ELSEVIER

Bull. Sci. math. 134 (2010) 207–214

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

Another proof for the equivalence between invariance of closed sets with respect to stochastic and deterministic systems ☆

Rainer Buckdahn [a], Marc Quincampoix [a], Catherine Rainer [a,*], Josef Teichmann [b]

[a] *Université de Bretagne Occidentale, Laboratoire de Mathématiques, Unité CNRS UMR 6205, Brest, France*

[b] *University of Technology Vienna, Institute of Mathematical Methods in Economics, Vienna, Austria*

Received 30 September 2007; accepted 12 November 2007

Available online 24 November 2007

Abstract

We provide a short and elementary proof for the recently proved result by G. da Prato and H. Frankowska that – under minimal assumptions – a closed set is invariant with respect to a stochastic control system if and only if it is invariant with respect to the (associated) deterministic control system.

MSC: 93E03; 60H10

Keywords: Invariance of closed sets; Stratonovich drift; Stochastic control system; Deterministic control system; Stochastic Taylor expansion

1. Introduction

We deal in this note with invariance of controlled stochastic differential systems. We consider a non-empty, closed subset $K \subset \mathbb{R}^n$ and ask for characterizations of invariance of K with respect to a controlled stochastic differential system

☆ The authors gratefully acknowledge the support from the RTN network HPRN-CT-2002-00281 (European Union) and from the FWF-grant Y 328 (Austrian Science Funds).

* Corresponding author.
E-mail address: Catherine.Rainer@univ-brest.fr (C. Rainer).

0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2007.11.003

$$dX_t = b(X_t, u_t)\,dt + \sigma(X_t, u_t)\,dW_t, \quad t \geqslant 0,$$
$$X_0 = x \in \mathbb{R}^n, \tag{1}$$

driven by a d-dimensional Brownian motion W.

Invariance of K here means that $P[X_t^{x,u} \in K] = 1$, for all $x \in K$, all times $t \geqslant 0$ and all admissible control processes u.

There exist already a lot of literature concerning invariance as well as the connected notion of viability; characterizations of both have been expressed through stochastic tangent cones [1,11], viscosity solutions of second-order partial differential equations (e.g. see [3,4,6,7,15]) or other approaches (e.g. see [8,14]).

A natural approach of the notion of invariance is to look at the associated controlled ordinary differential system:

$$x'(t) = \tilde{b}\big(x(t), u(t)\big) + \sigma\big(x(t), u(t)\big)v(t), \quad t \geqslant 0,$$
$$x(0) = x, \tag{2}$$

where $\tilde{b}(x,u)$ denotes the Stratonovich drift $\tilde{b}(x,u) = b(x,u) - \frac{1}{2}\sum_{i=1}^{d}\langle D_x\sigma^i(x,u), \sigma^i(x,u)\rangle$ and $v \in L^1_{\text{loc}}([0,\infty), \mathbb{R}^n)$. For the case without control it is well known that invariance with respect to (1) is equivalent to invariance with respect to the ordinary differential system (2) (see [2,10,16]). Recently G. da Prato and H. Frankowska [9] proved the result on the equivalence for controlled deterministic and stochastic systems under minimal assumptions on the involved parameters. Our aim here is to provide a *new, short and very elementary proof* of this intuitive equivalence result.

The intuition behind our main result stems from the local asymptotics of the stochastic systems, which correspond precisely to those of the deterministic system. Reading this insight, which is well-known in numerical analysis for the given stochastic differential system, in the correct way, leads us to the proof. A central step in our investigation is to show that stochastic as well as deterministic invariance is equivalent to invariance with respect to constant controls. This permits us to pass from deterministic invariance to stochastic invariance by the classical Wong–Zakai-approach to martingale problems (which can be seen as a sort of Euler–Mayurama-scheme, too). Concerning the other direction of the proof, the necessary conditions on the parameters follow naturally from a stochastic Taylor expansion.

As a crucial tool we apply optimization theory, since both invariance problems can be associated with problems of minimal distance to K. Hence we can also assert an equivalence between first and second order Hamilton–Jacobi–Bellman systems.

