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Bridges with pillars: a graphical calculus of knot algebra

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Abstract

The paper comprises a graphical calculus which is designed to deal with the Coxeter–Dynkin series of type E and some generalizations. Temperley–Lieb algebras of type E are defined as quotients of Hecke algebras and the module structure of the algebra associated to E_6 is determined. The graphical calculus is a refinement of the calculus for the ordinary Temperley–Lieb algebra: a planar strip is decomposed by the arcs of a diagram into domains and the domains are used to incorporate additional information into the figure. © 1997 Elsevier Science B.V.

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1. Introduction

This paper comprises a graphical calculus which is designed to deal with the Coxeter–Dynkin series of type E and some generalizations.

A Coxeter matrix (S, m) consists of a finite set S and a symmetric mapping

$$m : S \times S \to \mathbb{N} \cup \{\infty\}$$

with $m(s,s) = 1$ and $m(s,t) \geqslant 2$ for $s \neq t$. A Coxeter matrix (S,m) is often specified by its weighted Coxeter graph $\Gamma(S,m)$. It has S as its set of vertices and an edge with weight $m(s,t)$ connecting s and t whenever $m(s,t) \geqslant 3$. Usually, the weight $m(s,t) = 3$ is omitted in the notation. If $m(S \times S) \subset \{1,2,3\}$, we define the associated Temperley–Lieb algebra $T_d(S,m)$ as follows. Let $\mathcal{K}$ be a commutative ring and $d \in \mathcal{K}^*$

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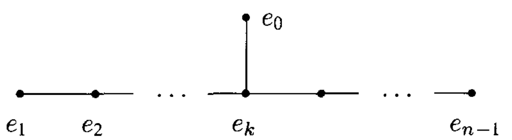
an invertible parameter. Then $T_d(S, m)$ is the associative algebra with 1 over $\mathcal{K}$ with generators $(e_s \mid s \in S)$ and relations

$$\begin{aligned} &e_s^2 = de_s, && \\ &e_s e_t = e_t e_s, && m(s,t) = 2, \\ &e_s e_t e_s = e_s, && m(s,t) = 3. \end{aligned} \tag{1}$$

Recall that the Hecke algebra $H_q(S, m)$ associated to the Coxeter matrix (S, m) and an invertible parameter $q \in \mathcal{K}$ is the associative algebra with 1 over $\mathcal{K}$ with generators $(x_s \mid s \in S)$ and relations $x_s^2 = (q-1)x_s + q$ and braid relations $x_s x_t \cdots = x_t x_s \cdots$ ($m(s,t)$ factors x_s, x_t alternating on each side). Suppose $p \in \mathcal{K}^*$, $q = p^2$, $d = p + p^{-1}$. Then the assignment $x_s \mapsto pe_s - 1$ yields a surjection $H_q(S, m) \to T_d(S, m)$ (see [3]). The classical Coxeter matrices (Coxeter graphs) of simple Lie groups yield finite dimensional Hecke algebras. It turns out that there are additional graphs which yield finite dimensional Temperley–Lieb algebras, e.g., those associated to the graphs $E_n(k)$ below.

In general, the algebras $T_d(S, m)$ are difficult to analyze, because of their definition by generators and relations. The purpose of this paper is to describe a graphical calculus which is adapted to Coxeter graphs of type E. We denote by $E_n(k)$ the graph with n vertices of the following shape.



We also use the standard notation of Lie theory

$$E_n(1) = A_n, \qquad E_n(2) = D_n, \qquad E_n(3) = E_n.$$

Note that $E_n(k)$ contains the linear subgraph A_{n-1} with $n-1$ vertices (e_0 omitted). Although the graphical calculus is designed for the finite dimensional algebras $T_d E_n(k)$, it has other uses as well.

The starting point for our calculus is the graphical notation of Kauffman [5, p. 100] for the standard Temperley–Lieb algebra $T_d A_n$ associated to the Coxeter graph A_n. A basis element of $T_d A_{n-1}$ over $\mathcal{K}$ consists of n disjoint arcs in $\mathbb{R} \times [0,1]$ with endpoint set $\{1, \ldots, n\} \times \{0,1\}$. We call such figures *$(n,n)$-bridges*. We use the decomposition of $\mathbb{R} \times [0,1]$ into planar domains produced by the arcs of a bridge. This decomposition is used to incorporate additional information into the figure: we single out certain regions by placing *pillars*. A $\mathcal{K}$-basis of the algebra $T_d E_n(k)$ will then, hopefully, consist of certain such *bridges with pillars*. For typographical reasons we use a bracket notation for bridges with pillars. The reader is advised to decode this into ordinary planar figures. The advantage of the graphical calculus is its semiglobal nature, compared with generators and relations. (This is similar to the difference between the global definition of the symmetric group by permutations and its definition by generators and relations from the Coxeter graph.) As an example of the use of the calculus we mention: a geometrically defined filtration on the set of figures allows a splitting of the resulting algebras into matrix components (in the generic semisimple case), and also allows a geometric construction of modules.

This paper presents the calculus and applies it to the basic example $E_6 = E_6(3)$. A typical result is:

Theorem A. *The algebra $T_dE_6(3)$ has rank* 662. *For generic parameters d in a field the algebra is semisimple and has simple modules $M(0)$, $M(1)$, $M(2)$, and $M(3)$ of rank* 1, 6, 20, *and* 15, *respectively.*

Since $A_5 \subset E_6$, we have an inclusion $T_dA_5 \subset T_dE_6$. We determine the decomposition of the simple T_dE_6-modules when restricted to T_dA_5. Recall that the algebra T_dA_5 has simple modules M_0, M_1, M_2, and M_3 of rank 1, 5, 9, and 5 (see, e.g., [4, 2.8] for the module theory of T_dA_n in general).

