Nonexpansive selections of metric projections in spaces of continuous functions

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Abstract

A subset A of a metric space X is said to be a nonexpansive proximinal retract (NPR) of X if the metric projection from X to A admits a nonexpansive selection. We study the structure of NPR’s in the space C(K) of continuous functions on a compact Hausdorff space K. The main results are a characterization of finite-codimensional and of finite-dimensional NPR subspaces of C(K) and a complete characterization of all NPR subsets of l∞.

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1. Introduction

A subset A of a metric space X is said to be proximinal if the metric projection of every point x ∈ X (i.e., the set PA(x) of points in A nearest to x) is nonempty. Proximinal sets, their structure and the existence of single-valued selections for the multi-valued metric projection have been the subject of a lot of research. Note that a continuous single-valued selection for the metric projection is a retraction of X onto A. Another family of retracts, the nonexpansive retracts (i.e., subsets A ⊂ X such that there is a nonexpansive retraction from X onto A), has also been the subject of intensive study.

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In this article, we combine these two properties and study sets $A \subset X$ for which there is a map from $X$ onto $A$ which is simultaneously a single-valued selection of the metric projection and a nonexpansive map. We call such sets nonexpansive proximinal retracts and the associated map will be called a nonexpansive proximinal retraction. (We shall abbreviate both as NPR.)

As the title suggests, our main interest in this article is when the containing space $X$ is a $C(K)$ space.

We use standard notation. In particular, we shall identify $C(K)^*$ with the space of regular Borel measures on $K$. We only consider real Banach spaces, although many of the results extend to the complex case.

We shall use without further explanation some basic properties of nonexpansive (not necessarily proximinal) retracts $A \subset X$. It is clear that such a set $A$ is closed. If $X$ is a convex subset of a normed space, then $A$ is metrically convex. Indeed, if $\varphi : X \to A$ is a nonexpansive retraction and if $x, y$ are two points in $A$, then the curve $\gamma(t) = \varphi((1-t)x + ty)$ (for $0 \leq t \leq 1$) connects $x$ and $y$ in $A$. By the nonexpansiveness of $\varphi$ and the triangle inequality this curve is a “metric segment”: $||\gamma(t) - \gamma(s)|| = |t - s| ||x - y||$.

In Section 2 we consider NPR subspaces of $C(K)$ spaces. We characterize their finite-codimensional and finite-dimensional NPR subspaces and formulate a conjecture on the characterization of a general NPR subspace of $C(K)$. The results are analogous to the results on linear selections for the metric projection, see for example [3,5], although the methods and proofs are, of course, different.

In Section 3 we consider the case of finite-dimensional $C(K)$ spaces, namely, the spaces $l_\infty^n$. For these spaces we give a complete characterization of NPR subsets (and not only subspaces as in Section 2): they are exactly the intersections of NPR half-spaces. In particular, it turns out that NPR subsets of $l_\infty^n$ are convex. We do not know if this is true in general $C(K)$ spaces, but we give an example showing that in general Banach spaces a NPR subset does not have to be convex.

In this section, we use the fact that $l_\infty^n$ is a hyperconvex space and apply the following theorem from [7]. For the sake of the reader, and since the article [7] uses a somewhat different terminology, we give the proof of the theorem, as well as basic information on hyperconvex spaces, in the Appendix.

**Theorem 1.1** (Espínola et al. [7]). A boundedly compact subset $A$ of a hyperconvex metric space $X$ is a NPR of $X$ if and only if $A$ is a NPR of $A \cup \{z\}$ for any $z \in X \setminus A$.

We finish the introduction with the comment that in many cases the existence of a nonexpansive retraction from a Banach space $X$ onto a closed subspace $E$ implies the existence of a norm-one linear projection on $E$. This is the case, for example when $E$ is reflexive, or is norm-one complemented in its second dual (see [4, Chapter 7]).

A simpler observation of this nature (explicitly stated in Aronszajn and Smith [2], but possibly even older), is that when $E$ is a proximinal one-codimensional subspace of $E$, then the metric projection admits a linear selection.

The existence of a linear norm-one projection gives some information on the geometry of $E$ that could be used to study its structure (although we shall not use such an approach in this article). But it should be noted that when $E$ is a NPR, then even if a norm-one linear projection $P$ does exist, $P$ is usually not proximinal. (A linear projection $P$ is a NPR iff it is bi-contractive, i.e., $||P|| = ||I - P|| = 1$.) Indeed, the one-dimensional subspace of $C(K)$ consisting of the constant functions is a NPR (take $S = K$ for a subspace of type II, see Section 2). Also by the Hahn–Banach theorem every one-dimensional subspace of a Banach space is the range of a norm-one
projection. But one checks easily that when \( K \) has at least three points, then this subspace is not the range of a linear bi-contractive projection.

2. NPR subspaces of spaces of continuous functions

We start by describing three types of canonical NPR subspaces of \( C(K) \):

Type I: Fix a clopen (closed and open) subset \( Z \subset K \) and put

\[
E_0^Z = \{ f \in C(K) : f|_Z \equiv 0 \}.
\]

A nonexpansive proximinal retraction onto \( E_0^Z \) is given by

\[
\varphi(f)(t) = \begin{cases} 
0 & \text{for } t \in Z, \\
 f(t) & \text{for } t \notin Z.
\end{cases}
\]

Type II: Fix a clopen subset \( S \subset K \) and put

\[
E_S = \{ f \in C(K) : f|_S \text{ is constant} \}.
\]

A nonexpansive proximinal retraction onto \( E_S \) is given by

\[
\varphi(f)(t) = \begin{cases} 
(\max_{s \in S} f(s) + \min_{s \in S} f(s))/2 & \text{for } t \in S, \\
 f(t) & \text{for } t \notin S.
\end{cases}
\]

Type III: Fix two disjoint clopen subsets \( S^1, S^2 \subset K \) and put

\[
E_{S^1,S^2} = \{ f \in C(K) : f|_{S^i} \text{ is constant and } f|_{S^1} = -f|_{S^2} \}.
\]

\( E_{S^1,S^2} \) is a NPR because the isometry \( T \) of \( C(K) \) onto itself given by

\[
Tf = \begin{cases} 
f & \text{on } S^1, \\
-f & \text{on } K \setminus S^1,
\end{cases}
\]

maps \( E_{S^1,S^2} \) onto the NPR subspace \( E_S \), where \( S = S^1 \cup S^2 \).

It is obvious that translates of these subspaces are also NPR’s. Also, when these subspaces are of codimension one (i.e., when the sets \( Z, S^1, S^2 \) reduce to single points and \( S \) to two points), then these retractions are actually linear. (This is true for \( E_0^Z \) without the restriction that it is one-codimensional.)

It should also be noted that a subspace of codimension one is a NPR iff the half-spaces it determines are NPR.

