,C)(v\otimes' u,\ g\otimes f) \tag{6.4}$$

natural in $f\in\mathscr{V}(A_-,B_-)$, $g\in\mathscr{V}(B_-,C_-)$, $u\in\mathscr{W}(A_+,B_+)$, $v\in\mathscr{W}(B_+,C_+)$. In this language, axioms (2.6)–(2.8) become the two equations:

$$\mathscr{A}(A,B)(\ell^{-1},\ell)(\mu^{B}_{AB}(1_{B_+},u;1_{B_-},f)(\eta_B,\xi))=\xi, \tag{6.5}$$

$$\mathscr{A}(A,B)(\imath^{-1},\imath)(\mu^{A}_{AB}(u,1_{A_+};f,1_{A_-})(\xi,\eta_A))=\xi, \tag{6.6}$$

and (dropping now and henceforth the primes on $\otimes'$) the commutativity of the diagram

$$\begin{array}{ccc} & \mathscr{A}(C,D)(w,h)\times\mathscr{A}(B,C)(v,g)\times\mathscr{A}(A,B)(u,f) & \\ \swarrow{\scriptstyle \mu^{C}_{B,D}\times 1} & & \searrow{\scriptstyle 1\times\mu^{B}_{A,C}} \\ \mathscr{A}(B,D)(w\otimes v,h\otimes g)\times\mathscr{A}(A,B)(u,f) & & \mathscr{A}(C,D)(w,h)\times\mathscr{A}(A,C)(v\otimes u,g\otimes f) \\ \downarrow{\scriptstyle \mu^{B}_{A,D}} & & \downarrow{\scriptstyle \mu^{C}_{A,D}} \\ \mathscr{A}(A,D)((w\otimes v)\otimes u,(h\otimes g)\otimes f) & \xrightarrow[\mathscr{A}(A,D)(a^{-1},a)]{\cong} & \mathscr{A}(A,D)(w\otimes(v\otimes u),h\otimes(g\otimes f)). \end{array} \tag{6.7}$$

6.3. For procategories $\mathscr{A}$ and $\mathscr{B}$ enriched from $\mathscr{V}$ to $\mathscr{W}$, a *functor* $T:\mathscr{A}\to\mathscr{B}$ is given by the same data as in Section 2.4, except that in place of (2.12) we now have a module morphism

$$T_{AB}:\mathscr{A}(A,B)\to\mathscr{B}(TA,TB):\mathscr{V}(A_-,B_-)\to\mathscr{W}(A_+,B_+), \tag{6.8}$$

consisting of components

$$T_{AB}(u,f):\mathscr{A}(A,B)(u,f)\to\mathscr{B}(TA,TB)(u,f) \tag{6.9}$$

for which the equation

$$T_{AA}(1_{A_+},1_{A_-})(\eta_A)=\eta_{TA} \tag{6.10}$$

holds and the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A}(B,C)(v,g)\times\mathscr{A}(A,B)(u,f) & \xrightarrow{T_{BC}(v,g)\times T_{AB}(u,f)} & \mathscr{B}(TB,TC)(v,g)\times\mathscr{B}(TA,TB)(u,f) \\ \downarrow{\scriptstyle \mu^{B}_{A,C}} & & \downarrow{\scriptstyle \mu^{TB}_{TA,TC}} \\ \mathscr{A}(A,C)(v\otimes u,\ g\otimes f) & \xrightarrow[T_{AC}(v\otimes u,\ g\otimes f)]{} & \mathscr{B}(TA,TC)(v\otimes u,\ g\otimes f). \end{array} \tag{6.11}$$

Functors $T:\mathscr{A}\to\mathscr{B}$ and $P:\mathscr{B}\to\mathscr{C}$ compose to give a functor $PT:\mathscr{A}\to\mathscr{C}$, where

$$(PT)_{AB}(u,f)=P_{AB}(u,f)\cdot T_{AB}(u,f); \tag{6.12}$$

and this associative composition has identities $1_{\mathscr{A}}:\mathscr{A}\to\mathscr{A}$, where $(1_{\mathscr{A}})_{AB}:\mathscr{A}(A,B)\to\mathscr{A}(A,B)$ is itself the identity module morphism.

6.4. For functors $T,S:\mathscr{A}\to\mathscr{B}$, a *natural transformation* $\alpha:T\to S$ is a function assigning to each pair A,B of objects of $\mathscr{A}$ a module morphism

$$\alpha_{AB}:\mathscr{A}(A,B)\to\mathscr{B}(TA,SB):\mathscr{V}(A_-,B_-)\to\mathscr{W}(A_+,B_+) \tag{6.13}$$

with components

$$\alpha_{AB}(u,f):\mathscr{A}(A,B)(u,f)\to\mathscr{B}(TA,SB)(u,f), \tag{6.14}$$

subject to the condition (cf. (3.3)) that the following diagram commute for all v, u, g, f:

$$\begin{array}{ccc}
\mathscr{A}(B,C)(v,g)\times\mathscr{A}(A,B)(u,f) & \xrightarrow{S_{BC}(v,g)\times\alpha_{AB}(u,f)} & \mathscr{B}(SB,SC)(v,g)\times\mathscr{B}(TA,SB)(u,f)\\
{\scriptstyle \alpha_{BC}(v,g)\times T_{AB}(u,f)}\downarrow & & \downarrow{\scriptstyle \mu^{SB}_{TA,SC}}\\
\mathscr{B}(TB,SC)(v,g)\times\mathscr{B}(TA,TB)(u,f) & \xrightarrow{\mu^{TB}_{TA,SC}} & \mathscr{B}(TA,SC)(v\otimes u,g\otimes f).
\end{array} \tag{6.15}$$

As in Section 3.1, we can equally describe a natural transformation $\alpha:T\to S$ by giving its *one-sided components* (or merely its *components*), which are the elements

$$\alpha_A=\alpha_{AA}(1_{A_+},1_{A_-})(\eta_A)\in\mathscr{B}(TA,SA)(1_{A_+},1_{A_-}); \tag{6.16}$$

indeed a family α_A for $A\in \mathrm{ob}\,\mathscr{A}$ so arises precisely when we have commutativity in

$$\begin{array}{ccc}
 & \mathscr{A}(A,B)(u,f)\xrightarrow{\alpha_B\times T_{AB}(u,f)}\mathscr{B}(TB,SB)(1_{B_+},1_{B_-})\times\mathscr{B}(TA,TB)(u,f) & \\
{\scriptstyle S_{AB}(u,f)\times\alpha_A}\swarrow & & \searrow{\scriptstyle \mu^{TB}_{TA,SB}}\\
\mathscr{B}(SA,SB)(u,f)\times\mathscr{B}(TA,SA)(1_{A_+},1_{A_-}) & & \mathscr{B}(TA,SB)(1_{B_+}\otimes u,1_{B_-}\otimes f)\\
{\scriptstyle \mu^{SA}_{TA,SB}}\downarrow & & \downarrow{\scriptstyle \mathscr{B}(TA,SB)(\ell^{-1},\ell)}\\
\mathscr{B}(TA,SB)(u\otimes 1_{A_+},f\otimes 1_{A_-}) & \xrightarrow[\mathscr{B}(TA,SB)(r^{-1},r)]{} & \mathscr{B}(TA,SB)(u,f),
\end{array} \tag{6.17}$$

and then $\alpha_{AB}(u,f)$ is the diagonal of (6.17).

Natural transformations $\alpha:T\to S$ and $\beta:S\to R$ have a "vertical" composite $\beta\cdot\alpha:T\to R$ whose components are given by

$$(\beta\cdot\alpha)_A=\mu^{SA}_{TA,RA}(1_{A_+},1_{A_-};1_{A_+},1_{A_-})(\beta_A,\alpha_A), \tag{6.18}$$

while natural transformations $\alpha:T\to S:\mathscr{A}\to\mathscr{B}$ and $\gamma:P\to Q:\mathscr{B}\to\mathscr{C}$ have a "horizontal composite" $\gamma\alpha:PT\to QS$ defined by taking the homomorphism $(\gamma\alpha)_{AB}:\mathscr{A}(A,B)\to\mathscr{B}(PTA,QSB)$ to be the composite

$$\mathscr{A}(A,B)\xrightarrow{\alpha_{AB}}\mathscr{B}(TA,SB)\xrightarrow{\beta_{TA,SB}}\mathscr{C}(PTA,QSB). \tag{6.19}$$

We leave the reader to verify that what we have described in Sections 6.2–6.4 is a (large) 2-category $\mathbf{PCaten}(\mathscr{V},\mathscr{W})$.