2. Main theorem

Let U be some compact metric space and, for $d, n \geqslant 1$, let b be a bounded and continuous map from $\mathbb{R}^n \times U$ to $\mathbb{R}^n$, Lipschitz in $x \in \mathbb{R}^n$ uniformly in $u \in U$, and σ a continuous map from $\mathbb{R}^n \times U$ to $\mathbb{R}^{n\times d}$, differentiable with respect to x, such that σ and $D_x\sigma$ are bounded and Lipschitz, both uniformly in u.

Let W be a d-dimensional Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t, t \geqslant 0)$ the filtration generated by W, satisfying the usual assumptions. We denote by $\mathcal{U}$ the set of all U-valued processes (u_t) that are progressively measurable w.r.t. $(\mathcal{F}_t, t \geqslant 0)$.

For $(u_t) \in \mathcal{U}$, we consider the controlled stochastic differential system:

$$dX_t = b(X_t, u_t)\,dt + \sigma(X_t, u_t)\,dW_t, \quad t \geqslant 0,$$
$$X_0 = x \in \mathbb{R}^n. \tag{3}$$

It is well known that under the above assumptions on b and σ, the system (3) has a unique strong solution, which we denote by $X^{x,u}$.

We associate to this system the usual second order operator: for $\varphi \in C^2(\mathbb{R}^n, \mathbb{R})$, $x \in \mathbb{R}^n$ and $u \in U$,

$$\mathcal{L}_{x,u}\varphi = \langle b(x,u), \varphi(x)\rangle + \frac{1}{2}\mathrm{tr}\big(D^2\varphi(x)\sigma(x,u)\sigma^*(x,u)\big).$$

We denote by $\tilde{b}$ the Stratonovich drift

$$\tilde{b}(x,u) = b(x,u) - \frac{1}{2}\sum_{i=1}^{d}\langle D_x\sigma^i(x,u), \sigma^i(x,u)\rangle,$$

where $\sigma^i(x,u)$ is the ith column of the matrix $\sigma(x,u)$.

Furthermore we consider a non-empty closed set $K \subset \mathbb{R}^n$. The notion of invariance of K with respect to (3) is defined as follows:

Definition 2.1. We say that K is *invariant with respect to* (3) if, for all $x \in K$, $u \in \mathcal{U}$, and $t \geqslant 0$, $P[X_t^{x,u} \in K] = 1$.

We also shall introduce the deterministic system:

$$\begin{aligned} &x'(t) = \tilde{b}\big(x(t), u(t)\big) + \sigma\big(x(t), u(t)\big)v(t), \quad t \geqslant 0, \\ &x(0) = x, \end{aligned} \tag{4}$$

driven by the deterministic control process $v(t) \in \mathcal{B} := L^1_{\mathrm{loc}}([0,+\infty), \mathbb{R}^d)$ and $u(t) \in \mathcal{A} := L^\infty([0,\infty), U)$. For given $x \in \mathbb{R}^n$, $u \in \mathcal{A}$ and $v \in \mathcal{B}$, the solution of (4) will be denoted by $x^{x,u,v}$. The associated first order operator is, for $\varphi \in C^1(\mathbb{R}^n)$,

$$\mathcal{L}'_{x,u}\varphi = \langle \tilde{b}(x,u), D\varphi(x)\rangle.$$

Definition 2.2. We say that a closed set K is invariant w.r.t. (4) if, for every $x \in K$, $u \in \mathcal{A}$ and $v \in \mathcal{B}$, $x^{x,u,v}(t) \in K$ for every $t \geqslant 0$.

For $\varphi : \mathbb{R}^n \to \mathbb{R}$, we denote by $\mathrm{Argmax}_K\varphi$ the set of $x \in K$ such that φ attains a maximum at x in K.

Our main result is the following theorem.