Theorem B. *The following isomorphisms hold for the restricted modules:*

$$\operatorname{res} M(0) \cong M_0,$$

$$\operatorname{res} M(1) \cong M_1 \oplus M_0,$$

$$\operatorname{res} M(2) \cong M_2 \oplus 2M_1 \oplus M_0,$$

$$\operatorname{res} M(3) \cong M_3 \oplus M_2 \oplus M_0.$$

We mention (but do not prove in this paper) the module structure of $T_dE_7(3)$:

Theorem C. *The algebra $T_dE_7(3)$ has in the generic case simple modules $N(4)$, $N(3)$, $N'(3)$, $N(2)$, $N'(2)$, $N(1)$, and $N(0)$ of rank* 15, 35, 35, 27, 18, 7, *and* 1, *respectively.*

In the statement of the next result we use the fact that the algebra T_dA_6 has simple modules N_0, N_1, N_2, N_3 of rank 1, 6, 14, 14.

Theorem D. *The restriction properties of these modules to $A = T_dA_6$ and $E = T_dE_6$ are:*

$$\operatorname{res}_E N(4) = M(3),$$

$$\operatorname{res}_A N(4) = N_0 \oplus N_3,$$

$$\operatorname{res}_E N(3) = M(2) \oplus M(3),$$

$$\operatorname{res}_A N(3) = N_0 \oplus N_1 \oplus N_2 \oplus N_3,$$

$$\operatorname{res}_E N(2) = M(2) \oplus M(1) \oplus M(0),$$

$$\operatorname{res}_E N'(2) = 3M(1),$$

$$\operatorname{res}_A N(2) = N_2 \oplus 2N_1 \oplus N_0,$$
$$\operatorname{res}_A N'(2) = 3N_1,$$
$$\operatorname{res}_E N(1) = M(1) \oplus M(0),$$
$$\operatorname{res}_A N(1) = N_1 \oplus N_0.$$

The module $N'(3)$ has the same restriction properties as $N(3)$.

In the next section we define the calculus and various related algebras. Section 3 presents some elementary examples. In Sections 4 and 5 we give a detailed description of the algebra related to $E_6(3)$. We conclude by mentioning other uses of the calculus.

2. Bridges with pillars

We remind the reader that an (n,n)-bridge consists of a system of n arcs in $\mathbb{R} \times [0,1]$ without crossings such that the set of its boundary points is $P_n := \{1,2,\ldots,n\} \times \{0,1\}$. We think of the arcs meeting $\mathbb{R} \times \{0,1\}$ transversely. Two bridges are equal if they connect the same points. An *upper* (*lower*) arc has both of its boundary points in $\mathbb{R} \times 1$ ($\mathbb{R} \times 0$). The upper and lower arcs are called *horizontal*, the others *vertical*. If a bridge b has k upper arcs, we call k its *horizontal edge number* and write $HE(b) = k$. The configuration which consists of the upper arcs of an (n,n)-bridge b is called the *upper n-bridge* of b. An (n,n)-bridge is determined by its upper and lower n-bridge. An upper n-bridge per se of horizontal edge number k can be specified by a system of k admissible bracket pairs with additional marks for the end points of the vertical arcs. Thus, the 9 upper 6-bridges b with $HE(b) = 2$ are in this notation:

$$(())||,\quad ()()||,\quad |(())|,\quad |()()|,\quad ||(()),\quad ||()(),\quad ()|()|,\quad ()||(),\quad |()|().$$

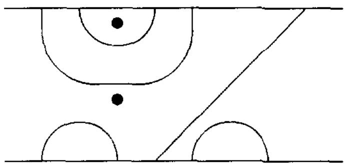
An (n,n)-bridge will, in a similar manner, be denoted by a pair of bracket systems with a fraction stroke (compare the figure below for the notation of a pillar bridge).

We need bridges with additional structure. The definition needs some preparation. The arcs of a bridge subdivide the strip $\mathbb{R} \times [0,1]$ into domains. There are one or two unbounded domains. An (n,n)-bridge yields $n+1$ domains. The *distance* between two bridge domains is the minimal numbers of arcs which a path from one domain to the other has to cross. The distance modulo two is a topological intersection number. The boundary curve of a domain consists of certain intervals in $\mathbb{R} \times \{0,1\}$ and some arcs of the bridge. The intervals are called the *feet* of the domain. A domain is determined by its feet. An *upper* (*lower*) domain has all its feet in $\mathbb{R} \times 1$ ($\mathbb{R} \times 0$). The remaining domains are called *vertical*.

Each domain D of a bridge has 4 *separation numbers*: $SNL(D)$, $SNR(D)$, $SNT(D)$, $SNB(D)$. Here L, R, T, B stands for left, right, top, bottom. The *left separation number* is the minimal number of arcs separating the domain from the left unbounded one. The *bottom separation number* is the minimal number of arcs separating the domain from $\mathbb{R} \times 0$. Thus a vertical domain is characterized by $SNB = SNT = 0$.

We fix a *base number* $k \in \{0, 1, 2, \ldots, [\frac{1}{2}n]\}$. A bridge then has a *basic domain* with foot $]\infty, 0] \times 1$ in case $k = 0$ or $[k, k+1] \times 1$ in case $k > 0$. A domain is called *even* (*odd*) if the distance to the basic domain is even (odd). This depends only on $k \bmod 2$; but later k itself will play a role.

Now the basic notion of our calculus! An *(n, n)-bridge with pillars and base number k* consists of an (n, n)-bridge together with a (possibly empty) subset of its even domains. We specify graphically the chosen domains by placing a point (a *pillar*) into the domain. For typographical reasons we use the bracket notation introduced above. We add a $\bullet$ in order to specify a pillar domain. The following figure explains the usage. It displays a $(5, 5)$-bridge with even base number and two pillar domains; also the bracket notation for its upper and lower bridge is given.



$$= \frac{((\bullet))\bullet|}{()|()} .$$

Observe that one may have a choice where to put certain $\bullet$ in the bracket notation, in this case we could also use $\frac{((\bullet))|}{()\bullet|()}$. Let $E^*(n, k)$ denote the set of (n, n)-bridges with pillars and base number k. Later we have to use a certain subset: a pillar bridge is called *reduced* if the distance of any two pillar domains is at least 4. Let $E(n, k)$ denote the set of reduced bridges in $E^*(n, k)$. We note that the distance between any two pillar domains is even.