Using these canonical NPR subspaces, we now describe more NPR subspaces. Let \( Z, \{S_i\}_{i=1}^n \) and \( \{S^1_j, S^2_j\}_{j=1}^m \) be a finite family of mutually disjoint clopen sets and put

\[
E = \{ f \in C(K) : f|_Z = 0, \text{ and } f|_{S_i}, f|_{S^1_j} = -f|_{S^2_j} \text{ are constant} \}
\]

\[
= E_0^Z \cap \bigcap_{i=1}^n E_{S_i} \cap \bigcap_{j=1}^m E_{S^1_j, S^2_j}.
\]

Then one checks easily that \( E \) is also a NPR (with the natural formula for the retraction).

Note that \( E \) is finite-dimensional iff the union of the disjoint sets \( Z, S_i, S^1_j, S^2_j \) has a finite complement in \( K \).

The main results of this section are the following two theorems.
Theorem 2.1. Let $E$ be a finite-codimensional NPR subspace of $C(K)$, then it has the form (1).

Theorem 2.2. Let $E$ be a finite-dimensional NPR subspace of $C(K)$, then it has the form (1).

We do not know whether the dimension restrictions in these theorems are really necessary. We conjecture they are not:

Conjecture 2.3. Every NPR subspace of a $C(K)$ space is of the form (1).

Theorems 2.1 and 2.2 show, in particular, that when $K$ is connected, then $C(K)$ has no finite-codimensional or finite-dimensional NPR subspaces except for the one-dimensional subspace consisting of the constant functions (i.e., $E_K$). If Conjecture 2.3 is true, then this is actually the only NPR subspace it has.

Before passing to the proof of Theorem 2.1, we first need some preparations.

Lemma 2.4. Let $E$ be a NPR subspace of $C(K)$ of finite codimension. Then

(i) Every measure in the annihilator $E^\perp$ is purely atomic.

(ii) If $k \in K$ is an atom of some measure $\mu \in E^\perp$, then $k$ is isolated in $K$.

Proof. Let $\varphi : C(K) \to E$ be the NPR.

Let $\eta_1, \ldots, \eta_n$ be a basis for $E^\perp$ and put $\eta = |\eta_1| + \cdots + |\eta_n|$. Denote the (countable) set of atoms of $\eta$ by $A \subset K$ and note that $A$ contains all the atoms of any $\mu \in B(E^\perp)$. Also, for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ so that $\eta(A) < \delta$ implies that $|\mu|(A) < \varepsilon$ for every $\mu \in B(E^\perp)$.

We are now ready for the proofs.

(i) Fix $\tau \in E^\perp$ and $\varepsilon > 0$. By the regularity of $\tau$ there are two disjoint compact sets $K^+$ and $K^-$ contained in the supports of the positive and negative parts $\tau^\pm$ of $\tau$, respectively, with $|\tau|(K^+ \cup K^-) > \|\tau\| - \varepsilon$. Let $\nu$ be the restriction of $\tau$ to $K^+ \cup K^-$ and let $f$ be a continuous function with $-1 \leq f \leq 1$ such that $f_{K^+} \equiv 1$ and $f_{K^-} \equiv -1$. Thus $|f| \equiv 1$ a.e. $-d\nu$ and $f \, d\nu$ is a nonnegative measure. Clearly $\|\nu - \tau\| < \varepsilon$.

Note that $\|\varphi(f) - f\| = d(f, E) \leq \|f\| = 1$, and thus $\varphi(f)(k) \geq 0$ on $K^+$, where $f(k) = 1$. Similarly $\varphi(f)(k) \leq 0$ on $K^-$. It follows that $\varphi(f) \, d\nu$ is also a nonnegative measure. Also $\|\varphi(f)\| = \|\varphi(f) - \varphi(0)\| \leq \|f\| = 1$.

Fix any point $k \in K^+ \setminus A$. Since $k$ is not an atom of $\eta$ we can find, by the equi-integrability of $B(E^\perp)$, an open neighborhood $V$ of $k$ with $\overline{V} \subset \{f > 1 - \varepsilon\}$ so that $|\mu|(V) < \varepsilon$ for every $\mu \in B(E^\perp)$. Let $0 \leq g \leq 1$ be a continuous function supported in $V$ so that $g(k) = 1$. It follows that

$$\|f - 2g\| \leq 1 + \varepsilon. \quad (2)$$

We claim that $\|2g - \varphi(2g)\| \leq 2\varepsilon$. Indeed, by the choice of $V$ we obtain that $\int_V g \, d|\mu| < \varepsilon$ for every $\mu \in B(E^\perp)$. Identifying $(C(K)/E)^*$ with $E^\perp$ and using the definition of the norm in $C(K)/E$, it follows that there is an $h \in E$ with $\|g - h\| \leq \varepsilon$. Thus $\|2g - \varphi(2g)\| = d(2g, E) \leq 2g - 2h\| \leq 2\varepsilon$.

Combining this estimate with (2), it follows that

$$|\varphi(f)(k) - 2| = |\varphi(f)(k) - 2g(k)| \leq \|\varphi(f) - \varphi(2g)\| + \|\varphi(2g) - 2g\| \leq \|f - 2g\| + 2\varepsilon \leq 1 + 3\varepsilon.$$
Thus $\varphi(f)(k) \geq 1 - 3\varepsilon$. Using also $f(k) = 1$ and $\|\varphi(f)\| \leq 1$ give that $|f(k) - \varphi(f)(k)| \leq 3\varepsilon$. Similarly, $|f(k) - \varphi(f)(k)| \leq 3\varepsilon$ when $k \in K^+ \cup K^-$. Since $v$ is supported in $K^+ \cup K^-$ and $f dv$ is nonnegative, it follows that

$$\int_{K \setminus A} \varphi(f) \, dv \geq \int_{K \setminus A} f \, dv - 3\varepsilon = |v|(K \setminus A) - 3\varepsilon.$$ 

Using $\tau \in E^\perp$, $\|v - \tau\| < \varepsilon$ and $\int_A \varphi(f) \, dv \geq 0$ (because, $\varphi(f) \, dv$ is a nonnegative measure), it follows that

$$0 = \int \varphi(f) \, d\tau \geq \int \varphi(f) \, dv - \varepsilon = \int_A \varphi(f) \, dv + \int_{K \setminus A} \varphi(f) \, dv - \varepsilon \geq |v|(K \setminus A) - 4\varepsilon.$$

Thus $|\tau|(K \setminus A) \leq |v|(K \setminus A) + \varepsilon \leq 5\varepsilon$. Letting $\varepsilon \to 0$ gives that $|\tau|(K \setminus A) = 0$, i.e., that $\tau$ is purely atomic.

(ii) Denote the atoms of $\eta$ by $A = \{k_j\}$. As observed earlier, the atoms of any $\mu \in B(E^\perp)$ are contained in $A$.