Theorem 2.1. *The following assertions are equivalent*:

a) *K is invariant with respect to* (3);

b) *For all $\varphi \in C^2$ and $x \in \mathrm{Argmax}_K\varphi$, it holds that*

$$\begin{cases} \sup_{u\in U}\mathcal{L}_{x,u}\varphi \leqslant 0, \\ \langle \sigma^i(x,u), D\varphi(x)\rangle = 0, \quad \forall i \in \{1,\dots,d\}, \quad \forall u \in U; \end{cases} \tag{5}$$

c) *For all $\varphi \in C^2$ and $x \in \mathrm{Argmax}_K\varphi$, it holds that*

$$\begin{cases} \sup_{u\in U}\mathcal{L}'_{x,u}\varphi(x) \leqslant 0, \\ \langle \sigma^j(x,u), D\varphi(x)\rangle = 0, \quad \forall i \in \{1,\dots,d\}, \quad \forall u \in U, \\ \textit{the matrix } A_{\varphi,x} = (a_{ij}) \textit{ with } a_{ij} = \langle \sigma^i(x,u), D_x\langle \sigma^j(\cdot,u), D\varphi(\cdot)\rangle(x)\rangle \\ \textit{is symmetric and semidefinite negative}; \end{cases} \tag{6}$$

d) *K is invariant with respect to* (4);
e) *For all $\varphi \in C^2$ and $x \in \mathrm{Argmax}_K \varphi$, it holds that*

$$\sup_{u\in U,\, v\in\mathbb{R}^d} \left\{\mathcal{L}'_{x,u}\varphi(x) + \left\langle \sigma(x,u)v, D\varphi(x)\right\rangle\right\} \leqslant 0. \tag{7}$$

Remark 2.1. Applying the notations from differential geometry $\tilde{b}_u\varphi(x) := \langle \tilde{b}(x,u), D\varphi(x)\rangle$ and $\sigma_u^i\varphi(x) := \langle \sigma^i(x,u), D\varphi(x)\rangle$, condition (6) can be rewritten as follows:

For all $u \in U$, it holds that

$$\begin{aligned} &\tilde{b}_u\varphi(x) \leqslant 0,\\ &\sigma_u^i\varphi(x) = 0, \quad \forall i \in \{1,\dots,d\},\\ &A_{\varphi,x} = \left(\sigma_u^i\sigma_u^j\varphi(x)\right)_{ij} \text{ is symmetric semidefinite negative.} \end{aligned}$$

The following lemma is crucial in the proof of the theorem.

Lemma 2.1. *Let $(W_t)_{t\geqslant 0}$ be a standard $\mathbb{R}^d$-valued Brownian motion issued from 0 and $(R_t)_{t\geqslant 0}$ a real stochastic process satisfying $\lim_{t\searrow 0}\frac{R_t}{t} = 0$ in probability.*

Let $(\alpha_i, 1\leqslant i\leqslant d) \in \mathbb{R}^d$, $(\beta_i, 1\leqslant i\leqslant d)\in\mathbb{R}^d$, $(\gamma_{ij}, (i,j)\in\{1,\dots,d\}^2, i\neq j)\in\mathbb{R}^{d^2-d}$ and $\delta\in\mathbb{R}$. Suppose that, for all $t\geqslant 0$, P-a.s.,

$$\sum_{i=1}^d \alpha_i W_t^i + \sum_{i=1}^d \beta_i (W_t^i)^2 + \sum_{1\leqslant i\neq j\leqslant d} \gamma_{ij}\int_0^t W_s^i\,dW_s^j + \delta t + R_t \leqslant 0. \tag{8}$$

Then it holds that
i) *$\alpha_i = 0$, for all $i\in\{1,\dots,d\}$*;
ii) *the matrix $A\in\mathbb{R}^{d\times d}$ defined by*

$$\begin{cases} A_{ij} = \gamma_{ij}, & \text{for } (i,j)\in\{1,\dots,d\}^2, \text{ with } i\neq j,\\ A_{ii} = 2\beta_i, & i\in\{1,\dots,d\}, \end{cases}$$

is symmetric and semidefinite negative;
iii) *$\delta\leqslant 0$.*

Proof. It is easy to see that

$$\sum_{i=1}^d \beta_i\frac{(W_t^i)^2}{\sqrt{t}} + \sum_{i\neq j}\gamma_{ij}\frac{\int_0^t W_s^i\,dW_s^j}{\sqrt{t}} + \delta\sqrt{t} + \frac{R_t}{\sqrt{t}} \xrightarrow{P} 0, \quad \text{as } t\searrow 0,$$

while,

$$\forall t\geqslant 0, \quad \sum_{i=1}^d \alpha_i\frac{W_t^i}{\sqrt{t}} \overset{(d)}{=} \sum_{i=1}^d \alpha_i W_1^i.$$

It follows that the left hand term of (8) divided by $\sqrt{t}$, say L_t, converges in distribution to $\sum_{i=1}^d \alpha_i W_1^i$. Now the assumption $P[L_t\leqslant 0] = 1$ for all $t>0$ implies that $P[\sum_{i=1}^d \alpha_i W_1^i\leqslant 0] = 1$, too. It follows that, necessarily $\alpha_1 = \dots = \alpha_d = 0$.