We recall the graphical definition of the Temperley–Lieb algebra $T_n = T_d A_{n-1}$ associated to the Coxeter graph A_{n-1} (see ([5, p. 100]), since we have to use it in the definition of our algebras. Let $\mathcal{K}$ be a commutative ring and $d \in \mathcal{K}$ a parameter. Additively, T_n is the free $\mathcal{K}$-module on the set of (n, n)-bridges. The multiplication is $\mathcal{K}$-bilinear. Thus it remains to define the product of two bridges S and T. Let $T \circ S$ denote the figure which is obtained by placing the T-strip above the S-strip and squeezing the result affinely into $\mathbb{R} \times [0, 1]$. In general, the figure $T \circ S$ is not a bridge, there may be circles in the interior of the strip. Suppose there exist $k(T, S)$ circles. Let $T \wedge S$ denote the bridge obtained by removing the circles. Then the product is defined by

$$T \cdot S = d^{k(T,S)} T \wedge S. \tag{2}$$

We now use pillar bridges to define other algebras in an analogous manner. The next section collects a few elementary examples which illustrate the following definitions.

2.1. *The algebra $TE^*(n, k)$*

It is additively the free $\mathcal{K}$-module on $E^*(n, k)$. The multiplication is again $\mathcal{K}$-bilinear. Its definition uses two further parameters $c, f \in \mathcal{K}$. Thus $TE^*(n, k)$ depends on c, d, f. In order to define the product of two pillar bridges we consider the underlying ordinary bridges and form $T \circ S$ as before. Then we look at the position of the pillars. The deviation from a pillar bridge can have several reasons. We correct the deviation by the Processes 2.1 and 2.2 below.

Process 2.1 (*Multiple pillars*). Suppose a domain of $T \circ S$ contains $a > 1$ pillars. We remove $a - 1$ pillars from the domain and multiply the result by c^{a-1}. Let $T \circ_{\mathrm{l}} S$ denote the resulting figure with pillars. Suppose $a(T, S)$ pillars are removed altogether by this process.

Process 2.2 (*Pillar circles*). Consider the circles of $T \circ_{\mathrm{l}} S$. If such a circle can be connected with a pillar in its interior without crossing of other circles (*pillar circle*), we remove circle and pillar and multiply the result by f. The other circles are treated with the parameter d as above. Suppose altogether there exist $b(T, S)$ pillar circles and $k(T, S)$ ordinary ones.

The above two processes yield a bridge $T \cdot S$ with pillars and base number k. The underlying ordinary bridge is $T \wedge S$. The product of T and S is defined by

$$c^{a(T,S)} f^{b(T,S)} d^{k(T,S)} T \cdot S.$$

The reader should draw a figure and verify the following product

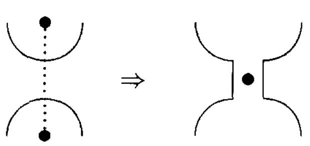
$$\frac{()\bullet|(())()}{|(\bullet())(\bullet())} \cdot \frac{()()|(\bullet())}{((\bullet))\bullet()|()} = c^3 f d^2 \frac{()\bullet|(())()}{((\bullet))()|()}.$$

In this paper, the algebras $TE^*(n, k)$ are technical tools. But they have some independent interest. For instance, $TE^*(n, 1)$ is related to the Temperley–Lieb algebra associated to the root system B_n, see [1].

2.2. *The algebra $TE(n, k)$*

It is additively the free $\mathcal{K}$-module on $E(n, k)$. In order to define the multiplication, we begin as for $E^*(n, k)$. But there is now another reason why the result may be wrong: if two pillar domains have distance 2. We correct by the next process.

Process 2.3 (*Reduction*). If two pillar domains have distance 2, we modify as in the following figure ($\bullet$ = pillar):

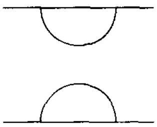
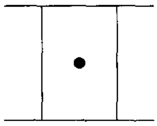


Thus, the two pillar domains are connected by a corridor, and there results a single new pillar domain. One has to show that reduction is well defined. Before doing this, we have to restrict the possible parameters, namely we set in $TE(n, k)$

$$c = d, \qquad f = 1.$$

The following example demonstrates why this is necessary. It also shows what reduction has to do with the defining relations (1) of a Temperley–Lieb algebra.

Example 2.4. We use the notation

 $= e_1,$  $= e_0.$

Then by Process 2.3 $e_0e_1e_0 = e_0$ and by Process 2.2 $e_1e_0e_1 = fe_1$. We use this to compute $(e_0e_1e_0)e_1 = e_0(e_1e_0e_1)$ in two ways and see that $f = 1$ is necessary. The requirement $c = d$ is unimportant and can always be achieved by a suitable parameter transformation (in an extension ring).

Reduction can be defined globally as follows. It is easy to see that the domains of a bridge are 2-cells. Let D be a pillarless domain of a bridge which has at least 2 adjacent pillar domains $P_1, \dots, P_k$. The intersection of the closures $\overline{D} \cap \overline{P}_j$ is an interval I_j. Choose a point $x \in D$ and connect it by an arc w_j in D to I_j. For $i \neq j$ the arcs w_i and w_j should only intersect in x, so that $w = \bigcup w_j$ is a star-like contractible complex. Let W be a closed regular neighbourhood of w in $\overline{D}$ such that $\overline{P}_j \cap W \subset I_j$. Then the interior of $W \cup \overline{P}_1 \cup \dots \cup \overline{P}_k$ is a new pillar domain. We apply this process successively to all domains of type D as above. The result is the reduced pillar bridge. The same result is obtained by a succession of moves of Process 2.3 in any order.