Assume $v \in B(E^\perp)$ has an atom at a nonisolated point, say, at $k_1$. Normalize so that $\|v\| = 1$, put $v(k_j) = v_j$, and assume that $v_1 > 0$. Fix $\varepsilon > 0$.

Choose $N$ so that $\sum_{j > N} |\mu(k_j)| < \varepsilon$ for every $\mu \in B(E^\perp)$ and let $V$ be a neighborhood $k_1$, such that $k_j \not\in V$ for $2 \leq j \leq N$. Let $-1 \leq f \leq 1$ be a continuous function with $f \equiv 1$ in $V$ and $f(k_j) = \text{sign}(v_j)$ for $2 \leq j \leq N$. As in part (i) we obtain that $\|\varphi(f)\| \leq 1$ and that $\varphi(f)(k_j)v_j \geq 0$ for every $j \leq N$.

Since $k_1$ is not isolated, every neighborhood $U \subset V$ of $k_1$ contains a point $k_U \neq k_1, \ldots, k_N$. Choose a continuous $0 \leq g \leq 1$ supported in $U$ with $g(k_U) = 1$ and $g(k_1) = 0$. Thus $g(k_j) = 0$ for $j \leq N$ and $\|f - 2g\| = 1$. Since

$$\left| \int g \, d\mu \right| = \left| \sum_{j > N} g(k_j)\mu(k_j) \right| \leq \sum_{j > N} |\mu(k_j)| < \varepsilon$$

for every $\mu \in B(E^\perp)$, it follows, as in part (i), that $|\varphi(f)(k_U) - 2| \leq 1 + 3\varepsilon$ and consequently that $\varphi(f)(k_U) \geq 1 - 3\varepsilon$. But the neighborhood $U$ was arbitrary, hence also $\varphi(f)(k_1) \geq 1 - 3\varepsilon$. Thus

$$0 = \int \varphi(f) \, dv = \varphi(f)(k_1)v_1 + \sum_{2 \leq j \leq N} \varphi(f)(k_j)v_j + \sum_{j > N} \varphi(f)(k_j)v_j.$$

But $\varphi(f)(k_1)v_1 \geq (1 - 3\varepsilon)v_1 > 0$, the first sum is nonnegative and the second is bounded in absolute value by $\varepsilon$. This is impossible when $\varepsilon$ is so small that $(1 - 3\varepsilon)v_1 > \varepsilon$. □

**Proof of Theorem 2.1.** We first observe that it is enough to prove the theorem under the additional assumption that $E$ is not contained in any “canonical” hyperplane or, equivalently

(*) $E^\perp$ does not contain any measure of the form $\delta_k$ or $\delta_k \pm \delta_l$.

Of course, under (*) we need to show that actually $E = C(K)$.

The reduction to this special case is obtained as follows: assume that there is a point $z \in K$ with $f(z) = 0$ for all $f \in E$. By the lemma $z$ is isolated in $K$, hence $F = \{f \in C(K) : f(z) = 0\}$ is isometric to $C(K \setminus \{z\})$ and $E \subset F$. The restriction of $\varphi$ to $F$ is a NPR from $F$ onto $E$. Similarly,
if there are isolated points \( k \neq l \) in \( K \) so that \( f(k) = f(l) \) (resp.. \( f(k) = -f(l) \)) for all \( f \in E \),
then \( E \) is contained in \( F = \{ f \in C(K) : f(k) = f(l) \} \) (resp.. \( f(k) = -f(l) \)),
which is isometric to \( C(K \setminus \{l\}) \), and again the restriction of \( \varphi \) to \( F \) is a NPR from \( F \) onto \( E \).

Making these reductions at most \( n \) times (where \( n \) is the codimension of \( E \)), yields the required reduction.

Before passing to the proof, we make the useful observation that when we are given a measure \( \mu = \sum \mu_i \delta_{k_i} \in E^\perp \) and a finite set \( J \) of indices, then we may assume that \( \mu_i \geq 0 \) for all \( j \in J \).

Indeed, assume that \( \mu_i < 0 \) for some \( j \in J \). Since \( k_j \) is isolated, the operator \( T \) that changes the sign of a function \( f \) at the point \( k_j \) is an isometry of \( C(K) \) onto itself with \( T^{-1} = T \). We can thus replace \( E \) by \( TE \), the retraction \( \varphi \) by \( T \circ \varphi \circ T \), and the atom \( \mu_j \) of \( \mu \) at \( k_j \) by \(-\mu_j \).

Assume now for contradiction that \( E \) satisfies (*) and that its codimension is \( n \geq 1 \). By Lemma 2.4 every \( \mu \in E^\perp \) is purely atomic and there is a countable set of isolated points \( \{k_j\} \) containing all the atoms of elements in \( E^\perp \).

Find a basis \( \mu_1, \ldots, \mu_n \) for \( E^\perp \) which, after possibly renumbering of the \( k_j \)’s, has the form

\[
\mu_i = \delta_{k_i} + \sum_{j > n} \mu_i^j \delta_{k_j} \quad \text{for } i \leq n.
\]

Fix \( \varepsilon > 0 \) and choose \( N > n \) so that \( \sum_{j > N} |\mu_i^j| < \varepsilon \) for all \( 1 \leq i \leq n \). The function \( f \) on \( K \)
defined by \( f(k_j) = 1 \) for \( 1 \leq j \leq N \) and \( f(k) = 0 \) otherwise is continuous because the \( k_j \)’s are isolated. As in Lemma 2.4, \( \varphi(f)(k_j) \geq 0 \) for all \( 1 \leq j \leq N \) and \( \|\varphi(f)\| \leq 1 \).

**Claim.** \( \sum_{j > N} |\mu_i^j| \leq 1 \) for all \( 1 \leq i \leq n \).

Assume that \( \max_{1 \leq i \leq n} \sum_{n < j \leq N} |\mu_i^j| \) is attained for \( i = 1 \), and we show that it is bounded by \( 1 \).
Since this holds for every large enough \( N \) the claim will follow.

Put \( \lambda_i = \sum_{n < j \leq N} \mu_i^j \). As noted above, we may assume that \( \mu_i^j \geq 0 \) for every \( n < j \leq N \), hence \( \lambda_1 = \sum_{n < j \leq N} \mu_1^j \geq 0 \). We may also assume that \( \lambda_i \geq 0 \) for every \( i \geq 2 \) (by replacing, if necessary, \( \mu_i \) by \(-\mu_i \) and changing the sign of \( \mu_i(k_i) \)).

With this notation we need to prove that \( \lambda_1 \leq 1 \), so assume for contradiction that \( \lambda_1 > 1 \) and define \( g \) by

\[
g(k) = \begin{cases} 
-\lambda_i & \text{for } k = k_i \text{ and } i \leq n, \\
1 & \text{for } k = k_j \text{ and } n < j \leq N, \\
0 & \text{otherwise}.
\end{cases}
\]

Once again \( g \) is continuous because the \( k_i \)’s are isolated. Also \( g \in E \) because the definition of \( g \) and the \( \lambda_i \)’s imply that \( \int g \, d\mu_i = 0 \) for all \( i \leq n \).