Using again the scaling property of Brownian motion, we have, for all $(i, j) \in \{1, \dots, d\}^2$ and for all $t \geqslant 0$, $\frac{1}{t}\int_0^t W_s^i\, dW_s^j \overset{(d)}{=} \int_0^1 W_s^i\, dW_s^j$. By the same arguments as above, we can deduce from (8) that, P-a.s.,

$$\sum_{i=1}^{d} \beta_i (W_1^i)^2 + \sum_{i \neq j} \gamma_{ij} \int_0^1 W_s^i\, dW_s^j + \delta \leqslant 0. \tag{9}$$

Let us focus now on a fixed arbitrary couple of indexes (i, j) with $i \neq j$. After conditioning by $\sigma(W_s^i, W_s^j, s \geqslant 0)$, we get from (9), P-a.s.,

$$\beta_i (W_1^i)^2 + \beta_j (W_1^j)^2 + \gamma_{ij} \int_0^1 W_s^i\, dW_s^j + \gamma_{ji} \int_0^1 W_s^j\, dW_s^i + \delta + \sum_{k \neq i,j} \beta_k^2 \leqslant 0. \tag{10}$$

Introducing the Levy area $L^{ij} = \int_0^1 W_s^i\, dW_s^j - \int_0^1 W_s^j\, dW_s^i$, we can write:

$$\gamma_{ij} \int_0^1 W_s^i\, dW_s^j + \gamma_{ji} \int_0^1 W_s^j\, dW_s^i = \frac{1}{2}(\gamma_{ij} + \gamma_{ji}) W_1^i W_1^j + \frac{1}{2}(\gamma_{ij} - \gamma_{ji}) L^{ij}.$$

If we substitute this in (10), it follows that, P-a.s.,

$$\frac{1}{2}(\gamma_{ij} - \gamma_{ji}) E\left[L^{ij} \mid W_1^i = W_1^j = 0\right] + \delta + \sum_{k \neq i,j} \beta_k^2 \leqslant 0.$$

But, even after conditioning by $W_1^i = W_1^j = 0$, the distribution of L^{ij} is symmetric and of unbounded support. Consequently it holds that $\gamma_{ij} = \gamma_{ji}$.

Since $(i, j) \in \{1, \dots, d\}^2$, $i \neq j$ was chosen arbitrarily, (9) becomes now, P-a.s.,

$$\sum_{i=1}^{d} \beta_i (W_1^i)^2 + \sum_{i<j} \gamma_{ij} W_1^i W_1^j + \delta \leqslant 0,$$

or, equivalently,

$$\frac{1}{2}\langle W_1, A W_1 \rangle + \delta \leqslant 0. \tag{11}$$

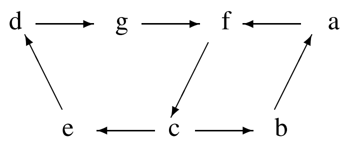
Since the support of W_1 is $\mathbb{R}^d$, ii) and iii) follow. □

Proof of the theorem. We consider the following two additional assertions, where $u \in U$ is identified with the deterministic constant control process $u_t = u$, $t \geqslant 0$. Notice that $X^{x,u}$ is defined by (1) and $x^{x,t,u,v}$ by (4).

f) For all $u \in U$, $x \in K$ and $t \geqslant 0$, $P[X_t^{x,u} \in K] = 1$.

g) For all $u \in U$, $x \in K$ and any admissible control $v \in \mathcal{B}$, the function $x^{x,u,v}(t)$ takes its values in K.