6.5. In fact, the $\mathbf{PCaten}(\mathscr{V},\mathscr{W})$ are the hom-2-categories for a (large) bi-2-category **PCaten**, whose composition 2-functors

$$\circ=\circ^{\mathscr{W}}_{\mathscr{V}\mathscr{U}}:\mathbf{PCaten}(\mathscr{W},\mathscr{U})\times\mathbf{PCaten}(\mathscr{V},\mathscr{W})\to\mathbf{PCaten}(\mathscr{V},\mathscr{U}) \tag{6.20}$$

we now define. We begin as with the definition of the functor (2.21). For procategories $\mathscr{A}:\mathscr{V}\to\mathscr{W}$ and $\mathscr{C}:\mathscr{V}\to\mathscr{U}$, the definition of $\mathscr{C}\circ\mathscr{A}$ follows (2.22)–(2.25) precisely (although, of course, the composite mentioned in (2.25) is that of modules, not of functors); in place of (2.26) we take $\eta_{(C,A)}\in(\mathscr{C}\circ\mathscr{A})((C,A),(C,A))(1_{C_+},1_{A_-})$ to be (see (6.3)) the image

$$\eta_{(C,A)}=[\eta_C,\eta_A] \tag{6.21}$$

of the pair (η_C,η_A) under the coprojection

$$\mathscr{C}(C,C)(1_{C_+},1_{C_-})\times\mathscr{A}(A,A)(1_{A_+},1_{A_-})$$
$$\xrightarrow{\mathrm{copr}_{1_{A_+}}}\int^{u}\mathscr{C}(C,C)(1_{C_+},u)\times\mathscr{A}(A,A)(u,1_{A_-});$$

and in place of (2.27) we take (see (6.4)) the family of functions

$$\int^{v,u}\mathscr{C}(C',C'')(k,v)\times\mathscr{A}(A',A'')(v,g)\times\mathscr{C}(C,C')(h,u)\times\mathscr{A}(A,A')(u,f)$$
$$\xrightarrow{\mu^{(C',A')}_{(C,A),(C'',A'')}}\int^{w}\mathscr{C}(C,C'')(k\otimes h,w)\times\mathscr{A}(A,A'')(w,g\otimes f) \tag{6.22}$$

whose composite with the (v,u)-coprojection into the domain coend is the composite of a middle-four-interchange isomorphism, the function $\mu^{C'}_{CC''}\times\mu^{A'}_{AA''}$, and the $(v\otimes u)$-coprojection. Given functors $T:\mathscr{A}\to\mathscr{B}$ and $S:\mathscr{C}\to\mathscr{D}$, we define $S\circ T:\mathscr{C}\circ\mathscr{A}\to\mathscr{D}\circ\mathscr{B}$ on objects as in (2.28), while the "effect on homs" is induced on coends by the functions

$$S_{CC'}(h,u)\times T_{AA'}(u,f). \tag{6.23}$$

It should now be clear how to modify (3.16) in order to define $\circ^{\mathscr{W}}_{\mathscr{V}\mathscr{U}}$ on 2-cells.

To give a module $\mathscr{A}(A,B):\mathscr{V}(A_-,B_-)\to\mathscr{W}(A_+,B_+)$ is equally to give a functor $\mathscr{W}(A_+,B_+)^{op}\times\mathscr{V}(A_-,B_-)\to\mathbf{Set}$, or again, to give a functor $\mathscr{A}^\flat(A,B):\mathscr{V}(A_-,B_-)\times\mathscr{W}(A_+,B_+)^{op}\to\mathbf{Set}$. That being so, it is immediate from (6.3)–(6.7) that to give a procategory $\mathscr{A}:\mathscr{V}\to\mathscr{W}$ is equally to give a category $\mathscr{A}^\flat:\mathscr{V}\times\mathscr{W}^{co}\to\Sigma\mathbf{Set}$. Again, by (6.8)–(6.11), to give a functor $T:\mathscr{A}\to\mathscr{B}$ is equally to give a functor $T^\flat:\mathscr{A}^\flat\to\mathscr{B}^\flat$;

while, by (6.13) and (6.15), to give a natural transformation $\alpha : \mathrm{T} \to \mathrm{S}$ is equally to give a natural transformation $\alpha^\flat : \mathrm{T}^\flat \to \mathrm{S}^\flat$. Moreover, these bijections respect the various compositions and identities which make up the 2-category $\mathbf{PCaten}(\mathscr{V}, \mathscr{W})$ and the 2-category $\mathbf{Caten}(\mathscr{V} \times \mathscr{W}^{\mathrm{co}}, \Sigma\mathbf{Set})$. However, we may not properly speak of the latter 2-category, since $\Sigma\mathbf{Set}$ is not a "bicategory" in our present sense: it is not a bicategory internal to **Set**. We may however consider a larger category **SET** of sets, in which **Set** is a category object, and form the tricategory **CATEN** of bicategory-objects in **SET**, related to **SET** as **Caten** is to **Set**. So what we have established is an isomorphism of 2-categories

$$\mathbf{PCaten}(\mathscr{V}, \mathscr{W}) \cong \mathbf{CATEN}(\mathscr{V} \times \mathscr{W}^{\mathrm{co}}, \Sigma\mathbf{Set}) \tag{6.24}$$

for $\mathscr{V}, \mathscr{W} \in \mathbf{Caten}$. Of course, **CATEN** has an "internal-hom" $\mathbf{CONV}(\mathscr{U}, \mathscr{Z})$ whenever the $\mathscr{U}(\mathrm{A}, \mathrm{B})$ lie in **Set** and the $\mathscr{Z}(\mathrm{C}, \mathrm{D})$ admit $\mathscr{K}$-colimits for $\mathscr{K}$ a category-object in **Set**; and in particular we have, for $\mathscr{W} \in \mathbf{Caten}$, an analogue

$$\mathscr{P}^* \mathscr{W} = \mathbf{CONV}(\mathscr{W}^{\mathrm{co}}, \Sigma\mathbf{Set}) \tag{6.25}$$

of $\mathscr{P}\mathscr{W}$. Now the analogue of Proposition 4.3 gives:

Proposition 6.5. *For* $\mathscr{V}, \mathscr{W} \in \mathbf{Caten}$, *there is an isomorphism of* 2-*categories*

$$\mathbf{PCaten}(\mathscr{V}, \mathscr{W}) \cong \mathbf{CATEN}(\mathscr{V}, \mathscr{P}^* \mathscr{W}) \tag{6.26}$$

sending the procategory $\mathscr{A} : \mathscr{V} \to \mathscr{W}$ *to the category* $\mathscr{A}^\# : \mathscr{V} \to \mathscr{P}^* \mathscr{W}$, *given on objects by*

$$\mathscr{A}^\#(\mathrm{A}, \mathrm{B})(\mathrm{f})(\mathrm{u}) = \mathscr{A}(\mathrm{A}, \mathrm{B})(\mathrm{u}, \mathrm{f})$$

and similarly on morphisms.

6.6. Many important bicategories are locally small; if we were content to restrict our attention to these, we could have established a result like Proposition 6.5 without going outside **Caten**. We first replace **Mod** by the bicategory **mod** of *small categories* and *small modules*: such a module $\mathrm{M} : \mathscr{A} \to \mathscr{B}$ being a functor $\mathrm{M} : \mathscr{B}^{\mathrm{op}} \times \mathscr{A} \to \mathbf{set}$. Then, for locally-small bicategories $\mathscr{V}$ and $\mathscr{W}$, a *small* protocategory $\mathscr{A} : \mathscr{V} \to \mathscr{W}$ is a small module $\mathscr{A}(\mathrm{A}, \mathrm{B}) : \mathscr{V}(\mathrm{A}_-, \mathrm{B}_-) \to \mathscr{W}(\mathrm{A}_+, \mathrm{B}_+)$. Proceeding as in Sections 6.2–6.5, we obtain a tricategory **pCaten** of locally small bicategories, small procategories, functors, and natural transformations. In place of (6.24) and (6.26) we have isomorphisms

$$\mathbf{pCaten}(\mathscr{V}, \mathscr{W}) \cong \mathbf{Caten}(\mathscr{V} \times \mathscr{W}^{\mathrm{co}}, \Sigma\mathbf{Set}) \cong \mathbf{Caten}(\mathscr{V}, \mathscr{P}\mathscr{W}) \tag{6.27}$$

for locally small $\mathscr{V}$ and $\mathscr{W}$.

6.7. The analogue of Proposition 5.3 for the higher universe gives us, in an obvious notation, an equivalence of 2-categories

$$\mathbf{CATEN}(\mathscr{V}, \mathscr{P}^* \mathscr{W}) \simeq \mathbf{LLA}(\mathscr{P}^* \mathscr{V}, \mathscr{P}^* \mathscr{W}) \tag{6.28}$$

for $\mathscr{V},\mathscr{W}\in$ **Caten**; composing this with the isomorphism (6.26) gives an equivalence

$$\mathbf{PCaten}(\mathscr{V},\mathscr{W})\simeq\mathbf{LLA}(\mathscr{P}^*\mathscr{V},\mathscr{P}^*\mathscr{W}) \tag{6.29}$$

for $\mathscr{V},\mathscr{W}\in$ **Caten**; similarly, when $\mathscr{V}$ and $\mathscr{W}$ here are also locally small, we have an equivalence

$$\mathbf{pCaten}(\mathscr{V},\mathscr{W})\simeq\mathbf{Lla}(\mathscr{P}^*\mathscr{V},\mathscr{P}^*\mathscr{W}). \tag{6.30}$$

Proposition 6.7. *The assignment* $\mathscr{V}\mapsto\mathscr{P}^*\mathscr{V}$ *extends to a biequivalence between* **PCaten** *and the subtricategory* of **CATEN** *consisting of the objects of the form* $\mathscr{P}^*\mathscr{V}$, *the morphisms which are local left adjoints, and all* 2*-cells and* 3*-cells. Similarly, the assignment* $\mathscr{V}\mapsto\mathscr{P}\mathscr{V}$ *extends to a biequivalence between* **pCaten** *and the subtricategory of* **Caten** *consisting of the objects of the form* $\mathscr{P}\mathscr{V}$, *the morphisms which are local left adjoints, and all* 2*-cells and* 3*-cells.*