The nonzero values of the function \( f - tg \) are \( 1 + t \lambda_i \) for \( i \leq n \) and \( 1 - t \). It follows from \( \lambda_1 > 1 \),
the maximality of \( \lambda_1 \) and from \( \lambda_i \geq 0 \) for all \( i \leq n \) that if \( t < 0 \) and if \( |t| \) is large enough, then

\[
\|f - tg\| = \max(|1 - t|, |1 + \lambda_i t|) = |1 + \lambda_i t| = -1 - \lambda_1 t.
\]

Combining this estimate with \( \varphi(tg) = tg \) and \( t < 0 \) it follows that if \( |t| \) is large enough, then

\[
-(\varphi(f)(k_1) + \lambda_1 t) = |\varphi(f)(k_1) + \lambda_1 t| = |(\varphi(f) - \varphi(tg))(k_1)|
\leq \|f - tg\| = -1 - \lambda_1 t
\]
and hence $\varphi(f)(k_1) \geq 1$. But this is impossible for small enough $\varepsilon$, because

$$0 = \int \varphi(f) \, d\mu_1 = \varphi(f)(k_1) + \sum_{n < j \leq N} \mu_1^j \varphi(f)(k_j) + \sum_{j > N} \mu_1^j \varphi(f)(k_j)$$

and $\varphi(f)(k_1) \geq 1$, the first sum is nonnegative (because $\varphi(f)(k_j)$ and $\mu_1^j$ are nonnegative for all $n < j \leq N$) and the third term is bounded in absolute value by $\varepsilon$ (because $\sum_{j > N} |\mu_1^j| < \varepsilon$ and $\|\varphi(f)\| \leq 1$). This proves the claim.

Combining the claim with the assumption (*), it follows that $|\mu_1^j| < 1$ for all $i \leq n$ and $j > n$, and that for each $i \leq n$ there is a $j > n$ with $\mu_1^j \neq 0$. Assume that $0 < \mu_1^{n+1} < 1$, say, and then assume also that $\mu_1^j \geq 0$ for $n + 2 \leq j \leq N$. We may also assume that $\mu_1^{n+1} \geq 0$ for every $i \geq 2$ (by replacing, if necessary, the measure $\mu_i$ by $-\mu_i$ and changing the sign of $\mu_i(k_i)$).

Let $f \in C(K)$ be as above (i.e. $f(k_j) = 1$ for $j \leq N$ and $f(k) = 0$ otherwise), then $\|\varphi(f)\| \leq 1$ and $\varphi(f)(k_j) \geq 0$ for $j \leq N$. Define $g \in E$ by $g(k_i) = -\mu_i^{n+1}$ for $i \leq n$, $g(k_{n+1}) = 1$ and $g(k) = 0$ otherwise.

The nonzero values of $f - tg$ are $1 + t\mu_i^{n+1}$ at $k_i$ for $i \leq n$, $1 - t$ at $k_{n+1}$ and the value 1. Since $0 \leq \mu_i^{n+1} < 1$ for every $i \leq n$, it follows that if $t > 0$ is large enough, then $\|f - tg\| = |1 - t| = t - 1$. Thus, if $t > 0$ is large enough, then

$$0 \leq t - \varphi(f)(k_{n+1}) = (\varphi tg - \varphi(f))(k_{n+1})$$

$$\leq \|\varphi tg - \varphi(f)\| \leq \|tg - f\| = t - 1$$

and hence $\varphi(f)(k_{n+1}) \geq 1$. But this is impossible for small enough $\varepsilon$ because

$$0 = \int \varphi(f) \, d\mu_1 = \sum_{1 \leq j \leq N: j \neq n+1} \mu_1^j \varphi(f)(k_j) + \mu_1^{n+1} \varphi(f)(k_{n+1}) + \sum_{j > N} \mu_1^j \varphi(f)(k_j),$$

where the first term is nonnegative, the second at least $\mu_1^{n+1} > 0$, and the third is bounded in absolute value by $\varepsilon$. \hfill \Box

**Corollary 2.5.** If $K$ is perfect (i.e., with no isolated points), then $C(K)$ does not admit any NPR subspace of finite codimension.

For the proof of Theorem 2.2 we shall need the following known lemma.

**Lemma 2.6.** Let $E$ be a subspace of $C(K)$ which is the range of a nonexpansive retraction $\psi : C(K) \to E$. Then $E^*$ is isometric to $L_1(\mu)$ for some measure $\mu$.

**Proof.** Lindenstrauss [8, Theorem 6.1, (2) $\Leftrightarrow$ (12)] proved that $E^*$ is isometric to $L_1(\mu)$ iff every collection of four mutually intersecting balls in $E$ with the same radius $r$ has a common intersection.

If $B_E(x_i, r)$ are the four balls in $E$, then the balls $B_{C(K)}(x_i, r)$ in $C(K)$ with the same centers and radius intersect in $C(K)$, because $C(K)^*$ is isometric to an $L_1(\mu)$ space. Choose a point $f$ in their intersection, then $\psi(f) \in \bigcap B_E(x_i, r)$. \hfill \Box

**Proof of Theorem 2.2.** Since $E$ is the range of a nonexpansive retraction of $C(K)$, it follows from Lemma 2.6 that $E^*$ is isometric to a finite-dimensional $L_1(\mu)$ space, i.e., to $l_1^n$. Thus $E$ is isometric to $l_1^n$. 


Let \( \{ f_i \}_{i=1}^n \subset E \) be a \( l_\infty^n \) basis for \( E \), i.e., \( \| \sum_{i \leq n} x_i f_i \| = \max_{i \leq n} |x_i| \) for all scalars \( \{x_i\}_{i \leq n} \).

It follows that the sets \( S_i = \{ t \in K : |f_i(t)| = 1 \} \) are nonempty and pairwise disjoint. (Actually \( S_i \) is disjoint from \( \{ t : f_j(t) \neq 0 \} \) whenever \( i \neq j \).) Also \( \sum_{i \leq n} |f_i(t)| \leq 1 \) for all \( t \in K \). Put \( S = \bigcup_{i \leq n} S_i \).

The theorem will follow once we show that \( f_i(t) = 0 \) for all \( i \) and for all \( t \notin S \). Indeed, take \( Z = K \setminus S \), the sets \( S_i \) for the \( i \)'s where \( f_i \) has a constant sign on \( S_i \), and \( S_i^1 = \{ t \in S_i : f_i(t) = 1 \} \) and \( S_i^2 = \{ t \in S_i : f_i(t) = -1 \} \) for the \( i \)'s where \( f_i \) attains both values \( \pm 1 \) on \( S_i \). The continuity of the \( f_i \)'s implies that all these sets are clopen.