The proof will be organized as follows:



- d) $\Rightarrow$ g) is trivial.
- g) $\Rightarrow$ f): We fix some $u \in U$ and consider a scheme which converges in distribution to the solution of the stochastic differential equation (1) with constant, deterministic control $u_t = u$. For the construction we apply the following limit theorem [12, Theorem 1, p. 698]: For all $t \geqslant 0$, we set $\xi_t = W_{t+1} - W_t$. The process $(\xi_t)_{t\geqslant 0}$ is strictly stationary and ergodic. We let $\eta_t^m = \sqrt{m}\xi_{mt}$, $t \geqslant 0, m \geqslant 1$, and put
$$Y_t^m = \int_0^t \eta_s^m \, ds, \quad t \geqslant 0.$$
Notice that Y^m is a stochastic process with differentiable trajectories. Furthermore, the process Y^m converges ω-wise, uniformly on compacts to W, as $m \to \infty$. Consequently, it converges also in distribution on pathspace. Theorem 1 from [12] tells now that the unique solution of
$$dX_t^m = \tilde{b}(X_t^m, u)\, dt + \sigma(X_t^m, u)\, dY_t^m, \quad X_0^m = x,$$
converges in distribution on pathspace to $X^{x,u}$. The conditions as stated in [12] on σ are slightly stronger than our assumptions, namely C^2 is required. However, the proof in [12] also holds for σ satisfying our $C^{1,1}$-assumptions. Certainly we cannot deduce by [12, Theorem 1] a rate of convergence for $X^m \to X^{x,u}$, but we also do not need such a rate for our purposes.
We know that, by assumption, with probability 1, $X_t^m \in K$ for all $x \in K$, $t \geqslant 0$ and $n \geqslant 1$, whence we obtain the result: Indeed, if $d_K(x)$ denotes the distance from $x \in \mathbb{R}^n$ to K, we have
$$E\big[d_K(X_t)\big] = \lim_{n\to\infty} E\big[d_K(X_t^m)\big] = 0, \quad \text{for all } t \geqslant 0.$$
- f) $\Rightarrow$ c): Consider a constant control $u_t \equiv u \in U$ and suppose that, for all $x \in K$ and $t \geqslant 0$, $P[X_t^{x,u} \in K] = 1$. Let $\varphi \in C^2$ and $x \in \mathrm{Argmax}_K \varphi$. Up to change φ outside of some open set including x, we can suppose that φ, $\|D\varphi\|$ and $\|D^2\varphi\|$ are bounded. We can apply the stochastic Taylor expansion formula ([13] or [5]): for all $t \geqslant 0$, P-a.s.,
$$\varphi(X_t^{x,u}) = \varphi(x) + \sum_{i=1}^d \sigma_u^i \varphi(x) W_t^i + \sum_{i=1}^d (\sigma_u^i)^2 \varphi(x) \frac{(W_t^i)^2}{2} + \sum_{i\neq j} \sigma_u^j \sigma_u^i \varphi(x) \int_0^t W_s^i \, dW_s^j + \tilde{b}_u \varphi(x) t + R_t,$$
where R_t satisfies $\frac{R_t}{t} \to 0$ in probability as $t \searrow 0$. We apply here the operator-notations $\sigma_u^i \varphi(x) = \langle \sigma(x,u), D\varphi(x)\rangle$ and $\tilde{b}_u \varphi(x) = \langle \tilde{b}(x,u), D\varphi(x)\rangle$.
Since K is invariant for the constant control u and since $x \in \mathrm{Argmax}_K \varphi$, we have P-a.s., for all $t \geqslant 0$, $\varphi(X_t^{x,u}) \leqslant \varphi(x)$. Thus, P-a.s., for any fixed $t \geqslant 0$,
$$\sum_{i=1}^d \sigma_u^i \varphi(x) W_t^i + \sum_{i=1}^d (\sigma_u^i)^2 \varphi(x) \frac{(W_t^i)^2}{2} + \sum_{i\neq j} \sigma_u^i \sigma_u^j \varphi(x) \int_0^t W_s^i \, dW_s^j + \tilde{b}_u \varphi(x) t + R_t \leqslant 0.$$

Now we can apply Lemma 2.1 and get exactly the claim.