Proof. The principle being the same in both statements, it suffices to prove only the second. It is a matter of showing that equivalences (6.30) are compatible with the compositions in **pCaten** and **Caten**. For this, suppose the procategories $\mathscr{A}:\mathscr{V}\to\mathscr{W}$ and $\mathscr{C}:\mathscr{W}\to\mathscr{U}$ are taken to the locally left-adjoint categories $\mathscr{B}:\mathscr{P}\mathscr{V}\to\mathscr{P}\mathscr{W}$ and $\mathscr{D}:\mathscr{P}\mathscr{W}\to\mathscr{P}\mathscr{U}$; this means $\mathscr{B}\circ\mathscr{Y}_{\mathscr{V}}\cong\mathscr{A}^{\#}$ and $\mathscr{D}\circ\mathscr{Y}_{\mathscr{W}}\cong\mathscr{C}^{\#}$. We need to see that $\mathscr{C}\circ\mathscr{A}$ is taken to $\mathscr{D}\circ\mathscr{B}$; so we must see that $\mathscr{D}\circ\mathscr{B}\circ\mathscr{Y}_{\mathscr{V}}\cong(\mathscr{C}\circ\mathscr{A})^{\#}$, or, in other words, that $\mathscr{D}\circ\mathscr{A}^{\#}\cong(\mathscr{C}\circ\mathscr{A})^{\#}$. On objects this is clear since the spans for $\mathscr{A}$ and $\mathscr{A}^{\#}$ are equal, as are those for $\mathscr{C}$ and $\mathscr{D}$, and those for $\mathscr{C}\circ\mathscr{A}$ and $(\mathscr{C}\circ\mathscr{A})^{\#}$. On homs it follows from the fact that **mod** is biequivalent to the sub-2-category of **Cat** consisting of the **set**-valued presheaf categories and the left-adjoint functors; more explicitly,

$$\begin{aligned}
&(\mathscr{C}\circ\mathscr{A})^{\#}((C,A),(D,B))(f)(h)\\
&\quad=(\mathscr{C}\circ\mathscr{A})((C,A),(D,B))(h,f)\\
&\quad=\int^{u}\mathscr{C}(C,D)(h,u)\times\mathscr{A}(A,B)(u,f)=\int^{u}\mathscr{C}^{\#}(C,D)(u)(h)\times\mathscr{A}^{\#}(A,B)(f)(u)\\
&\quad\cong\int^{u}\mathscr{D}(C,D)(\mathscr{W}(C_-,D_-)(-,u))(h)\times\mathscr{A}^{\#}(A,B)(f)(u),
\end{aligned}\tag{6.31}$$

this last since $\mathscr{D}\circ\mathscr{Y}_{\mathscr{W}}\cong\mathscr{C}^{\#}$. However, the left-adjoint $\mathscr{D}(C,D)$ is the left Kan extension of its restriction to the representables, so that, for any $F:\mathscr{W}(C_-,D_-)^{op}\to\mathbf{Set}$, we have

$$\mathscr{D}(C,D)F\cong\int^{u}\mathscr{D}(C,D)(\mathscr{W}(C_-,D_-)(h,u))\times Fu,$$

so that (6.31) is isomorphic to $\mathscr{D}(C,D)(\mathscr{A}^{\#}(A,B)(f))(h)$, which is $(\mathscr{D}\circ\mathscr{A}^{\#})(f)(h)$, as desired.

The remaining details are left to the reader. □

6.8. Remark. Let $\mathfrak{M}$ denote a monoidal bicategory as defined in [14, Definition 2.6] and studied in [12]. It is possible to construct a tricategory $\mathfrak{M}$-**Caten**. In the case where $\mathfrak{M} = \mathbf{Cat}$ (with the cartesian monoidal structure), this reduces to **Caten**. In the case where $\mathfrak{M} = \mathbf{Mod}$ (with the cartesian product as the tensor product), $\mathfrak{M}$-**Caten** contains **PCaten** as a full subtricategory: the objects of $\mathfrak{M}$-**Caten** are *probicategories* (see [10, p. 63; 11]), not merely bicategories. In general, the objects of $\mathfrak{M}$-**Caten** are $\mathfrak{M}$-*bicategories*: the definition mimics that of bicategories except that the homs are objects of $\mathfrak{M}$ rather than categories.

6.9. There is an inclusion

$$\mathbf{Caten} \to \mathbf{PCaten} \tag{6.32}$$

which is the identity on objects and uses the inclusion $\mathbf{Cat} \to \mathbf{Mod}$ to interpret every category $\mathscr{A}:\mathscr{V} \to \mathscr{W}$ as a procategory.

Proposition 6.9. *Suppose that idempotents split in all the hom-categories of the bicategory* $\mathscr{W}$ (*that is, that* $\mathscr{W}$ *is locally* "*cauchy complete*"). *A procategory* $\mathscr{A}:\mathscr{V} \to \mathscr{W}$ *has a right adjoint in* **PCaten** *if and only if it is isomorphic to a pseudofunctor.*

Proof. By an argument similar to the proof of Proposition 2.7 we see that $\mathscr{A}:\mathscr{V} \to \mathscr{W}$ has a right adjoint in **PCaten** if and only if the span ob $\mathscr{A}$ has a right adjoint, the composition μ and identity η are invertible as module morphisms, and each hom-module $\mathscr{A}(A, B)$ has a right adjoint in **Mod**. This last means, since idempotents split in $\mathscr{W}(A_+, B_+)$, that $\mathscr{A}(A, B)$ is isomorphic to a functor. So $\mathscr{A}$ is essentially in **Caten**. □

6.10. Examples. (a) Among the objects of **PCaten** is $\Sigma\mathbf{set}$, and Proposition 6.5 gives

$$\mathbf{PCaten}(\mathbf{1}, \Sigma\mathbf{set}) \cong \mathbf{CATEN}(\mathbf{1}, \mathscr{P}^*(\Sigma\mathbf{set})) \cong \mathscr{P}^*(\Sigma\mathbf{set})\text{-}\mathbf{CAT};$$

moreover (6.25) and (4.24) give

$$\mathscr{P}^*(\Sigma\mathbf{set}) = \mathbf{CONV}((\Sigma\mathbf{set})^{co}, \Sigma\mathbf{Set}) \cong \Sigma[\mathbf{set}^{op}, \mathbf{Set}].$$

Thus,

$$\mathbf{PCaten}(\mathbf{1}, \Sigma\mathbf{set}) \cong [\mathbf{set}^{op}, \mathbf{Set}]\text{-}\mathbf{CAT},$$

where the monoidal structure on $[\mathbf{set}^{op}, \mathbf{Set}]$ is the cartesian one.

(b) A set X can be seen as a discrete category, or again as a discrete bicategory: in each case the set of objects is X, while all morphisms and 2-cells are identities. For sets X and Y seen as bicategories, to give a procategory $\mathscr{A}:X \to Y$ is by (6.24) to give a category $X \times Y \to \Sigma\mathbf{Set}$; and this is easily seen to amount to the giving of a (classical) category $|\mathscr{A}|_{x,y}$ for each $(x, y) \in X \times Y$, or again to give a span $(X \leftarrow |\mathscr{A}| \to Y)$ in **Cat**. In fact, the two tricategories obtained by restricting the objects of both **PCaten** and Span(**Cat**) to sets are biequivalent.

(c) For each bicategory $\mathscr{V}$ there is a functor $\mathscr{I}:\text{ob}\,\mathscr{V} \to \mathscr{V}$ which is the identity on objects; as in Example 2.3(b), this $\mathscr{I}$ can be regarded as a category enriched from ob $\mathscr{V}$

to $\mathscr{V}$. Yet there is also a procategory $\mathscr{J}:\mathscr{V}\to \mathrm{ob}\,\mathscr{V}$. Here again $\mathrm{ob}\,\mathscr{J}$ is the identity span of $\mathrm{ob}\,\mathscr{V}$. We need to define the module $\mathscr{J}(V,W):\mathscr{V}(V,W)\to \mathrm{ob}\,\mathscr{V}(V,W)$ for each V,W; however $\mathrm{ob}\,\mathscr{V}(V,W)$ is empty unless $V=W$, and $\mathrm{ob}\,\mathscr{V}(V,V)$ is a singleton; so a module $\mathscr{V}(V,V)\to \mathrm{ob}\,\mathscr{V}(V,V)$ amounts to a functor $\mathscr{V}(V,V)\to \mathbf{Set}$; we take $\mathscr{J}(V,V)$ to be the functor $\mathscr{V}(V,V)(1_V,-):\mathscr{V}(V,V)\to\mathbf{Set}$ represented by the identity of V:

$$\mathscr{J}(V,V)(e)=\mathscr{V}(V,V)(1_V,e).$$

Now $\eta_V\in\mathscr{J}(V,V)(1_V)$ is the identity 2-cell of 1_V, and the natural transformation

$$\mu^V_{VV}(e',e):\mathscr{J}(V,V)(e')\times\mathscr{J}(V,V)(e)\to\mathscr{J}(V,V)(e'\otimes e)$$

takes $(\sigma':1_V\Rightarrow e',\sigma:1_V\Rightarrow e)$ to the composite $1_V\xrightarrow{\cong}1_V\otimes 1_V\xrightarrow{\sigma'\otimes\sigma}e'\otimes e$.

In fact, as the reader will easily verify, $\mathscr{J}$ is just the right adjoint of $\mathscr{I}$, whose existence is guarenteed by Proposition 6.9.