Thus, assume for contradiction that there is a \( f_1 \notin S \) so that \( f_1(t_1) \neq 0 \), say.

Put \( I = \{ i : f_i(t_1) \neq 0 \} \). Replacing \( f_1 \) by \( -f_1 \) if necessary, we may assume that \( f_1(t_1) > 0 \) for all \( i \in I \).

Pick \( 1 > \eta > f_1(t_1) \) and set \( T = \{ |f_1(t)| \geq \eta \} \). Then \( T \) contains \( S_1 \) and is disjoint from \( (\bigcup_{i \notin 1} S_i) \cup \{ t_1 \} \). Using Tietze's theorem, find \( f \in C(K) \) with \( \| f \| = 1 \) so that

\[
    f(t) = \begin{cases} 
        f_1(t) & \text{for } t \in T, \\
        f_i(t) & \text{for } t \in S_i; \ 1 \neq i \in I, \\
        -1 & \text{for } t = t_1 
    \end{cases}
\]

and expand \( \varphi(f) = \sum x_i f_i \). We claim that \( x_1 = 0 \).

Indeed, fix \( i \in I \) and \( t \in S_i \). Then \( \| \varphi(f) - f \| = d(f, E) \leq \| f \| = 1 \) and \( f(t) = f_i(t) = \pm 1 \), together with \( f_j(t) = 0 \) for \( j \neq i \) imply that \( x_i > 0 \). Since \( f_i(t_1) > 0 \) for all \( i \in I \) by our normalization and since \( x_i > 0 \), we obtain that \( \varphi(f)(t_1) = \sum_{i \in I} x_i f_i(t_1) \geq 0 \) and is strictly positive if one of the \( x_i \)'s is nonzero. But then \( f(t_1) = -1 \) and \( \| \varphi(f)(t_1) - f(t_1) \| \leq 1 \) implies that necessarily \( \varphi(f)(t_1) \leq 0 \), hence \( \varphi(f)(t_1) = 0 \) and \( x_i = 0 \) for all \( i \in I \). In particular \( x_1 = 0 \) as claimed.

Fix \( \lambda > 1 \) and \( s \in S_1 \). Then \( x_1 = 0 \) and \( f_i(s) = 0 \) for all \( i \neq 1 \) imply that

\[
\| \varphi(f) - \varphi(\lambda f_1) \| = \left\| \sum_{i \neq 1} x_i f_i - \lambda f_1 \right\| \geq \sum_{i \neq 1} x_i f_i(s) - \lambda f_1(s) \right\| = |0 - \lambda f_1(s)| = \lambda.
\]

We finish the proof by showing that \( \| f - \lambda f_1 \| < \lambda \) for big enough \( \lambda \), contradicting the nonexpansiveness of \( \varphi \). To this end we distinguish two cases:

If \( t \in T \), then \( f(t) = f_1(t) \), hence

\[
|f(t) - \lambda f_1(t)| = |f_1(t) - \lambda f_1(t)| = \lambda - (\lambda - 1) < \lambda.
\]

If \( t \notin T \), then \( |f_1(t)| \leq \eta \), hence

\[
|f(t) - \lambda f_1(t)| \leq |f(t)| + \lambda |f_1(t)| \leq 1 + \lambda \eta < \lambda
\]

provided \( \lambda > 1/(1 - \eta) \). \( \square \)

**Corollary 2.7.** If \( K \) is connected, then \( C(K) \) does not admit any NPR subspace of finite dimension except for the one-dimensional subspace \( E_K \) of type II.
3. NPR subsets of $l^n_\infty$

The main result of this section is a complete characterization of NPR subsets of $l^n_\infty$.

**Theorem 3.1.** A subset $A \subset l^n_\infty$ is a NPR iff it is the intersection of NPR half-spaces.

We also give some results in general Banach spaces and make some comments on the structure of NPR’s in general $C(K)$ spaces. We start with some preliminary preparations.

**Lemma 3.2.** Let $A$ be a convex NPR in a Banach space $X$ and assume that the affine subspace $E$ spanned by $A$ is finite-dimensional. Then

(i) $E$ is a NPR of $X$.

(ii) Let $z$ be a smooth point of the relative boundary of $A$ in $E$ and let $V$ be the supporting hyperplane of $A$ in $E$ at the point $z$. Then $V^+$, the half-space of $E$ determined by $V$ and containing $A$, is a NPR of $X$.

**Proof.** Let $\varphi : X \to A$ be the NPR and assume, as we may, that $0 \in A$. Direct computation shows that for each $\lambda > 0$ the map $\varphi_\lambda(x) = \lambda \varphi(x/\lambda)$ is a NPR from $X$ onto $\lambda A$, and for each fixed $x$ the function $\lambda \to \varphi_\lambda(x)$ is bounded by $\|x\|$ (because $\varphi_\lambda(0) = 0$).

Since $E$ is finite-dimensional, there is a $E$-valued Banach limit $\text{LIM}$ on bounded function from $\mathbb{R}^+$ to $E$. One checks easily that $\psi(x) = \text{LIM}_{\lambda \to \infty} \varphi_\lambda(x)$ is a NPR from $X$ onto the closure $Y$ of $\bigcup \{\lambda A : \lambda > 0\}$.

To prove (i) assume that 0 is in the relative interior of $A$ in $E$. It then follows that $Y = E$.

To prove (ii) assume that the smooth point is $z = 0$. It follows from the smoothness that $Y = V^+$. □

**Remark.** The assumption that $E$ is finite-dimensional could, of course, be replaced by weaker conditions. What we really need is that closed balls in $E$ are compact under some topology $T$ so that the norm is lower semi-continuous with respect to $T$. (For example, the $o^*$-topology when $E$ happens to be a dual space.) We shall use, however, only the finite-dimensional case.

**Lemma 3.3.** Let $A \subseteq l^n_\infty$ be a NPR in $l^n_\infty$. Then $A$ is convex.

**Proof.** Denote the NPR on $A$ by $\varphi : l^n_\infty \to A$.

Observe first that whenever a point $v = (v_1, \ldots, v_n) \in l_\infty$ attains its norm in all its coordinates, i.e., when $|v_j|$ is constant, then the linear segment connecting $v$ and $-v$ is the only metric segment between them.

We shall show that whenever there is a point $x \in A$ so that also $-x \in A$, then $0 \in A$. The general case follows by translation.

Choose $x$ so that it attains its norm in $k$ coordinates, and so that $k$ is maximal among all the points $y \in A$ with $-y \in A$. We shall show that $k = n$, and this will prove the lemma: Since $A$ is a NPR, any two points in $A$ are connected in $A$ by a metric segment, and by the observation above $k = n$ implies that the metric segment connecting $x$ and $-x$ is a linear segment. Hence $0 \in A$.