The reader should draw a figure and follow the prescription above in the following example. The (12,12)-bridge

$$\frac{\bullet|(\bullet())|\bullet((\bullet)(\bullet))}{|(\bullet)|((\bullet((\bullet))))}$$

has 4 domains $D_1,\ D_2,\ D_3,\ D_4$ of the type D just considered, adjacent to 4, 3, 2, 2 pillar domains, respectively. The reduction process yields

$$\frac{\bullet()()()()()()}{()()()()()()}.$$

2.3. *The algebra $TE_{n,k}$*

It is here where the actual value of k matters. We consider the subalgebra $TE_{n,k}$ of $TE(n,k)$ which is generated by ordinary bridges (i.e., bridges without pillars) and a single further bridge e_0 with only vertical arcs and with a single pillar between the kth and $(k+1)$st arc. Additively, $TE_{n,k}$ is the free $\mathcal{K}$-module with basis a certain subset $E_{n,k}$ of $E(n,k)$. This definition is taylored to give the following result. The algebra $T_dE_n(k)$ has been defined in the introduction.

Theorem 2.5. *There is a canonical surjective homomorphism of algebras*

$$\gamma_{n,k} : T_d E_n(k) \to TE_{n,k}.$$

Proof. We have to specify the image of the generators e_j. They will be denoted with the same symbol. For $j \geqslant 1$ we use the ordinary bridge with $HE = 1$, which connects j and $j+1$ on top and on the bottom. The image of e_0 is the vertical bridge with a pillar between the kth and $(k+1)$st string. (Example 2.4 displays the case $n = 2$, $k = 1$.) One has to verify that $\gamma_{n,k}$ respects the relations (1). This is easy. As far as e_0 is involved, Processes 2.2 and 2.3 are relevant. □

We show in Section 5 that $\gamma_{6,3}$ is an isomorphism. It is conjectured that $\gamma_{n,k}$ is always an isomorphism. It is known that $\gamma_{n,1}$ and $\gamma_{n,2}$ are isomorphisms. For the latter see [3].

2.4. *The algebra $TE^*_{n,k}$*

For completeness we define $TE^*_{n,k}$ to be the subalgebra of $TE^*(n,k)$ generated by ordinary bridges and the bridge e_0 with only vertical arcs and a pillar at position k. Additively, it is the free $\mathcal{K}$-module with basis a certain subset $E^*_{n,k}$ of $E^*(n,k)$.

3. Examples

We present some examples of (n,n)-bridges with basic number k for small values of n and k.

Example 3.1. $n = 2$, $k = 1$. The pillar bridges are:

$$\begin{array}{cccccc} \frac{||}{||} & \frac{|\bullet|}{||} & \frac{()}{()} & \frac{(\bullet)}{()} & \frac{()}{(\bullet)} & \frac{(\bullet)}{(\bullet)} \\ 1 & e_0 & e_1 & e_0e_1 & e_1e_0 & e_0e_1e_0 \end{array}.$$

The right most element is not reduced. The algebra $TE^*(2,1)$ is generated by e_0 and e_1 with relations

$$e_0^2 = ce_0, \qquad e_1^2 = de_1, \qquad e_1e_0e_1 = fe_1.$$

This is an algebra of type TB_2; it was studied in [1–3]. In the reduced case $TE(2,1)$ we have the same generators but the relations are now

$$e_0^2 = de_0, \qquad e_1^2 = de_1, \qquad e_1e_0e_1 = e_1, \qquad e_0e_1e_0 = e_0.$$

This is the algebra T_dA_2. In this case we have equalities $TE^*(2,1) = TE^*_{2,1}$ and $TE(2,1) = TE_{2,1}$.

Example 3.2. $n = 2$, $k = 0$. The pillar bridges are:

$$\begin{array}{cccccc} \frac{||}{||} & \frac{\bullet||}{||} & \frac{||\bullet}{||} & \frac{()}{()} & \frac{\bullet()}{()} & \frac{\bullet||\bullet}{||} \\ 1 & e_0 & f_0 & e_1 & e_0e_1 & e_0f_0 \end{array}.$$

They form a basis of $TE^*(2,0)$. The algebra

$$TE(2,0) = TE_{2,0}$$

has the basis $1, e_0, e_1, e_0e_1 = e_1e_0$ and is isomorphic to $T_dA_1 \otimes T_dA_1$.

Example 3.3. $n = 3$, $k = 1$. The algebra $TE_{3,1}$ has the following basis.

$$\begin{array}{ccccccc}
\frac{|||}{|||} & \frac{()|}{()|} & \frac{|()}{|()} & \frac{()|}{|()} & \frac{|()}{()|} & \frac{|\bullet||}{|||} & \frac{(\bullet)|}{()|} \\
1 & e_1 & e_2 & e_1e_2 & e_2e_1 & e_0 & e_0e_1 \\[1em]
\frac{()|}{(\bullet)|} & \frac{(\bullet)|}{|()} & \frac{()|}{|\bullet()} & \frac{|\bullet()}{()|} & \frac{|()}{(\bullet)|} & \frac{|\bullet()}{|()} & \frac{()|\bullet}{()|} \\
e_1e_0 & e_0e_1e_2 & e_1e_2e_0 & e_0e_2e_1 & e_2e_1e_0 & e_0e_2 = e_2e_0 & e_1e_0e_2e_1
\end{array}$$

This are 14 elements. The generators e_0, e_1, e_2 satisfy the relations of T_dA_3, hence $TA_3 \cong TE_{3,1}$.

The unreduced algebra $TE^*(3,1)$ has the additional elements

$$\begin{array}{ccccccc}
\frac{|\bullet||\bullet}{|||} & \frac{(\bullet)|}{(\bullet)|} & \frac{|\bullet()}{(\bullet)|} & \frac{(\bullet)|}{|\bullet()} & \frac{(\bullet)|\bullet}{()|} & \frac{()|\bullet}{(\bullet)|} & \frac{(\bullet)|\bullet}{(\bullet)|}. \\
 & e_0e_1e_0 & e_0e_2e_1e_0 & e_0e_1e_2e_0 & e_0e_1e_0e_2e_1 & e_1e_0e_2e_1e_0 & e_0e_1e_0e_2e_1e_0
\end{array}$$

The first element only appears in $TE^*(3,1)$. Hence there are six further elements in $TE^*_{3,1}$. The generators e_0, e_1, e_2 satisfy the relations

$$e_0^2 = ce_0, \qquad e_1^2 = de_1, \qquad e^2 = de_2, \qquad e_1e_0e_1 = fe_1, \qquad e_0e_2 = e_2e_0.$$

This are relations for an algebra TB_3 which was studied in [3].