(d) Examples (b) and (c) have the consequence that each procategory (and hence every category) $\mathscr{A}:\mathscr{V}\to\mathscr{W}$ has a "family of underlying categories" $\mathscr{A}_{VW}$. For we have the composite

$$\mathrm{ob}\,\mathscr{V}\xrightarrow{\mathscr{I}}\mathscr{V}\xrightarrow{\mathscr{A}}\mathscr{W}\xrightarrow{\mathscr{J}}\mathrm{ob}\,\mathscr{W}$$

in **PCaten** and hence a span $|\mathscr{J}\circ\mathscr{A}\circ\mathscr{I}|:\mathrm{ob}\,\mathscr{V}\to\mathrm{ob}\,\mathscr{W}$ in **Cat**. The objects of the category $|\mathscr{J}\circ\mathscr{A}\circ\mathscr{I}|$ are easily seen to be the objects of $\mathscr{A}$, while there is an arrow $f:A\to B$ in $|\mathscr{J}\circ\mathscr{A}\circ\mathscr{I}|$ only when $A_-=B_-$ and $A_+=B_+$, in which case f is an element of the set $\mathscr{A}(A,B)(1_{A_+},1_{A_-})$. We write $\mathscr{A}_{VW}$ for the full subcategory of $|\mathscr{J}\circ\mathscr{A}\circ\mathscr{I}|$ consisting of those objects A with $A_-=V$ and $A_+=W$, so that $\mathscr{A}_{VW}(A,B)=\mathscr{A}(A,B)(1_W,1_V)$.

In fact, we have a trifunctor $\mathrm{ob}:\mathbf{PCaten}\to\mathrm{Span}(\mathbf{Cat})$ whose effect on homs is the pseudofunctor

$$\begin{aligned}\mathbf{PCaten}(\mathscr{I},\mathscr{J}):\mathbf{PCaten}(\mathscr{V},\mathscr{W})&\to\mathbf{PCaten}(\mathrm{ob}\,\mathscr{V},\mathrm{ob}\,\mathscr{W})\\&\to\mathrm{Span}(\mathbf{Cat})(\mathrm{ob}\,\mathscr{V},\mathrm{ob}\,\mathscr{W}).\end{aligned}$$

Thus each functor $T:\mathscr{A}\to\mathscr{B}:\mathscr{V}\to\mathscr{W}$ gives an ordinary functor $T_{VW}:\mathscr{A}_{VW}\to\mathscr{B}_{VW}$ and each natural transformation $\alpha:T\to S:\mathscr{A}\to\mathscr{B}:\mathscr{V}\to\mathscr{W}$ gives a natural transformation $\alpha_{VW}:T_{VW}\to S_{VW}$.

6.11. The monoidal structure on **Caten** extends to **PCaten**, where every object gains a dual. For it is clear that we can form the cartesian product $\mathscr{A}\times\mathscr{B}:\mathscr{V}\times\mathscr{U}\to\mathscr{W}\times\mathscr{X}$ of procategories $\mathscr{A}:\mathscr{V}\to\mathscr{W}$ and $\mathscr{B}:\mathscr{U}\to\mathscr{X}$ by taking the product of the spans on objects and the product of the modules on homs; this easily extends also to functors and natural transformations. As before the associativity and unit constraints are obvious.

Proposition 6.11. *For any bicategories $\mathscr{U},\mathscr{V},\mathscr{W}$, there is a pseudonatural isomorphism of 2-categories*

$$\mathbf{PCaten}(\mathscr{U}\times\mathscr{V},\mathscr{W})\cong\mathbf{PCaten}(\mathscr{U},\mathscr{V}^{co}\times\mathscr{W}).$$

Proof. The isomorphism is immediate from (6.24); we leave the reader to verify its pseudonatural character. □

6.12. Proposition 6.11 should be compared with Proposition 4.3, whose proof depended on the autonomy of the monoidal bicategory **Span**:

$$\mathbf{Span}(X \times Y, Z) \cong \mathbf{Span}(X, Y \times Z) \tag{6.33}$$

and the closedness of the monoidal bicategory **Cat**:

$$\mathbf{Cat}(\mathscr{A} \times \mathscr{B}, \mathscr{C}) \cong \mathbf{Cat}(\mathscr{A}, [\mathscr{B}, \mathscr{C}]). \tag{6.34}$$

Extending Remark 6.8, we point out that the analogue of Proposition 4.3 can be proved with any closed monoidal $\mathfrak{M}$ in place of **Cat**. In particular, this works for $\mathfrak{M} = \mathbf{Mod}$; indeed the situation is better because **Mod** is autonomous: so that applying (6.33) at the object level of a category $\mathscr{A} : \mathscr{U} \times \mathscr{V} \to \mathscr{W}$, and (6.2) at the level of homs, we are led to the bicategory $\mathscr{V}^{\mathrm{co}} \times \mathscr{W}$ as internal hom in **PCaten**, without any requirement of local cocompleteness on $\mathscr{W}$.

6.13. Expanding on Example 6.10(b), we shall show how to regard procategories as special spans between bicategories. Let us begin with a procategory $\mathscr{A} : \mathscr{V} \to \mathscr{W}$ and construct a bicategory $\mathscr{E}$ and functors (= strict morphisms of bicategories)

$$\mathscr{V} \overset{()_-}{\leftarrow} \mathscr{E} \overset{()_+}{\rightarrow} \mathscr{W}. \tag{6.35}$$

The objects of $\mathscr{E}$ are the objects of $\mathscr{A}$. The hom-category $\mathscr{E}(A, B)$ is the two-sided category of elements (in the sense of [22]) of the module (= profunctor) $\mathscr{A}(A, B) : \mathscr{V}(A_-, B_-) \to \mathscr{W}(A_+, B_+)$; so a morphism $(u, a, f) : A \to B$ in $\mathscr{E}$ consists of $u \in \mathscr{W}(A_+, B_+), f \in \mathscr{V}(A_-, B_-)$ and $a \in \mathscr{A}(A, B)(u, f)$, and a 2-cell $(\xi, \sigma) : (u, a, f) \Rightarrow (u', a', f')$ consists of 2-cells $\xi : u \Rightarrow u'$ in $\mathscr{W}$ and $\sigma : f \Rightarrow f'$ in $\mathscr{V}$ for which

$$\mathscr{A}(A, B)(\xi, 1)(a') = \mathscr{A}(A, B)(1, \sigma)(a).$$

Horizontal composition $\otimes : \mathscr{E}(B, C) \times \mathscr{E}(A, B) \to \mathscr{E}(A, C)$ is given by

$$(v, b, g) \otimes (u, a, f) = (v \otimes u, \mu^{B}_{AC}(v, u; g, f)(b, a), g \otimes f),$$

$$(\zeta, \tau) \otimes (\xi, \sigma) = (\zeta \otimes \xi, \tau \otimes \sigma). \tag{6.36}$$

The identity morphism of A is $(1_{A_+}, \eta_A, 1_{A_-})$. The associativity and unit constraints are uniquely determined by the condition that we have functors as displayed in (6.35), where

$$(\xi, \sigma) : (u, a, f) \;\Rightarrow\; (u', a', f') : A \to B$$

in $\mathscr{E}$ goes to $\sigma : f \Rightarrow f' : A_- \to B_-$ in $\mathscr{V}$ under $()_-$, and goes to $\xi : u \Rightarrow u' : A_+ \to B_+$ in $\mathscr{W}$ under $()_+$.

Conversely, any span (6.35) of functors between bicategories, for which each span

$$\mathscr{W}(A_+, B_+) \overset{()_+}{\leftarrow} \mathscr{E}(A, B) \overset{()_-}{\rightarrow} \mathscr{V}(A_-, B_-) \tag{6.37}$$

of functors between categories is a 2-sided discrete fibration from $\mathscr{W}(A_+, B_+)$ to $\mathscr{V}(A_-, B_-)$ (in the sense of [22]), is isomorphic to one constructed as above from a procategory.

7. Modules

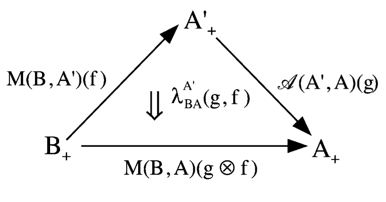
We would expect there to be a good notion of module $M : \mathscr{A} \to \mathscr{B} : \mathscr{V} \to \mathscr{W}$ between two-sided enriched categories. For categories enriched in a bicategory on one side, the definition and properties can be found in [24,6]. Indeed, equipped with the convolution construction of Proposition 4.3, we have a mechanism for turning the one-sided theory into the two-sided. However the two-sided definition is itself quite natural, and leads to new phenomena such as the behaviour of modules under the composition of two-sided enriched categories. We also need to keep in mind that our enriched categories here are generalized lax functors, so that modules give generalized transformations between lax functors; observe the increase in generality from the enriched functors of Example 2.5(b) between such categories, to the enriched modules of Example 7.4(a) below.