Assume for contradiction that $k < n$. We may assume that $\|x\| = 1$ and that $x = (a_1, \ldots, a_n)$ with $a_j \geq 0$ and $a_1 = \cdots = a_k = 1$. Put $\max \{a_j : j > k\} = \alpha < 1$ and $x_t = (t, \ldots, t, a_{k+1}, \ldots, a_n)$ for $\alpha \leq t \leq 1$. Note that $x_\alpha$ attains its norm ($\|x_\alpha\| = \alpha$) in at least $k + 1$ coordinates. We claim
that \( x_\varepsilon \in A \), and a similar argument will show that \(-x_\varepsilon \in A \). This contradicts the maximality of \( k \).

Assume the claim is false. Since \( A \) is closed there is an \( \varepsilon > 0 \) so that \( B(x_\varepsilon, \varepsilon) \cap A = \emptyset \). Let \([z,s]\) be the maximal interval so that \( B(x_\varepsilon, \varepsilon) \cap A = \emptyset \) for all \( z \leq t < s \).

Since \( \|x_t - x\| = 1 - t \) and \( x \in A \), it follows that if \( B(x_\varepsilon, \varepsilon) \cap A = \emptyset \), then \( \varepsilon < 1 - t \). Taking the supremum over \( z \leq t < s \) gives that \( s \leq 1 - \varepsilon \). Also \( d(x_\varepsilon, A) = \varepsilon \) implies that \( \varphi(x_\varepsilon) \in B(x_\varepsilon, \varepsilon) \cap A \).

Observe also that if \( y \in B(x_\varepsilon, \varepsilon) \cap A \), then there is an \( i \leq k \) so that \( y_i = \varepsilon + s \). Indeed, \( s - \varepsilon \leq y_j \leq s + \varepsilon \) for all \( j \leq k \). Since \( y \notin B(x_\varepsilon, \varepsilon) \) for \( z \leq t < s \), then the two conditions \( y \in B(x_\varepsilon, \varepsilon) \) and \( (x_\varepsilon)_j = (x_t)_j \) for \( j > k \) imply that there is an \( i \leq k \) so that either \( y_i > t + \varepsilon \) or \( y_i < t - \varepsilon \). But the latter is impossible because combining \( y_i < t - \varepsilon \) with \( y_i > s - \varepsilon \) would contradict \( t < s \). Letting \( t \to s \) gives \( y_i \geq s + \varepsilon \) and proves the observation.

Applying the observation above to \( y = \varphi(x_\varepsilon) \in B(x_\varepsilon, \varepsilon) \cap A \), choose \( i \leq k \) so that \( \left( \varphi(x_\varepsilon) \right)_i = s + \varepsilon \). Then

\[
\| \varphi(x_\varepsilon) - \varphi(-x) \| = \| \varphi(x_\varepsilon) - (-x) \| \geq \left( \varphi(x_\varepsilon) - (-x) \right)_i = s + \varepsilon + 1
\]

but on the other hand

\[
\| x_\varepsilon - (-x) \| = \max \left( s + 1, 2 \max \{ a_j : j > k \} \right) = s + 1
\]

because \( 2 \max \{ a_j : j > k \} = \varepsilon + 2 < 1 + s \). This contradicts the nonexpansiveness of \( \varphi \) and proves the lemma. \( \Box \)

We do not know if NPR’s in infinite-dimensional \( C(K) \) spaces are necessarily convex. The following example shows, however, that NPR’s do not have to be convex in general Banach spaces.

**Example 3.4.** Let \( E \) be the two-dimensional Banach space whose unit ball is the regular hexagon with vertices at \((\pm 2/\sqrt{3}, 0); (\pm 1/\sqrt{3}, \pm 1)\). Let \( A \subset E \) be the (nonconvex) union of the two rays emanating from the origin and passing through \((1/\sqrt{3}, \pm 1)\). One checks directly that if \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \), then \( \| (x_1, x_2) - (y_1, y_2) \| \geq |x_2 - y_2| \) and that \( \| (x_1, x_2) - (y_1, y_2) \| = |x_2 - y_2| \) whenever \( x, y \in A \). It follows that the horizontal projection \( \varphi(x) = (|x_2|/\sqrt{3}, x_2) \) from \( E \) onto \( A \) is nonexpansive, and one checks directly that \( \varphi(x) \) is a nearest point in \( A \) to \( x \).

**Proof of Theorem 3.1.** Assume that \( A \) is a NPR in \( l^m_\infty \) and we show that it is the intersection of NPR half-spaces.

Let \( E \) be the affine subspace of \( l^m_\infty \) spanned by \( A \) and we may assume that \( 0 \in A \), i.e., that \( E \) is a linear subspace. By Lemma 3.3 the set \( A \) is convex, hence part (i) of Lemma 3.2 applies and \( E \) is a NPR. By Theorem 2.2 \( E \) is isometric to \( l^k_\infty \) for some \( k \leq n \). Moreover, the explicit form (1) of NPR subspaces implies that \( E \) is the intersection of NPR hyperplanes in \( l^m_\infty \).

Since the smooth points of the relative boundary of \( A \) in \( E \) are dense in this boundary, it follows that every \( y \in E \setminus A \) can be separated from \( A \) by a hyperplane in \( E \) which supports \( A \) in a relatively smooth point. By part (ii) of Lemma 3.2 the half-space determined by this hyperplane is a NPR in \( E \), and the special form (1) of \( E \) implies that it is the intersection of \( E \) with a NPR half-space of \( l^k_\infty \). Thus \( A \) is, indeed, the intersection of NPR half-spaces.

Conversely, assume that \( A \) is an intersection of NPR half-spaces in \( l^m_\infty \). The special form (1) of the NPR hyperplanes in \( l^m_\infty \) is applied through the following claim:
Claim. Let \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) be two points in \( A \) and let \( t \geq 0 \). Define a new point \( c = (c_1, c_2, \ldots, c_n) = c(a, b, t) \) by

\[
c_i = \begin{cases} 
    a_i & \text{if } |a_i - b_i| \leq t, \\
    b_i - t & \text{if } a_i < b_i - t, \\
    b_i + t & \text{if } a_i > b_i + t.
\end{cases}
\]

Then also \( c \in A \).

To prove the claim we may assume that \( A \) is just one NPR half-space and we check separately each of the three types of NPR half-spaces. Thus assume, for example, that \( A = \{ x = (x_1, \ldots, x_n) : x_1 + x_2 \leq 1 \} \) is a half-space of type II and we make a case by case check that \( c_1 + c_2 \leq \max(a_1 + a_2, b_1 + b_2) \leq 1 \). For example, assume that \( c_1 = a_1 \) and \( c_2 = b_2 + t \). Then \( b_2 + t < a_2 \), and hence \( c_1 + c_2 < a_1 + a_2 \). The other cases, as well as checking the other types of half-spaces are similar.

We shall apply the claim for a pair of points satisfying \( \|b\| \geq \|a\| = 1 \) and with \( t = \|b\| - 1 \). Then certainly \( |c| \geq \|b\| - t = 1 \) and actually \( |c| = 1 \). Indeed, assume for example that \( a_1 < b_1 - t \).