Example 3.4. $n = 4$, $k = 2$. We display the reduced elements in the following manner. There are 14 pillarless $(4,4)$-bridges. For each bridge we draw the possible pillars, but assemble all possibilities in a single figure. This yields all bridges with one pillar. Under the figure we write the number of the corresponding basis elements.

$$\begin{array}{ccccccc}
\frac{||\bullet||}{||||} & \frac{()\bullet||\bullet}{()||} & \frac{\bullet|(\bullet)|\bullet}{|(\bullet)|} & \frac{\bullet||\bullet()}{||()} & \frac{()\bullet||\bullet}{|(\bullet)|} & \frac{\bullet|(\bullet)|\bullet}{()||} & \frac{\bullet|(\bullet)|\bullet}{||()} \\
2 & 3 & 5 & 3 & 4 & 4 & 4 \\
\frac{\bullet||\bullet()}{|(\bullet)|} & \frac{()\bullet||\bullet}{||()} & \frac{\bullet||\bullet()}{()||} & \frac{()\bullet()}{()()} & \frac{((\bullet))}{()\bullet()} & \frac{()\bullet()}{((\bullet))} & \frac{((\bullet))\bullet}{((\bullet))} \\
4 & 3 & 3 & 2 & 3 & 3 & 4+1
\end{array}$$

There exists a single reduced element with 2 pillars. This accounts for the $+1$ in the second row. Altogether we obtain 48 elements.

The generators

$$\frac{\|\bullet\|}{\|\|} = e_0, \qquad \frac{()\|}{()\|} = e_1, \qquad \frac{|()|}{|()|} = e_2, \qquad \frac{\|()}{\|()} = e_3$$

satisfy the relations of the Temperley–Lieb algebra T_dD_4. The algebra T_dD_4 has rank

$$\operatorname{rank} TA_3 + \tfrac{1}{2}\operatorname{rank} TB_4 - 1 = 14 + 35 - 1 = 48,$$

as was shown in [3]. One checks that the displayed bridges are contained in $TE_{4,2}$. Hence $TE_{4,2} \cong T_dD_4$.

4. The algebra $TE_{6,3}$

As an application of the calculus we study the algebra $TE_{6,3}$ and its module structure. Unfortunately, a lot of case by case checking is involved. It is also assumed that the reader knows how to deal graphically with the ordinary Temperley–Lieb algebra and its module theory. This will not be reviewed here. The algebra $TE_{6,3}$ is generated by the ordinary $(6,6)$-bridges and the vertical pillar bridge $|||\bullet||| = e_0$. We use the following tools.

Property 4.1 (*Invariants of bridges*).

(1) The *horizontal edge number HE*. In our case it is contained in $\{0, 1, 2, 3\}$.

(2) The number $VP(b)$ of *vertical pillar domains* of a bridge b. Recall that a domain is called vertical if it has boundary points in both $\mathbb{R} \times 0$ and $\mathbb{R} \times 1$. The element e_0 is vertical.

(3) The *filtration* of a bridge b is defined to be $F(b) = HE(b) + VP(b)$. We see in a moment that $0 \leqslant F(b) \leqslant 3$.

(4) The *separation numbers* SNT, SNB, SNL, SNR of a domain. We shall see that $0 \leqslant SNB + SNT \leqslant 3$ and $SNL,\ SNR \in \{1, 3\}$ holds.

We state properties of these invariants.

Property 4.2. The separation numbers mod 2 are topological invariants. Under multiplication SNR and SNL cannot increase. Since e_0 has left and right separation number 3, we see that $SNL, SNR \in \{1, 3\}$.

Property 4.3. For each domain of a bridge either $SNT = 0$ or $SNB = 0$. A separation of a domain from $\mathbb{R} \times 0$ is achieved by an upper horizontal arc. Therefore SNT equals at most the maximal possible HE. In our case SNT can assume the values 0, 1, 2, 3.

Property 4.4. The HE cannot decrease under multiplication. If a vertical pillar domain exists, then it is separated from left and right infinity by one or three vertical arcs. Hence there exist 6, 4, 2, or 0 vertical arcs. These cases lead to the distribution of pillars as displayed in Table 1. The reduction condition says that further pillars are impossible.

Table 1

Pillar	HE	F
$\vert\vert\vert\bullet\vert\vert\vert$	0	1
$\vert\bullet\vert\vert\vert$	1	2
$\vert\vert\vert\bullet\vert$	1	2
$\vert\bullet\vert$	2	3

Table 2

$2\frac{(\bullet)}{}$ 45	$2\frac{(\bullet)\vert\vert}{\vert\vert}$ 45	$2\frac{(\bullet)\vert\vert\vert\vert}{\vert\vert\vert\vert}$ 5	190
$\frac{\vert\bullet\vert}{\vert\vert}$ 81	$\frac{\vert\bullet\vert\vert\vert}{\vert\vert\vert\vert}$ 25	$\frac{\vert\bullet\vert\vert\vert\vert}{\vert\vert\vert\vert\vert}$ 1	107
$2\frac{(((\bullet)))}{}$ 5	$2\frac{\vert\vert(\bullet)}{\vert\vert}$ 45	$2\frac{\vert\vert(\bullet)\vert\vert}{\vert\vert\ \vert\vert}$ 5	110
$2\frac{\vert((\bullet))}{\vert}$ 9	$\frac{\vert\vert\vert\bullet\vert}{\vert\vert\vert\vert}$ 25	$\frac{\vert\vert\vert\bullet\vert\vert\vert}{\vert\vert\vert\ \vert\vert\vert}$ 1	44
$2\frac{(((\bullet)))}{(\bullet)}$ 9	$2\frac{\vert((\bullet))\vert}{(\bullet)\vert\vert}$ 5	$2\frac{\vert\vert(\bullet)}{(\bullet)\vert\vert}$ 25	68
$\frac{(((\bullet)))}{(((\bullet)))}$ 1	$2\frac{\vert\vert(\bullet)}{\vert((\bullet))\vert}$ 5	$\frac{\vert((\bullet))\vert}{\vert((\bullet))\vert}$ 1	12
$2\frac{(\bullet)\vert\vert(\bullet)}{\vert\vert}$ 9	$2\frac{(\bullet)\vert\vert\vert\bullet\vert}{\vert\vert\vert\vert}$ 5	$2\frac{\vert\vert(\bullet)\vert\vert}{(\bullet)\vert\vert}$ 1	30