7.1. Suppose $\mathscr{A}, \mathscr{B}$ are categories enriched from $\mathscr{V}$ to $\mathscr{W}$. A *module* $M : \mathscr{A} \nrightarrow \mathscr{B}$ consists of the following data:

(i) for objects A of $\mathscr{A}$ and B of $\mathscr{B}$, a functor

$$M(B, A) : \mathscr{V}(B_-, A_-) \to \mathscr{W}(B_+, A_+);$$

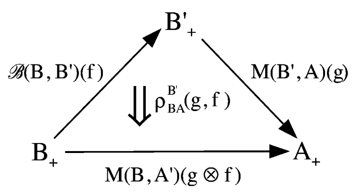
(ii) for objects A, A' of $\mathscr{A}$ and B of $\mathscr{B}$, 2-cells



(7.1)

in $\mathscr{W}$, natural in $f \in \mathscr{V}(B_-, A'_-)$ and $g \in \mathscr{V}(A'_-, A_-)$;

(iii) for objects A of $\mathscr{A}$ and B, B' of $\mathscr{B}$, 2-cells



(7.2)

in $\mathscr{W}$, natural in $f \in \mathscr{V}(B_-, B'_-)$ and $g \in \mathscr{V}(B'_-, A_-)$;

which are to be such that the five diagrams (7.3)–(7.7) commute.

$$
\begin{array}{ccc}
(\mathscr{A}(A',A)(g)\otimes\mathscr{A}(A'',A')(g'))\otimes M(B,A'')(f) & \xrightarrow{\mu^{A'}_{A''A}(g,g')\otimes 1} & \mathscr{A}(A'',A)(g\otimes g')\otimes M(B,A'')(f) \\
{\scriptstyle a}\Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \lambda^{A''}_{BA}(g\otimes g',f)} \\
\mathscr{A}(A',A)(g)\otimes(\mathscr{A}(A'',A')(g')\otimes M(B,A'')(f)) & & M(B,A)((g\otimes g')\otimes f) \\
{\scriptstyle 1\otimes\lambda^{A''}_{BA'}(g',f)}\Big\downarrow & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle M(B,A)(a)} \\
\mathscr{A}(A',A)(g)\otimes M(B,A')(g'\otimes f) & \xrightarrow[\lambda^{A'}_{BA}(g,g'\otimes f)]{} & M(B,A)(g\otimes(g'\otimes f)),
\end{array}
\tag{7.3}
$$

$$
\begin{array}{ccc}
\mathscr{A}(A,A)(1_{A_-})\otimes M(B,A)(f) & \xrightarrow{\lambda^{A}_{BA}(1_{A_-},f)} & M(B,A)(1_{A_-}\otimes f) \\
{\scriptstyle \eta_A\otimes 1}\Big\uparrow & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle M(B,A)(\ell)} \\
1_{A_+}\otimes M(B,A)(f) & \xrightarrow[\cong]{\ell} & M(B,A)(f),
\end{array}
\tag{7.4}
$$

$$
\begin{array}{ccc}
(M(B'',A)(g)\otimes\mathscr{B}(B',B'')(f'))\otimes\mathscr{B}(B,B')(f) & \xrightarrow{\rho^{B''}_{B'A}(g,f')\otimes 1} & M(B',A)(g\otimes f')\otimes\mathscr{B}(B,B')(f) \\
{\scriptstyle a}\Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \rho^{B'}_{BA}(g\otimes f',f)} \\
M(B'',A)(g)\otimes(\mathscr{B}(B',B'')(f')\otimes\mathscr{B}(B,B')(f)) & & M(B,A)((g\otimes f')\otimes f) \\
{\scriptstyle 1\otimes\mu^{B'}_{BB''}(f',f)}\Big\downarrow & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle M(B,A)(a)} \\
M(B'',A)(g)\otimes\mathscr{B}(B,B'')(f'\otimes f) & \xrightarrow[\rho^{B''}_{BA}(g,f'\otimes f)]{} & M(B,A)(g\otimes(f'\otimes f)),
\end{array}
\tag{7.5}
$$

$$
\begin{array}{ccc}
M(B,A)(f)\otimes\mathscr{B}(B,B)(1_{B_-}) & \xrightarrow{\rho^{B}_{BA}(f,1_{B_-})} & M(B,B)(f\otimes 1_{B_-}) \\
{\scriptstyle 1\otimes\eta_B}\Big\uparrow & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle M(B,A)(r)} \\
M(B,A)(f)\otimes 1_{B_+} & \xrightarrow[r]{\cong} & M(B,A)(f),
\end{array}
\tag{7.6}
$$

$$
\begin{array}{ccc}
(\mathscr{A}(A',A)(h)\otimes M(B',A')(g))\otimes\mathscr{B}(B,B')(f) & \xrightarrow{\lambda^{A'}_{B'A}(h,g)\otimes 1} & M(B',A)(h\otimes g)\otimes\mathscr{B}(B,B')(f) \\
{\scriptstyle a}\Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \rho^{B'}_{BA}(h\otimes g,f)} \\
\mathscr{A}(A',A)(h)\otimes(M(B',A')(g)\otimes\mathscr{B}(B,B')(f)) & & M(B,A)((h\otimes g)\otimes f) \\
{\scriptstyle 1\otimes\rho^{B'}_{BA'}(g,f)}\Big\downarrow & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle M(B,A)(a)} \\
\mathscr{A}(A',A)(h)\otimes M(B,A')(g\otimes f) & \xrightarrow[\lambda^{A'}_{BA}(h,g\otimes f)]{} & M(B,A)(h\otimes(g\otimes f)).
\end{array}
\tag{7.7}
$$

7.2. Suppose $M, N : \mathscr{A} \nrightarrow \mathscr{B}$ are modules, for categories $\mathscr{A}, \mathscr{B} : \mathscr{V} \to \mathscr{W}$. A *module morphism* $\alpha : M \to N$ is a family of natural transformations

$$\alpha_{BA} : M(B, A) \;\Rightarrow\; N(B, A) : \mathscr{V}(B_-, A_-) \to \mathscr{W}(B_+, A_+)$$

for $A \in \mathrm{ob}\,\mathscr{A}$ and $B \in \mathrm{ob}\,\mathscr{B}$, for which the two diagrams (7.8) and (7.9) commute:

$$\begin{array}{ccc}
\mathscr{A}(A', A)(g) \otimes M(B, A')(f) & \xrightarrow{\lambda^{A'}_{BA}(g,f)} & M(B, A)(g \otimes f) \\
{\scriptstyle 1 \otimes \alpha_{BA'}(f)} \Big\downarrow & & \Big\downarrow {\scriptstyle \alpha_{BA}(g \otimes f)} \\
\mathscr{A}(A', A)(g) \otimes N(B, A')(f) & \xrightarrow{\lambda^{A'}_{BA}(g,f)} & N(B, A)(g \otimes f),
\end{array} \qquad (7.8)$$

$$\begin{array}{ccc}
M(B', A)(g) \otimes \mathscr{B}(B, B')(f) & \xrightarrow{\rho^{B'}_{BA}(g,f)} & M(B, A)(g \otimes f) \\
{\scriptstyle \alpha_{B'A}(g) \otimes 1} \Big\downarrow & & \Big\downarrow {\scriptstyle \alpha_{BA}(g \otimes f)} \\
N(B', A)(g) \otimes \mathscr{B}(B, B')(f) & \xrightarrow{\rho^{B'}_{BA}(g,f)} & N(B, A)(g \otimes f).
\end{array} \qquad (7.9)$$

There is an obvious composition of module morphisms, and we obtain a category $\mathrm{Mod}(\mathscr{A}, \mathscr{B})$ whose objects are modules $M : \mathscr{A} \nrightarrow \mathscr{B}$.

7.3. Suppose $M : \mathscr{A} \nrightarrow \mathscr{B}$, $N : \mathscr{B} \nrightarrow \mathscr{C}$, $L : \mathscr{A} \nrightarrow \mathscr{C}$ are modules, for categories $\mathscr{A}, \mathscr{B}, \mathscr{C} : \mathscr{V} \to \mathscr{W}$. A *form*

$$\sigma : (N, M) \Rightarrow L : \mathscr{A} \nrightarrow \mathscr{C} : \mathscr{V} \to \mathscr{W}$$

is a family of 2-cells

$$\sigma^{B}_{CA}(g, f) : M(B, A)(g) \otimes N(C, B)(f) \Rightarrow L(C, A)(g \otimes f) : C_+ \to A_+,$$

natural in $f \in \mathscr{V}(C_-, B_-)$ and $g \in \mathscr{V}(B_-, A_-)$, for which the three diagrams (7.10)–(7.12) commute:

$$\begin{array}{ccc}
(\mathscr{A}(A', A)(h) \otimes M(B, A')(g)) \otimes N(C, B)(f) & \xrightarrow{\lambda^{A'}_{BA}(h,g) \otimes 1} & M(B, A)(h \otimes g) \otimes N(C, B)(f) \\
{\scriptstyle a} \Big\downarrow \cong & & \Big\downarrow {\scriptstyle \rho^{B}_{CA}(h \otimes g, f)} \\
\mathscr{A}(A', A)(h) \otimes (M(B, A')(g) \otimes N(C, B)(f)) & & L(C, A)((h \otimes g) \otimes f) \\
{\scriptstyle 1 \otimes \rho^{B}_{CA'}(g,f)} \Big\downarrow & & \cong \Big\downarrow {\scriptstyle L(C,A)(a)} \\
\mathscr{A}(A', A)(h) \otimes L(C, A')(g \otimes f) & \xrightarrow[\lambda^{A'}_{CA}(h, g \otimes f)]{} & L(C, A)(h \otimes (g \otimes f)),
\end{array}$$