Then \( c_1 = b_1 - t = b_1 - \|b\| + 1 \leq 1 \), and clearly \( c_1 = b_1 - t > a_1 \geq -\|a\| = -1 \), hence \( |c_1| \leq 1 \). Similar estimates show that \( |c_i| \leq 1 \) for all \( i \) and in the other cases as well, hence \( |c| \leq 1 \).

Moreover, with \( |a|, \|b\| \) and \( t \) as above the estimate \( |b - c| \leq t = \|b\| - |c| \) together with the triangle inequality give that \( |b - c| = \|b\| - |c| \). Hence

\[
|c| - |a - c| \leq |c| - (|a - b| + |b - c|) = \|b\| - \|b - a\|. \tag{3}
\]

We now prove by induction on the dimension \( n \) that \( A \) is a NPR.

Since \( l^n_{\infty} \) is hyperconvex, Theorem 1.1 (which is proved in the Appendix) implies that it suffices to prove that for every \( z \in l^n_{\infty} \setminus A \) the set \( A \) is a NPR in \( A \cup \{z\} \). We thus need to find a point \( a \in A \), which is nearest to \( z \) in \( A \), and such that \( \|a - b\| \leq \|z - b\| \) for every \( b \in B \).

We may assume that \( z = 0 \) and that \( \text{dist}(0,A) = 1 \). Since \( A \) is convex, its intersection with the unit ball \( B \) of \( l^n_{\infty} \) is contained in a face of \( B \). We may assume that the face is \( B \cap H \), where \( H = \{ x : x_1 = 1 \} \). In particular \( \|b\| \geq 1 \) for all \( b \in A \). Let \( R : l^n_{\infty} \rightarrow H \) be the nonexpansive retraction \( R(x_1, x_2, \ldots) = (1, x_2, \ldots) \) and note that \( R(0) = e_1 \).

Since \( H \) is a translate of \( l^n_{\infty} - 1 \), the induction hypothesis implies that there is a NPR \( \phi : H \rightarrow H \cap A \). Put \( a = \phi(e_1) = (\phi(R))(0) \) and note that \( a \in H \cap B \). Hence, \( \|a\| = 1 \) and it is a nearest point in \( A \) to \( z = 0 \).

To show that \( \|a - b\| \leq \|0 - b\| = \|b\| \) for every \( b \in A \) (or, equivalently, that \( \|b\| - \|b - a\| \geq 0 \)), let \( c = c(a, b, t) \) with \( t = \|b\| - 1 \) be as above. Then \( |c| = 1 \) and \( c \in A \) imply that it is in the face of \( B \) determined by \( H \), i.e., \( c \in B \cap A \subseteq H \cap A \) and hence \( (\phi(R))(c) = \phi(c) = c \).

Then \( \|c - a\| = \|\phi(R)(c) - (\phi(R))(0)\| \leq \|c - 0\| = \|c\| \), because \( \phi \) is nonexpansive. Combined with (3) this gives \( \|b\| - \|b - a\| \geq \|c\| - \|a - c\| \geq 0 \) as required. \( \Box \)

Remarks. (i) Lemmas 3.2 and 3.3 hold also when \( A \) is a NPR of a neighborhood \( B \) of \( A \) (rather than the whole space \( l^n_{\infty} \)). It follows that if \( A \subseteq l^n_{\infty} \) is a NPR of such a neighborhood \( B \), then \( A \) is the intersection of NPR half-spaces and, in particular a NPR of all of \( l^n_{\infty} \).

(ii) Lemmas 3.2 and 3.3 also remain true when \( A \) is a NPR of a NPR subset \( B \subseteq l^n_{\infty} \). We leave it to the reader to check that this, indeed, follows from the special form of such a set \( B \) as an intersection of NPR half-spaces of \( l^n_{\infty} \). Thus a NPR subset \( A \subseteq B \) of a NPR set \( B \subseteq l^n_{\infty} \) is a NPR in \( l^n_{\infty} \). This is no longer true in general Banach spaces.
Example 3.5. Let $A$ be the nonconvex NPR subset in the two-dimensional space $E$ of Example 3.4. Denote the hexagon by $H$.

Let $D$ be the unit disk in $\mathbb{R}^2$. Then $D$ is the inscribed disk in $H$. Let $F$ be the three-dimensional space whose unit ball $B$ is the convex hull of $H$ and $\{(x_1, x_2, \pm 1) : (x_1, x_2) \in D\}$. Denote by $P : F \to E$ the projection given by $P(x_1, x_2, x_3) = (x_1, x_2, 0)$. One checks easily that $\|P\| = \|I - P\| = 1$, and thus $P$ is a NPR from $F$ onto $E$.

We shall show that $A$ is not a NPR of $F$. In fact the metric projection from $F$ to $A$ does not admit any continuous selection.

Consider the points $x_t = \left(\frac{2}{\sqrt{3}} + |t|, \sqrt{3}t, 1\right)$, and let $B_t = B(x_t, 1)$ be the closed ball in $F$ of radius 1 and center $x_t$. Then the intersection $B_t \cap E$ is the translated disk $\left(\frac{2}{\sqrt{3}} + |t|, \sqrt{3}t\right) + D$, which touches $A$ in a unique point whenever $t \neq 0$. This point is the nearest point in $A$ to $x_t$. As $t \to 0^+$ and $t \to 0^-$ we get two different limit points: the two points where the disk $\left(\frac{2}{\sqrt{3}}, 0\right) + D$ touches $A$. (These two points are exactly the nearest points in $A$ to $x_0$.) Thus the metric projection from $F$ to $A$ does not admit a selection which is continuous at $x_0$.

Remarks. We make a few comments on the analogs of Theorem 3.1 in general $C(K)$ spaces.

(i) If $A$ is a finite intersection of NPR half-spaces in any $C(K)$ space, then it is a NPR. Indeed the explicit form (1) of NPR hyperplanes implies that there is a finite clopen subset $S \subset K$ of cardinality $n$, say, and a subset $B \subset C(S) = l_\infty^n$, which is an intersection of NPR hyperplanes in $C(S)$, so that $A = \{f \in C(K) : f|_S \in B\}$. By Theorem 3.1 there is a NPR $\psi : C(S) \to B$, and then the map $\varphi : C(K) \to A$, given by $\varphi(f)(k) = \psi(f|_S)(k)$ when $k \in S$ and $\varphi(f)(k) = f(k)$ otherwise, is a NPR on $A$.