In Table 2 we display in bracket notation the possible configurations of pillar domains and vertical arcs. The configurations have to be filled up with ordinary horizontal arcs. Under each symbol we have specified the number of fillings. Each such number is the product of the upper and lower fillings. They are obtained from well known combinatorics of bridges. Certain cases in the table appear twice, namely if interchange of top and bottom yields different configurations. This accounts for the 2 in front of a symbol. As an example, consider the left most entry in the first row. There are 5 6-bridges with

3 horizontal arcs; they are the possible "denominators". There are 9 possibilities to fill these 5 bridges with a pillar such that $SNL = 1$; they are the possible "numerators". The next entry in this line has 9 denominators and 5 numerators. The last one has 5 denominators. The right most column in the table gives the total number of cases in the corresponding row. We have to add the 132 ordinary bridges without pillars. Altogether we obtain

$$132 + 190 + 107 + 110 + 44 + 68 + 12 + 30 = 703.$$

But not all cases are possible in $TE_6(3)$! For one, the right separation numbers have to be at most 3. This eliminates the right most cases in the first two rows, hence 11 cases.

Moreover, the cases in the last row are impossible. This has the following reason. We have to investigate which figures are generated by ordinary bridges and e_0. The cases in question have to use e_0 at least twice. (Recall: left multiplication is placed graphically on top of the figure.) From a lower pillar domain we can obtain at most a vertical one by multiplication. In the first case (left in the last row of the table) we have, in the course of the multiplication with generators, to multiply a bridge with an upper pillar domain with e_0. But then the reduction condition is not satisfied, and therefore a second pillar domain does not occur.

Therefore a second pillar domain can only appear, if the upper foot

has distance at least 4 from the already existing pillar domain. This can only be a lower pillar domain. In cases 2 and 3 of the last row the HE can be at most 1. Hence a fourfold separation is impossible. What we have seen until now is

Property 4.5. The algebra $TE_{6,3}$ has rank at most 662.

In order to show that the rank is exactly 662 one could try to check that all remaining figures are actually possible. This is a matter of patience, if one does not use further structural investigations. But it turns out that the module and ideal structure of $TE_{6,3}$ gives better insight. We will show in the remaining part of this section:

Theorem 4.6. *The algebra $TE_{6,3}$ is semisimple for generic parameters d in a field $\mathcal{K}$ and has in that case simple modules of rang* 1, 6, 20, *and* 15.

We point out $662 = 1^2 + 6^2 + 20^2 + 15^2$. Note that the ranks are binomial coefficients $\binom{6}{n}$. In the nongeneric case we too have modules of the specified rank, since the modules are constructed with help of the graphical calculus. We assume from now on that the parameter $d \in \mathcal{K}$ is invertible. We also assume known a geometric treatment of the module theory for the ordinary Temperley–Lieb algebra $T_d A_5$.

Again we use bracket notation for the upper pillar bridges. A bullet indicates the pillar domain. The standard bridges with horizontal edge number 0, 1, 2, 3 are denoted

Table 3

((()))	()(())	(())()	()()()	(()())
((($\bullet$)))	()($\bullet$())	(()$\bullet$)()	()($\bullet$)()	(() $\bullet$ ())
($\bullet$(()))	($\bullet$)(())	(())($\bullet$)	($\bullet$)()()	()()($\bullet$)

β_0, β_1, β_2, β_3; the upper and lower bridges of these pillarless bridges are, by definition, $|||\,|||$, $||()||$, $|(())|$, $((()))$. The simple modules will be the left ideals generated by the β_j modulo bridges of higher filtration. This uses:

Lemma 4.7. *Multiplication cannot decrease the filtration.*

Proof. A vertical pillar domain can only be removed by multiplication with an ordinary bridge if its vertical boundary points become connected. But then the HE increases.

If multiplication by e_0 decreases the HE, then this happens through a reduction process which produces a vertical pillar domain. □

Theorem 4.8. *The element β_3 generates a left ideal $M(3)$ of rank* 15 *and a two-sided ideal $L(3)$ of rank* 15^2. *The ideal $L(3)$ is the direct sum of* 15 *left ideals isomorphic to $M(3)$. The upper bridges of a basis of $M(3)$ are displayed in Table* 3.

Proof. The first row contains the upper bridges of the pillarless elements of $M(3)$. The second row contains the elements which are obtained from the first row by left multiplication with e_0. The third row contains the elements

$$e_3z, \quad e_4e_3x, \quad e_2e_3y, \quad e_3x, \quad e_3y,$$

where x, y and z are the second, third, and fifth element of the second row. Further elements are impossible; this is seen by considering reduction and separation number.

By Lemma 4.7, the bridges with maximal filtration 3 generate a two-sided ideal which contains $M(3)$.