(7.10)

$$
\begin{array}{ccc}
(\mathrm{M}(\mathrm{B}',\mathrm{A})(\mathrm{h}) \otimes \mathscr{B}(\mathrm{B},\mathrm{B}')(\mathrm{g})) \otimes \mathrm{N}(\mathrm{C},\mathrm{B})(\mathrm{f}) & \xrightarrow{\rho^{\mathrm{B}'}_{\mathrm{BA}}(\mathrm{h},\mathrm{g})\otimes 1} & \mathrm{M}(\mathrm{B},\mathrm{A})(\mathrm{h}\otimes \mathrm{g}) \otimes \mathrm{N}(\mathrm{C},\mathrm{B})(\mathrm{f}) \\
{\scriptstyle a}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \sigma^{\mathrm{B}}_{\mathrm{CA}}(\mathrm{h}\otimes \mathrm{g},\mathrm{f})} \\
\mathrm{M}(\mathrm{B}',\mathrm{A})(\mathrm{h}) \otimes (\mathscr{B}(\mathrm{B},\mathrm{B}')(\mathrm{g}) \otimes \mathrm{N}(\mathrm{C},\mathrm{B})(\mathrm{f})) & & \mathrm{L}(\mathrm{C},\mathrm{A})((\mathrm{h}\otimes \mathrm{g})\otimes \mathrm{f}) \\
{\scriptstyle 1\otimes\lambda^{\mathrm{B}}_{\mathrm{CB}'}(\mathrm{g},\mathrm{f})}\downarrow & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathrm{L}(\mathrm{C},\mathrm{A})(a)} \\
\mathrm{M}(\mathrm{B}',\mathrm{A})(\mathrm{h}) \otimes \mathrm{N}(\mathrm{C},\mathrm{B}')(\mathrm{g}\otimes \mathrm{f}) & \xrightarrow[\sigma^{\mathrm{B}'}_{\mathrm{CA}}(\mathrm{h},\mathrm{g}\otimes \mathrm{f})]{} & \mathrm{L}(\mathrm{C},\mathrm{A})(\mathrm{h}\otimes(\mathrm{g}\otimes \mathrm{f})),
\end{array}
\tag{7.11}
$$

$$
\begin{array}{ccc}
(\mathrm{M}(\mathrm{B},\mathrm{A})(\mathrm{h}) \otimes \mathrm{N}(\mathrm{C}',\mathrm{B})(\mathrm{g})) \otimes \mathscr{C}(\mathrm{C},\mathrm{C}')(\mathrm{f}) & \xrightarrow{\sigma^{\mathrm{B}}_{\mathrm{C}'\mathrm{A}}(\mathrm{h},\mathrm{g})\otimes 1} & \mathrm{L}(\mathrm{C}',\mathrm{A})(\mathrm{h}\otimes \mathrm{g}) \otimes \mathscr{C}(\mathrm{C},\mathrm{C}')(\mathrm{f}) \\
{\scriptstyle a}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \rho^{\mathrm{C}'}_{\mathrm{CA}}(\mathrm{h}\otimes \mathrm{g},\mathrm{f})} \\
\mathrm{M}(\mathrm{B},\mathrm{A})(\mathrm{h}) \otimes (\mathrm{N}(\mathrm{C}',\mathrm{B})(\mathrm{g}) \otimes \mathscr{C}(\mathrm{C},\mathrm{C}')(\mathrm{f})) & & \mathrm{L}(\mathrm{C},\mathrm{A})((\mathrm{h}\otimes \mathrm{g})\otimes \mathrm{f}) \\
{\scriptstyle 1\otimes\rho^{\mathrm{C}'}_{\mathrm{CB}}(\mathrm{g},\mathrm{f})}\downarrow & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathrm{L}(\mathrm{C},\mathrm{A})(a)} \\
\mathrm{M}(\mathrm{B},\mathrm{A})(\mathrm{h}) \otimes \mathrm{N}(\mathrm{C}',\mathrm{B})(\mathrm{g}\otimes \mathrm{f}) & \xrightarrow[\sigma^{\mathrm{B}}_{\mathrm{C}'\mathrm{A}}(\mathrm{h},\mathrm{g}\otimes \mathrm{f})]{} & \mathrm{L}(\mathrm{C},\mathrm{A})(\mathrm{h}\otimes(\mathrm{g}\otimes \mathrm{f})).
\end{array}
\tag{7.12}
$$

We write $\mathrm{For}(\mathrm{N},\mathrm{M};\mathrm{L})$ for the set of forms $\sigma:(\mathrm{N},\mathrm{M}) \Rightarrow \mathrm{L}$. In the obvious way, this defines a functor

$$\mathrm{For} : \mathrm{Mod}(\mathscr{B},\mathscr{C})^{\mathrm{op}} \times \mathrm{Mod}(\mathscr{A},\mathscr{B})^{\mathrm{op}} \times \mathrm{Mod}(\mathscr{A},\mathscr{C}) \to \mathbf{Set}. \tag{7.13}$$

The functoriality of $\mathrm{For}(\mathrm{N},\mathrm{M};\mathrm{L})$ in the variables M and N is given by *substitution*; module morphisms $\alpha:\mathrm{M}' \to \mathrm{M}$ and $\beta:\mathrm{N}' \to \mathrm{N}$ can be substituted into a form $\sigma:(\mathrm{N},\mathrm{M}) \Rightarrow \mathrm{L}$ to yield a form $\sigma(\beta,\alpha):(\mathrm{N}',\mathrm{M}') \Rightarrow \mathrm{L}$. This is part of a general calculus of substitution of forms in forms.

A representing object for the functor

$$\mathrm{For}(\mathrm{N},\mathrm{M};-):\mathrm{Mod}(\mathscr{A},\mathscr{C}) \to \mathbf{Set}$$

is called a *tensor product of* N *and* M *over* $\mathscr{B}$ and is denoted by $\mathrm{N} \otimes_{\mathscr{B}} \mathrm{M}$ (or simply $\mathrm{N} \otimes \mathrm{M}$); then there is an isomorphism

$$\mathrm{Mod}(\mathscr{A},\mathscr{C})(\mathrm{N} \otimes_{\mathscr{B}} \mathrm{M},\mathrm{L}) \cong \mathrm{For}(\mathrm{N},\mathrm{M};\mathrm{L}) \tag{7.14}$$

which is natural in N and is induced by composition with a *universal form*

$$\upsilon:(\mathrm{N},\mathrm{M}) \Rightarrow \mathrm{N} \otimes_{\mathscr{B}} \mathrm{M}.$$

When tensor products over $\mathscr{B}$ exist, there is a unique way of extending to morphisms the assignment $(\mathrm{N},\mathrm{M}) \mapsto \mathrm{N} \otimes_{\mathscr{B}} \mathrm{M}$ which turns $\otimes_{\mathscr{B}}$ into a functor

$$\otimes_{\mathscr{B}}:\mathrm{Mod}(\mathscr{B},\mathscr{C}) \times \mathrm{Mod}(\mathscr{A},\mathscr{B}) \to \mathrm{Mod}(\mathscr{A},\mathscr{C}) \tag{7.15}$$

and makes the isomorphisms (7.14) natural in both N and M.

For each $\mathscr{B}:\mathscr{V} \to \mathscr{W}$ there is an *identity module* $\mathrm{I}_{\mathscr{B}}:\mathscr{B} \nrightarrow \mathscr{B}$, given by $\mathrm{I}_{\mathscr{B}}(\mathrm{B},\mathrm{A}) = \mathscr{B}(\mathrm{B},\mathrm{A})$, with the left and right actions given by the μ for $\mathscr{B}$. The tensor products $\mathrm{I}_{\mathscr{B}} \otimes \mathrm{M}$ and $\mathrm{N} \otimes \mathrm{I}_{\mathscr{B}}$ always exist, being given by M and N respectively, to within coherent isomorphisms.

Module morphisms can be considered to be forms in one variable M, while the forms above involve the two variables M and N. It is also possible to define forms

$$\tau:(\mathrm{K},\mathrm{N},\mathrm{M}) \;\Rightarrow\; \mathrm{L}$$

in three variables $\mathrm{M}:\mathscr{A} \nrightarrow \mathscr{B}$, $\mathrm{N}:\mathscr{B} \nrightarrow \mathscr{C}$ and $\mathrm{K}:\mathscr{C} \nrightarrow \mathscr{D}$, where $\mathrm{L}:\mathscr{A} \nrightarrow \mathscr{D}$. In the case where $\mathrm{For}(\mathrm{K},\mathrm{N},\mathrm{M};-)$ is representable, we are led to a ternary tensor product $\mathrm{K} \otimes \mathrm{N} \otimes \mathrm{M}$. Substitution of universal forms leads to forms

$$(\mathrm{K},\mathrm{N},\mathrm{M}) \Rightarrow (\mathrm{K} \otimes \mathrm{N}) \otimes \mathrm{M} \quad \text{and} \quad (\mathrm{K},\mathrm{N},\mathrm{M}) \Rightarrow \mathrm{K} \otimes (\mathrm{N} \otimes \mathrm{M})$$

and hence to a canonical span

$$(\mathrm{K} \otimes \mathrm{N}) \otimes \mathrm{M} \leftarrow \mathrm{K} \otimes \mathrm{N} \otimes \mathrm{M} \to \mathrm{K} \otimes (\mathrm{N} \otimes \mathrm{M}). \tag{7.16}$$

Proposition 7.3. *Suppose $\mathscr{V}$ is locally small and $\mathscr{W}$ is locally cocomplete. If* $\mathrm{ob}\,\mathscr{B}$ *is small then every pair of modules* $\mathrm{M}:\mathscr{A} \nrightarrow \mathscr{B}$, $\mathrm{N}:\mathscr{B} \nrightarrow \mathscr{C}$ *has a tensor product* $\mathrm{N} \otimes \mathrm{M}$. *If further* $\mathrm{ob}\,\mathscr{C}$ *is small and* $\mathrm{K}:\mathscr{C} \nrightarrow \mathscr{D}$, *then the ternary tensor product* $\mathrm{K} \otimes \mathrm{N} \otimes \mathrm{M}$ *exists and both of the arrows in span* (7.16) *are invertible. There is a bicategory* **Moden**$(\mathscr{V},\mathscr{W})$ *whose objects are categories* $\mathscr{A}:\mathscr{V} \to \mathscr{W}$ *with* $\mathrm{ob}\,\mathscr{A}$ *small, whose hom categories are the* $\mathrm{Mod}(\mathscr{A},\mathscr{B})$, *and whose horizontal composition is tensor product of modules.*