(ii) An infinite intersection of NPR hyperplanes does not have to be a NPR. For example, assume that $K$ contains a convergent sequence $\{k_n\}$ of isolated points with limit $k$, and take $E_n = \{f \in C(K) : f(k_{2n}) = 0\}$. Then $E = \cap E_n$ does not admit any nonexpansive retraction $\varphi$ (not even necessarily NPR). Indeed, let $e$ be the constant function 1. Since $f(k) = 0$ for every $f \in E$, it follows that $\varphi(e)(k) = 0$ and we can find $n$ such that $|\varphi(e)(k_{2n+1})| < \frac{1}{2}$. Let $g$ be the (continuous) function taking the value 1 at $k_{2n+1}$ and 0 elsewhere. Then $g \in E$ and $\|e - 2g\| = 1$, yet

$$\|\varphi(e) - \varphi(2g)\| = \|\varphi(e) - 2g\| \geq |\varphi(e)(k_{2n+1}) - 2g(k_{2n+1})| > 3/2.$$  

(iii) It is also false that a NPR subset of an infinite-dimensional $C(K)$ needs to be an intersection of NPR half-spaces. Indeed, for any $K$ the set $C(K)^+ = \{f \in C(K) : f \geq 0\}$ is a NPR with associated retraction $\varphi(f) = \max\{f, 0\}$. But when $K$ is connected $C(K)$ admits no NPR hyperplane whatsoever.

Similarly, Ubhaya [9] proved (among other results) that the set of nondecreasing continuous functions on $C(0, 1)$ is a NPR. He also showed that for each fixed $M > 0$ and $0 < \varepsilon \leq 1$, the set of all $f \in C(0, 1)$, such that $|f(x) - f(y)| \leq M|x - y|^\varepsilon$ is a NPR. Once again, $C(0, 1)$ admits no NPR hyperplane because the interval is connected.

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Appendix A. Proof of Theorem 1.1

Recall that a metric space $X$ is called hyperconvex if every family $\{B(x_i, r_i)\}_{i \in I}$ of balls in $X$ satisfying $d(x_i, x_j) \leq r_i + r_j$ has a common intersection. An equivalent condition is that when $Y$ is any metric space containing $X$, then there is a nonexpansive retraction from $Y$ onto $X$. The systematic study of hyperconvex spaces and the relations between intersection properties of balls and extensions of maps was initiated by Aronszajn and Panitchpakdi [1]. See Espínola and Khamsi [6] for details on hyperconvex spaces.

Hyperconvex Banach spaces are exactly the $C(K)$ spaces with $K$ an extremally disconnected compact Hausdorff space. In particular, the finite-dimensional hyperconvex Banach spaces are exactly the spaces $l_\infty^n$.

We now turn to the proof of Theorem 1.1, which is Theorem 4.1 in [7]. (The formulation in [7] is for compact sets $A$, but the proof holds for boundedly compact sets.) Lemma A.1 below and the proof of Theorem 1.1 combine the proofs of Theorem 2.1, Lemma 2.2 and Theorem 4.1 in [7]. (Subsets $A$ satisfying the conclusion of the following lemma were called in [7] weakly externally hyperconvex.)

**Lemma A.1.** Let $A$ be a subset of a hyperconvex metric space $X$ so that for every $y \in X$ there is a NPR from $A \cup \{y\}$ onto $A$. Then for every family of mutually intersecting balls $\{B_i\}_{i \in I}$ with centers in $A$ and for every point $z \in X \setminus A$ so that $B_i \cap B(z, d(z, A)) \neq \emptyset$ for every $i$, the intersection $(\bigcap_i B_i) \cap B(z, d(z, A)) \cap A$ is nonempty.

**Proof.** Put $B_i = B(x_i, r_i)$, where $x_i \in A$, and set $r_z = d(z, A)$.

By hyperconvexity the intersection $B = (\bigcap_i B_i) \cap B(z, r_z)$ is nonempty, and we need to show it intersects $A$. Since $B \subset B(z, r_z)$, we actually need to show that it intersects $A_1 = A \cap B(z, r_z)$. Choose $a \in A_1$ and $b \in B$ with $d(a, b) < \frac{3}{2}d(A_1, B)$ and put $d(a, b) = 2d$. We shall prove that $d = 0$.

One checks easily that the balls $B(a, d)$, $B(b, d)$ and $B(z, r_z - d)$ are mutually intersecting. By the hyperconvexity of $X$ there is a point $y$ with

$$y \in B(a, d) \cap B(b, d) \cap B(z, r_z - d).$$

Let $\varphi : A \cup \{y\} \to A$ be a NPR and note first that $\varphi(y) \in A_1$. Indeed, we only need to check that $\varphi(y) \in B(z, r_z)$, but

$$d(\varphi(y), z) \leq d(\varphi(y), y) + d(y, z) \leq d(y, A) + r_z - d \leq r_z$$

because $d(y, A) \leq d(y, a) \leq d$.

Next we show that there is a point $x \in B$ with $d(\varphi(y), x) \leq d$. Indeed, $d(\varphi(y), z) \leq r_z$, the estimate

$$d(\varphi(y), x_i) = d(\varphi(y), \varphi(x_i)) \leq d(y, x_i) \leq d(y, b) + d(b, x_i) \leq d + r_i$$

and the fact that $B_i \cap B(z, r_z) \neq \emptyset$ for all $i$ imply, by the hyperconvexity of $X$, that the balls $B(\varphi(y), d)$, $B(x_i, r_i)$ and $B(z, r_z)$ have a common intersection, i.e., that there is a point $x \in B(z, r_z) \cap (\bigcap_i B(x_i, r_i)) = B$ with $d(x, \varphi(y)) \leq d$. 


It follows that $2d = d(a, b) < \frac{3}{2}d(A_1, B) \leq \frac{3}{2}d(\varphi(y), x) \leq \frac{3d}{2}$ and hence $d = 0$. □

**Proof of Theorem 1.1.** We shall show that for every finite set $F \subset X \setminus A$ there is a NPR from $A \cup F$ onto $F$. The theorem then follows by a standard compactness argument, using the compactness of bounded sets in $A$.

The proof is by induction on the cardinality of $F$. Choose $z \in F$ such that $d(z, A) = \max_{y \in F} d(y, A)$ and set $G = F \setminus \{z\}$. Let $\varphi : A \cup G \to A$ be a NPR. The family of balls

$$\{B(x, d(x, z)) : x \in A\}; \quad \{B(\varphi(y), d(y, z)) : y \in G\}; \quad B(z, d(z, A))$$

satisfies the conditions of Lemma A.1, where the assumption that $d(z, A)$ is maximal is used to check that $d(\varphi(y), z) \leq d(y, z) + d(z, A)$ for $y \in G$. Indeed, $d(\varphi(y), z) \leq d(\varphi(y), y) + d(y, z)$ and $d(\varphi(y), y) = d(y, A) \leq d(z, A)$. That the other pairs of balls intersect follows immediately from the triangle inequality or the nonexpansiveness of $\varphi$.

By the lemma there is a point $a \in A$ in the intersection of all these balls, and we extend $\varphi$ to a map on $A \cup F$ by defining $\varphi(z) = a$. □

**References**