We reflect the basis bridges of $M(3)$ in $\mathbb{R} \times \frac{1}{2}$ (interchange of top and bottom). The left ideals generated by these reflected elements are isomorphic to $M(3)$. An isomorphism is obtained by right multiplication with a suitable element: if b is a basis element, then there exists a bridge c such that $bc = \lambda\beta_3$ with invertible $\lambda \in \mathcal{K}$; moreover $\beta_3 b = \mu\beta_3$ with invertible $\mu \in \mathcal{K}$. The sum of these left ideals contains $L(3)$. The sum is direct, since the basis sets of the ideals are pairwise disjoint. The element β_3 generates $L(3)$, since it generates the above left ideals. □

Theorem 4.9. *Let $M(2)$ denote the left ideal generated by β_2 modulo the submodule generated by bridges of higher filtration. The module $M(2)$ has rank* 20 *and is generated by bridges with lower bridge $|(())|$ and filtration* 2. *The two-sided ideal generated by β_2 modulo $L(3)$ is an algebra of rank* 20^2 *and is the direct sum of* 20 *ideals isomorphic to $M(2)$.*

Table 4

$\|((\bullet))\|$	$\|()()\|$	$\|()\|(\bullet)$	$(\bullet)\|()\|$	$(\bullet)\|\|(\bullet)$
$\|\|(\bullet)(\bullet)$	$(\bullet)(\bullet)\|\|$	$(\bullet())\|\|$	$\|\|(()\bullet)$	

Proof. By definition, the module $M(2)$ is generated by the bridges as stated in the theorem. One has to check that there 20 of them. The possible pillar bridges without vertical pillar domain are in condensed notation displayed in Table 4.

Condensed notation means: all possible occurences of pillars are assembled in the same figure. It remains to verify that they are generated by the 9 ordinary bridges among them and e_0. Left multiplica tion of ordinary bridges with e_0 produces

$$|((\bullet))|,\ ||(\bullet)(),\ ()(\bullet)||,\ (()\bullet)||,\ ||(\bullet()).$$

Finally, we multiply the last two with e_3, e_4 and e_5.

We now finish the proof as for Theorem 4.8. □

Finally, we use a similar procedure for β_0 and β_1. The first one yields the trivial module $M(0)$, the second one the 6-dimensional module $M(1)$. We mention in passing that $M(1)$ is obtainable from the 6-dimensional reflection representation of the Hecke algebra (see [3]).

This finishes the discussion of the module theory. We do not discuss in detail conditions for semisimplicity. By Theorem 2.5 and the homomorphism φ of the Introduction, $TE_{n,k}$ is a quotient of the Hecke algebra $H_qE_n(k)$, provided $q = p^2$, $d = p + p^{-1}$. Thus $TE_{n,k}$ is certainly semisimple when this holds for the Hecke algebra. The simplicity of the modules $M(j)$ follows from the next theorem if one assumes known the theory of T_dA_n-modules. The argument for $M(3)$ is as follows (notation as in Theorem 4.10). Suppose $M(3)$ has a decomposition $A \oplus B$ into submodules. Then $\operatorname{res} A$ or $\operatorname{res} B$ contains the uniquely determined submodule M_3, say $\beta_3 \in A$. But β_3 generates $M(3)$. The other modules are handled similarly. □

We have the subalgebra $T_dA_5 \subset TE_{6,3}$ generated by the ordinary bridges. From the structure theory of Temperley–Lieb algebras we know (see, e.g., [4, 2.8]): the algebra T_dA_5 has in the generic case simple modules M_0, M_1, M_2, and M_3 of rank 1, 5, 9, and 5, respectively. We denote the restriction of a $TE_{6,3}$-module M to TA_5 simply by $\operatorname{res} M$.

Theorem 4.10. *The following isomorphisms hold:*

$$\operatorname{res} M(0) \cong M_0,$$

$$\operatorname{res} M(1) \cong M_1 \oplus M_0,$$

$$\operatorname{res} M(2) \cong M_2 \oplus 2M_1 \oplus M_0,$$

$$\operatorname{res} M(3) \cong M_3 \oplus M_2 \oplus M_0.$$

Table 5

$\|\|(\bullet)()$	$\|\|()(\bullet)$	$\|\|(()\bullet)$	$()\|\|(\bullet)$	$\|()\|(\bullet)$
$()(\bullet)\|\|$	$(\bullet)()\|\|$	$(\bullet())\|\|$	$(\bullet)\|\|()$	$(\bullet)\|()\|$
$\|((\bullet))\|$				

Proof. The case $M(0)$ is trivial. We start with $M(3)$ and consider the bridges of filtration 3 described above. The ordinary bridges yield a submodule M_3 of rank 5. We study $M(3)/M_3$ and consider in it the element

$$z = \frac{()(\bullet)()}{()()()}.$$

If we multiply this element from the right by e_0 we obtain β_2. Multiplication from the left with ordinary bridges yields all ordinary upper $(2,2)$-bridges, and there are 9. Therefore z generates in $M(3)/M_3$ a 9-dimensional module isomorphic to M_2. Its quotient module has a basis $(((\bullet)))$.

We treat $M(2)$ in a similar manner. We have the submodule M_2 of ordinary bridges. The remaining 11 basis elements are displayed in Table 5.

We claim that the rows yield modules M_1, M_1, M_0, respectively. If we use the lower bridge $||()()$ in the first row and multiply with e_0 from the right we obtain the upper bridges with $HE = 1$. Similarly for the second row by using the lower bridge $()()||$.

The case $M(1)$ finally is clear, since we have the submodule of ordinary bridges M_1. □

5. Comparison with the algebra $T_dE_6(3)$

The algebra $T_dE_6(3)$ is the Temperley–Lieb algebra associated to the Coxeter graph $E_6(3)$ defined by generators and relations, see the introduction. By construction, we have a surjective homomorphism $T_dE_6(3) \to TE_{6,3}$, see Theorem 2.5. We already know the rank of $TE_{6,3}$. In order to show that the homomorphism is injective we derive a normal form for words in $e_0, \dots, e_5$ and verify that there are 662 normal forms. We have different cases according to the appearances of e_0 in the words.

Case 5.1. We have the subalgebra T_dA_5 of rank 132 generated by $e_1, \dots, e_5$.