Proof. It follows from each of [24,6,12] that this proposition is true for the one-sided $\mathscr{W}$-enriched case; that is, where $\mathscr{V} = \mathbf{1}$. By Proposition 4.3 we have the locally cocomplete bicategory **Conv**$(\mathscr{V},\mathscr{W})$ with horizontal composition $\mathrm{P}\bar{\otimes}\mathrm{Q}$ given by (4.19). We can therefore apply the one-sided case with $\mathscr{W}$ replaced by **Conv**$(\mathscr{V},\mathscr{W})$. In the notation of Section 4.3, it is easy to see that modules $\mathrm{M}:\mathscr{A} \nrightarrow \mathscr{B}:\mathscr{V} \to \mathscr{W}$ translate precisely to modules $\bar{\mathrm{M}}:\bar{\mathscr{A}} \nrightarrow \bar{\mathscr{B}}$ between **Conv**$(\mathscr{V},\mathscr{W})$-categories; furthermore, this translation extends to forms. So the proposition really follows from the one-sided case and we have

$$\mathbf{Moden}(\mathscr{V},\mathscr{W}) \cong \mathbf{Conv}(\mathscr{V},\mathscr{W})\text{-}\mathbf{Mod}, \tag{7.17}$$

wherein we are reverting on the right side to the more usual name $\mathscr{U}$-**Mod** of **Moden**$(1,\mathscr{U})$. However, for the sake of completeness, we shall describe the tensor product $\mathrm{N} \otimes \mathrm{M}$. For $\mathrm{A} \in \mathrm{ob}\,\mathscr{A}$ and $\mathrm{C} \in \mathrm{ob}\,\mathscr{C}$ we form the coequalizer $(\mathrm{N} \otimes \mathrm{M})(\mathrm{C},\mathrm{A})$

of the pair of arrows

$$\sum_{B,B'} M(B',A)\bar{\otimes}\mathscr{B}(B,B')\bar{\otimes}N(C,B) \underset{1\otimes\lambda}{\overset{\rho\otimes 1}{\rightrightarrows}} \sum_{B} M(B,A)\bar{\otimes}N(C,B). \tag{7.18}$$

The left action for $N\otimes M$ is induced by the left action for M, while the right action for $N\otimes M$ is induced by the right action for N. The isomorphism (7.14) is easily deduced. The construction of $K\otimes N\otimes M$ should now be clear. $\square$

7.4. Examples. (a) Suppose $\mathscr{A},\mathscr{B}:\mathscr{V}\to\mathscr{W}$ are lax functors (see Example 2.3(b)). Recall [3] that a (lax natural) transformation $\tau:\mathscr{B}\to\mathscr{A}$ is given by data as displayed below.

$$\begin{array}{ccc} \mathscr{B}(B) & \xrightarrow{\ \tau_B\ } & \mathscr{A}(B) \\ {\scriptstyle \mathscr{B}(f)}\big\downarrow & \Downarrow \tau_f & \big\downarrow{\scriptstyle \mathscr{A}(f)} \\ \mathscr{B}(A) & \xrightarrow[\ \tau_A\ ]{} & \mathscr{A}(A). \end{array} \tag{7.19}$$

Given such a transformation τ, we can define a module $M:\mathscr{A} \nrightarrow \mathscr{B}$ by letting the functor

$$M(B,A):\mathscr{V}(B,A)\to\mathscr{W}(\mathscr{B}(B),\mathscr{A}(A))$$

take $f:B\to A$ to the lower leg $\tau_A\otimes\mathscr{B}(f)$ of the above square, and letting the actions

$$\lambda_{BA}^{A'}(g,f):\mathscr{A}(g)\otimes M(B,A')(f)\to M(B,A)(g\otimes f),$$

$$\rho_{BA}^{B'}(g,f):M(B',A)(g)\otimes\mathscr{B}(f)\to M(B,A)(g\otimes f)$$

be the composites

$$\mathscr{A}(g)\otimes\tau_{A'}\otimes\mathscr{B}(f)\xrightarrow{\tau_g\otimes 1}\tau_A\otimes\mathscr{B}(g)\otimes\mathscr{B}(f)\xrightarrow{1\otimes\mu_{BA}^{A'}(g,f)}\tau_A\otimes\mathscr{B}(g\otimes f),$$

$$\tau_A\otimes\mathscr{B}(g)\otimes\mathscr{B}(f)\xrightarrow{1\otimes\mu_{BA}^{B'}(g,f)}\tau_A\otimes\mathscr{B}(g\otimes f),$$

where we have omitted the obvious associativity constraints. The verification that M is indeed a module is routine.

(b) Suppose $S:\mathscr{A}\to\mathscr{X}$, $T:\mathscr{B}\to\mathscr{X}$ are functors between categories enriched from $\mathscr{V}$ to $\mathscr{W}$. There is a module $\mathscr{X}(T,S):\mathscr{A}\nrightarrow\mathscr{B}$ defined by taking

$$\mathscr{X}(T,S)(B,A)=\mathscr{X}(TB,SA):\mathscr{V}(B_-,A_-)\to\mathscr{W}(B_+,A_+)$$

with left and right actions

$$\begin{aligned} \mathscr{A}(A',A)(g)\otimes\mathscr{X}(TB,SA')(f) &\xrightarrow{S_{A'A}(g)\otimes 1}\mathscr{X}(SA',SA)(g)\otimes\mathscr{X}(TB,SA')(f) \\ &\xrightarrow{\mu_{TB,SA}^{SA'}}\mathscr{X}(TB,SA)(g\otimes f), \\ \mathscr{X}(TB'\,SA)(g)\otimes\mathscr{B}(B,B')(f) &\xrightarrow{1\otimes T_{BB'}(f)}\mathscr{X}(TB',SA)(g)\otimes\mathscr{X}(TB,TB')(f) \\ &\xrightarrow{\mu_{TB,SA}^{TB'}}\mathscr{X}(TB,SA)(g\otimes f). \end{aligned}$$

More generally, for functors $S:\mathscr{A}\to\mathscr{X}$, $T:\mathscr{B}\to\mathscr{Y}$ and a module $M:\mathscr{X}\nrightarrow\mathscr{Y}$, there is a module $M(T,S):\mathscr{A}\nrightarrow\mathscr{B}$ given by $M(T,S)(B,A)=M(TB,SA)$, the actions of $M(T,S)$ being those of M. As particular cases, we put

$$S_* = \mathscr{X}(1_{\mathscr{X}},S):\mathscr{A}\nrightarrow\mathscr{X} \quad \text{and} \quad S^* = \mathscr{X}(S,1_{\mathscr{X}}):\mathscr{X}\nrightarrow\mathscr{A}, \tag{7.20}$$

and note that we always have the ternary tensor product

$$T^*\otimes M\otimes S_* = M(T,S) \tag{7.21}$$

independently of any size or cocompleteness conditions. Taking S and T to be identity functors, we see from (7.20) that the modules $\mathscr{X}(1_{\mathscr{X}},1_{\mathscr{X}})$ are the identity modules $I_{\mathscr{X}}$ of Section 7.3; we henceforth write simply $1_{\mathscr{X}}$ rather than $(1_{\mathscr{X}})_*$ or $I_{\mathscr{X}}$. For any functor $S:\mathscr{A}\to\mathscr{X}$, we have a module morphism

$$\eta_S:1_{\mathscr{A}} \;\Rightarrow\; S^*\otimes S_* \tag{7.22}$$

consisting of the natural transformations $S_{AB}:\mathscr{A}(A,B)\Rightarrow\mathscr{X}(SA,SB)$. We also have a form

$$\varepsilon_S:(S_*,S^*) \;\Rightarrow\; 1_{\mathscr{X}} \tag{7.23}$$

consisting of the family of 2-cells

$$\mu^{SA}_{XY}(g,f):\mathscr{X}(SA,Y)(g)\otimes\mathscr{X}(X,SA)(f) \;\Rightarrow\; \mathscr{X}(X,Y)(g\otimes f).$$

Similarly, there are forms

$$\varepsilon_S\otimes S_*:(S_*,S^*\otimes S_*)\Rightarrow S_* \quad \text{and} \quad S^*\otimes\varepsilon_S:(S^*\otimes S_*,S^*)\Rightarrow S^* \tag{7.24}$$

consisting of the obvious families of 2-cells μ. The module adjointness $S_*\dashv S^*$ is expressed in our present multilinear context by the identities:

$$(\varepsilon_S\otimes S_*)(1_{S_*},\eta_S)=1_{S_*}, \quad (S^*\otimes\varepsilon_S)(\eta_S,1_{S^*})=1_{S^*}. \tag{7.25}$$

7.5. We now extend the definition of the composition (2.21) of enriched categories to modules between these. Take modules

$$M:\mathscr{A}\nrightarrow\mathscr{B}:\mathscr{V}\to\mathscr{W} \quad \text{and} \quad N:\mathscr{C}\nrightarrow\mathscr{D}:\mathscr{W}\to\mathscr{U}.$$

There is a module $N\circ M:\mathscr{C}\circ\mathscr{A}\nrightarrow\mathscr{D}\circ\mathscr{B}:\mathscr{V}\to\mathscr{U}$ defined by taking the functor $(N\circ M)((D,B),(C,A))$ to be the composite

$$\mathscr{V}(B_-,A_-)\xrightarrow{M(B,A)}\mathscr{W}(B_+,A_+)=\mathscr{W}(D_-,C_-)\xrightarrow{N(D,C)}\mathscr{U}(D_+,C_+) \tag{7.26}$$

with the left action

$$\lambda:(\mathscr{C}\circ\mathscr{A})((C',A'),(C,A))(g)\otimes(N\circ M)((D,B),(C',A'))(f)$$
$$\to(N\circ M)((D,B),(C,A))(g\otimes f)$$

given by the composite

$$\mathscr{C}(C',C)(\mathscr{A}(A',A)(g)) \otimes N(D,C')(M(B,A')(f))$$
$$\xrightarrow{\lambda} N(D,C)(\mathscr{A}(A',A)(g) \otimes M(B,A')(f))$$
$$\xrightarrow{N(D,C)(\lambda)} N(D,C)(M(B,A)(g \otimes f))$$

and with the right action

$$\rho : (N \circ M)((D',B'),(C,A))(g) \otimes (\mathscr{D} \circ \mathscr{B})((D,B),(D',B'))(f)$$
$$\to (N \circ M)((D,B),(C,A))(g \otimes f)$$

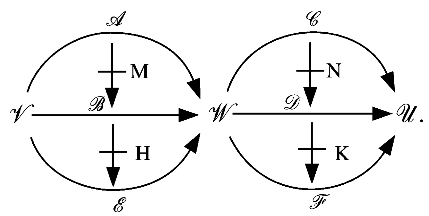
given by the composite

$$N(D',C)(M(B',A)(g)) \otimes \mathscr{D}(D,D')(\mathscr{B}(B,B')(f))$$
$$\xrightarrow{\rho} N(D,C)(M(B',A)(g) \otimes \mathscr{B}(B,B')(f))$$
$$\xrightarrow{N(D,C)(\rho)} N(D,C)(M(B,A)(g \otimes f)).$$

Given two module morphisms $\alpha : M \to M'$ and $\beta : N \to N'$, we obtain a module morphism $\beta \circ \alpha : N \circ M \to N' \circ M'$ by defining $(\beta \circ \alpha)_{(D,B)(C,A)}$ to be the horizontal composite of α_{BA} and β_{DC}. Indeed, we obtain a functor

$$- \circ - : \mathrm{Mod}(\mathscr{C},\mathscr{D}) \times \mathrm{Mod}(\mathscr{A},\mathscr{B}) \to \mathrm{Mod}(\mathscr{C} \circ \mathscr{A}, \mathscr{D} \circ \mathscr{B}). \tag{7.27}$$

Now, consider the diagram (7.28) of modules, along with further modules $P : \mathscr{A} \nrightarrow \mathscr{E}$ and $L : \mathscr{C} \nrightarrow \mathscr{F}$.



(7.28)

There is a function

$$- \circ - : \mathrm{For}(K,N;P) \times \mathrm{For}(H,M;L) \to \mathrm{For}(K \circ H, N \circ M; P \circ L) \tag{7.29}$$

taking forms $\tau : (K,N) \to P$ and $\sigma : (H,M) \to L$ to the form $\tau \circ \sigma : (K \circ H, N \circ M) \to P \circ L$ defined by taking

$$(\tau \circ \sigma)^{(D,B)}_{(F,E),(C,A)} : (N \circ M)((D,B),(C,A))(g) \otimes (K \circ H)((F,E),(D,B))(f)$$
$$\to (P \circ L)((F,E),(C,A))(g \otimes f)$$

to be the composite

$$\begin{aligned}&N(D,C)(M(B,A)(g)) \otimes K(F,D)(H(E,B)(f)) \xrightarrow{\tau^{D}_{F,C}(M(B,A)(g),H(E,B)(f))}\\ &\quad P(F,C)(M(B,A)(g) \otimes H(E,B)(f)) \xrightarrow{P(F,C)(\sigma^{B}_{E,A}(g,f))} P(F,C)(L(E,A)(g \otimes f)).\end{aligned} \tag{7.30}$$

Indeed, the functions (7.28) are natural in all six variables. Consequently, if the tensor products $H \otimes M$ and $K \otimes N$ exist, we can take $P = H \otimes M$ and $L = K \otimes N$ in (7.28) and evaluate at the universal forms to obtain a form

$$\varpi^{MN}_{HK} : (K \circ H, N \circ M) \to (K \otimes N) \circ (H \otimes M), \tag{7.31}$$

called the *middle-four-interchange constraint*. There are various naturality and coherence conditions satisfied by the family of forms (7.31); however, we shall content ourselves with the special, yet important, case where N and K are identities. We obtain the following process of change of base for modules:

Proposition 7.5. *Consider a locally small bicategory $\mathscr{V}$ and locally cocomplete bicategories $\mathscr{W}$ and $\mathscr{U}$. Each category $\mathscr{C} : \mathscr{W} \to \mathscr{U}$ determines a lax functor*

$$F = \mathscr{C} \circ - : \mathbf{Moden}(\mathscr{V}, \mathscr{W}) \to \mathbf{Moden}(\mathscr{V}, \mathscr{U}) \tag{7.32}$$

given on objects by $F\mathscr{A} = \mathscr{C} \circ \mathscr{A}$, and on hom-categories by fixing the first variable of (7.27) *at the identity module of $\mathscr{C}$; furthermore, the arrows $F_{0;\mathscr{A}}$ are invertible* (*so that* F *is what we call* normal) *and the arrows $F_{2;HM} : (\mathscr{C} \circ H) \otimes (\mathscr{C} \circ M) \to \mathscr{C} \circ (H \otimes M)$ are induced by instances of* (7.31). *For a functor $S : \mathscr{A} \to \mathscr{B} : \mathscr{V} \to \mathscr{W}$, there are canonical module isomorphisms*

$$(\mathscr{C} \circ S)_* \cong \mathscr{C} \circ S_* \quad \textit{and} \quad (\mathscr{C} \circ S)^* \cong \mathscr{C} \circ S^*.$$

7.6. In lectures in the early 1970s, Bénabou pointed out that the construction by Grothendieck of a fibration $\mathscr{E} \to \mathscr{C}$ from a pseudofunctor $F : \mathscr{C}^{op} \to \mathbf{Cat}$ can be generalized to the construction of an arbitrary functor $\mathscr{E} \to \mathscr{C}$ from a normal lax functor $F : \mathscr{C}^{op} \to \mathbf{Mod}$; both processes are invertible up to isomorphism. More generally, suppose we have bicategories $\mathscr{V}$ and $\mathscr{W}$ with $\mathscr{W}$ locally cocomplete. Consider a normal lax functor $F : \mathscr{V}^{op} \to \mathscr{W}$-**Mod**. Recall from [23] (although a duality is introduced here because of our conventions on order of composition) that there is a canonical pseudofunctor $\mathscr{I} : \mathscr{W}^{op} \to \mathscr{W}$-**Mod** taking $W \in \mathscr{W}$ to the $\mathscr{W}$-category $\mathscr{I}(W)$ whose only object is W and whose hom $\mathscr{I}(W)(W, W)$ is the identity arrow of W; on hom-categories $\mathscr{I}$ is the obvious isomorphism

$$\mathscr{W}(W', W) \cong (\mathscr{W}\text{-}\mathbf{Mod})(\mathscr{I}(W), \mathscr{I}(W')),$$

so actually $\mathscr{I}$ is a local equivalence. By Proposition 2.7, $\mathscr{I} : \mathscr{W}^{op} \to \mathscr{W}$-**Mod** has a right adjoint $\mathscr{J} : \mathscr{W}$-**Mod** $\to \mathscr{W}^{op}$ in **CATEN**. Thus, we obtain a category $\mathscr{J} \circ F : \mathscr{V}^{op} \to \mathscr{W}^{op}$ which, using the duality principle of Section 2.9, gives a category $\mathscr{A} : \mathscr{V} \to \mathscr{W}$.

This process can be inverted up to isomorphism as follows. Take any category $\mathscr{A}:\mathscr{V}\to\mathscr{W}$. By Proposition 7.5 we obtain a normal lax functor $\mathscr{A}\circ - :\mathscr{V}$-**Mod** $\to$ $\mathscr{W}$-**Mod** which composes with the pseudofunctor $\mathscr{I}:\mathscr{V}^{\mathrm{op}}\to\mathscr{V}$-**Mod** to give a normal lax functor $\mathrm{F}:\mathscr{V}^{\mathrm{op}}\to\mathscr{W}$-**Mod**. (In this presentation of the inverse construction, the apparent need for $\mathscr{V}$ to be locally cocomplete, in order to speak of $\mathscr{V}$-**Mod**, is not real.)

If under this correspondence the categories $\mathscr{A}:\mathscr{V}\to\mathscr{W}$ and $\mathscr{C}:\mathscr{W}\to\mathscr{U}$ correspond to the normal lax functors $\mathrm{F}:\mathscr{V}^{\mathrm{op}}\to\mathscr{W}$-**Mod** and $\mathrm{G}:\mathscr{W}^{\mathrm{op}}\to\mathscr{U}$-**Mod**, then the composite $\mathscr{C}\circ\mathscr{A}:\mathscr{V}\to\mathscr{U}$ corresponds to the composite of F and G after we make the identifications $(\mathscr{W}\text{-}\mathbf{Mod})^{\mathrm{op}}=\mathscr{W}^{\mathrm{op}}\text{-}\mathbf{Mod}$ and $(\mathscr{W}\text{-}\mathbf{Mod})\text{-}\mathbf{Mod}=\mathscr{W}\text{-}\mathbf{Mod}$ (see [23]).

Now suppose that $\mathscr{W}$ is a small bicategory and $\mathscr{V}$ is any bicategory. Each lax normal functor $\mathrm{F}:\mathscr{V}^{\mathrm{op}}\to\mathscr{P}^*\mathscr{W}$-**Mod** corresponds to a category $\mathscr{A}^{\#}:\mathscr{V}\to\mathscr{P}^*\mathscr{W}$ and hence, using Proposition 6.5, to a procategory $\mathscr{A}:\mathscr{V}\to\mathscr{W}$. Taking the viewpoint of Section 6.12 on procategories, we obtain a span (6.35) of bicategories. The Bénabou case is obtained by taking $\mathscr{W}$ to be **1** and $\mathscr{V}$ to be locally discrete.

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