Case 5.2. We determine the normal form of monomials of the form $\alpha e_0 \beta$ with $\alpha, \beta \in T_dA_5$. Monomial means: an arbitrary word in the symbols e_j, possibly with a further coefficient from $\mathcal{K}$. We only consider monomials which cannot be shortened by the defining relations and which have α of minimal length in its equivalence class modulo relations. The minimality of α means:

(1) $\alpha = 1$ or

(2) α contains e_3 and finishes on the right with e_3. Left to α there are no generators commuting with e_3, i.e., no e_1 and e_5.

Table 6

<table>
<tr><th>Upper bridge</th><th>Word</th><th>Number of β</th></tr>
<tr><td>||()||</td><td>3</td><td>90</td></tr>
<tr><td>|||()||</td><td>43</td><td>48</td></tr>
<tr><td>|()|||</td><td>23</td><td>48</td></tr>
<tr><td>||||()</td><td>543</td><td>20</td></tr>
<tr><td>()||||</td><td>123</td><td>20</td></tr>
<tr><td>|()()|</td><td>243</td><td>20</td></tr>
<tr><td>|()|()</td><td>2543</td><td>6</td></tr>
<tr><td>()|()|</td><td>1243</td><td>6</td></tr>
<tr><td>|(())|</td><td>3243</td><td>20</td></tr>
<tr><td>||()()</td><td>32543</td><td>6</td></tr>
<tr><td>()()||</td><td>13243</td><td>6</td></tr>
<tr><td>()||()</td><td>12543</td><td>1</td></tr>
<tr><td>||(())</td><td>432543</td><td>6</td></tr>
<tr><td>(())||</td><td>213243</td><td>6</td></tr>
<tr><td>()()()</td><td>132543</td><td>1</td></tr>
<tr><td>()(())</td><td>1432543</td><td>1</td></tr>
<tr><td>(())()</td><td>2132543</td><td>1</td></tr>
<tr><td>(()())</td><td>21432543</td><td>1</td></tr>
<tr><td>((()))</td><td>321432543</td><td>1</td></tr>
</table>

We use the fact that T_dA_5 can be described by bridges. Thus, we talk about bridges instead of words or monomials. Bridges with α satisfying these conditions have lower bridges $||()||$, $|(())|$, $((()))$. The number of α's with these lower bridges is 5, 9, 5, respectively. They are displayed in Table 6. The second column in this table gives the word in the generators, with 243 as shorthand for $e_2e_4e_3$. The third column gives the number of possible β's. We explain this in a moment. Elements α which are related by the reflection $e_j \leftrightarrow e_{6-j}$ yield the same number. We now go through the cases of Table 6 and derive an upper bound for the possible β's. In the following discussion we eliminate those β's which obviously lead to words which can be shortened.

Case $\alpha = e_3$. The β which begin with e_3 lead to words which can be shortened by the relation $e_3e_0e_3 = e_3$. The β's which begin with e_3 have an upper bridge which contains $||()||$. These are in bijection with the 42 upper 10-bridges. There remain $132 - 42 = 90$ cases.

Case $\alpha = e_4e_3$. No beginning with $||()||$ or $|||()|$. These are disjoint cases. Hence there remain $132 - 2 \cdot 42 = 48$.

Case $\alpha = e_5e_4e_3$. No beginning with $||()||$, $|||()|$ or $||||()$. The first and third case have as intersection the bridges which begin with $||()()$. These are 14 in number. There remain $132 - 3 \cdot 42 + 14 = 20$.

Case $\alpha = e_2e_5e_4e_3$. No beginning with $|()|||$, $||()||$, $|||()|$ or $||||()$. The first and third case have intersection $|()()|$; the second and fourth case have intersection $||()()$; the first and fourth case have intersection $|()()|()$. There are no further intersections. There remain $132 - 4 \cdot 42 + 3 \cdot 14 = 6$.

The remaining α's contain all symbols $e_1, \ldots, e_5$ and therefore β has to be 1. The sum of all these cases is 440.

Case 5.3. Monomials which contain two e_0: $\alpha e_0 \beta e_0 \gamma$. In order that $\beta e_0 \gamma$ be minimal with minimal β, the element β has, by Case 5.2, to terminate with $\beta = ||()||$, $|(())|$, or $((()))$. Hence β has to be one of the bridges β_2 and β_3 of Section 2. In the second case $\alpha = \gamma = 1$.

In the first case therefore, by Case 5.2, either $\alpha = 1$ and 20 cases f or γ, or $\gamma = 1$ and 20 cases for α, minus the intersection case $\alpha = \gamma = 1$. Altogether we obtain 40 cases.

If $\alpha \neq 1$ and $\gamma \neq 1$, then γ has to begin with e_1, e_5 and α has to end with e_1, e_5; and these cases have to be different. Hence we obtain altogether $2N^2$ possibilities, where N is the number of γ which begin with e_1 but not with e_2, e_3, e_4 and which have no presentation beginning with e_5. These are the five bridges which begin with $()|||$. Thus we obtain 50 further cases.

Altogether we now have the upper bound $132 + 440 + 40 + 50 = 662$ on normal forms of words. Since e_0 can occur at most twice, there are no further cases. We already have the surjection $T_dE_6(3) \to TE_{6,3}$ onto an algebra of rank 662. Thus we found normal forms of words.

6. Concluding remarks

As one might guess from the shape of the Coxeter graphs, the following isomorphisms hold $TE_{n,1} \cong TA_n$ and $TE_{n,2} \cong TD_n$. The structure of TD_n was determined in [3]. The calculus of the present paper adds a geometric interpretation to the algebra of [3]. In [1] we defined and studied the Temperley–Lieb algebra TB_n associated to the Coxeter graph B_n. With suitable parameters, the algebras $TE^*_{n,1}$ and TB_n are isomorphic. The algebra TB_n uses bridges which are symmetric with respect to a reflection in $0 \times \mathbb{R}$. One can also consider pillar bridges which have this symmetry. A certain algebra of this type is related to the graph F_4. Finally, one could use pillars of different type; this has applications to affine root systems. The usual closing procedure (braids to links) yields when applied to pillar bridges a semigeometric definition of Markov traces on the family $TE_{n,k}$ for fixed k.

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