- c) $\Rightarrow$ b) becomes trivial as soon we write

$$b_u\varphi(x)+\frac{1}{2}\mathrm{tr}\big(D^2\varphi(x)\sigma(x,u)\sigma^*(x,u)\big)=\frac{1}{2}A_{\varphi,x}+\tilde{b}_u\varphi(x).$$

- b) $\Rightarrow$ a): The proof is adapted from the equivalent result about viability in [6]. It is easy to see that, if b) holds, then the map $f: x\mapsto 1-\mathbb{1}_{K(x)}$ is a supersolution of

$$\sup_{u\in U}\mathcal{L}_{x,u}f(x)=0.$$

We consider now a constant $C\geqslant 1$ and an uniformly continuous application g from $\mathbb{R}^n$ to $[0,1]$ that satisfies $g(x)=0$ if and only if $x\in K$. Since, for all $x\in\mathbb{R}^n$, $g(x)\leqslant Cf(x)$, f is also a supersolution of the following Hamilton–Jacobi–Bellman equation

$$\sup_{u\in U}\mathcal{L}_{x,u}f(x)+g(x)-Cf(x)=0. \tag{12}$$

But we know that the unique solution V with polynomial growth of (12) can be represented as

$$V(x)=\sup_{u\in\mathcal{U}}E\Bigg[\int_0^\infty e^{-Cs}g(X_s^{x,u})\,ds\Bigg].$$

By the comparison theorem, we then have

$$V(x)\leqslant f(x),\quad x\in\mathbb{R}^n.$$

For $x\in K$, this implies that, for all $u\in\mathcal{U}$, for all $t\geqslant 0$, $P[X_t^{x,u}\in K]=1$.

- a) $\Rightarrow$ f) is trivial.
- c) $\Rightarrow$ e) is trivial.
- e) $\Rightarrow$ d) could be deduced from b) $\Rightarrow$ a) if v would take its values in a compact space and if b and σ would be replaced by suitable functions. Let us clarify this point: We fix $x\in K$, $v\in\mathcal{B}$ and $u\in\mathcal{A}$. We wish to prove that

$$x^{x,u,v}(t)\in K,\quad \text{for all } t\geqslant 0.$$

For any integer $n\geqslant 0$, we can define the control $t\mapsto v_n(t):=\pi_n(v(t))$, where π_n denotes the projection onto $\overline{B(0,n)}$.
By standard estimates, the sequence x^{x,u,v_n} converges to $x^{x,u,v}$ uniformly on every compact intervals $[0,T]$. Obviously x^{x,u,v_n} is solution to the following control system

$$\begin{aligned}x'(t)&=\widetilde{b}\big(x(t),u(t)\big)+\sigma\big(x(t),u(t)\big)v(t),\\ &\quad u(t)\in U, v(t)\in\overline{B(0,n)}, x(0)=x,\end{aligned} \tag{13}$$

with (u,v) taking values in the compact set $U\times\overline{B(0,n)}$. Hence we can apply the already proved relation b) $\Rightarrow$ a) to the control system (13) with $b(x,u)$ replaced by $\widetilde{b}(x,u)+\sigma(x,u)v$, σ replaced by 0 and the control u replaced by (u,v). In this case the relation (5) reduces to

$$\sup_{u\in U,\,v\in\overline{B(0,n)}}\big\{\mathcal{L}'_{x,u}\varphi(x)+\big\langle\sigma(x,u)v,D\varphi(x)\big\rangle\big\}\leqslant 0.$$

Consequently, we deduce from e) that $x^{x,u,v_n}(t)\in K$, for all $t\geqslant 0$. By passing to the limit with respect to n, we obtain that $x^{x,u,v}(t)\in K$ for all $t\geqslant 0$. Our claim is proved. □

References

[1] J.-P. Aubin, G. da Prato, The viability theorem for stochastic differential inclusions, Stochastic Anal. Appl. 16 (1998) 1–15.
[2] J.-P. Aubin, H. Doss, Characterization of stochastic viability of any nonsmooth set involving its generalized contingent curvature, Stochastic Anal. Appl. 21 (2003) 955–981.
[3] M. Bardi, P. Goatin, Invariant sets for controlled degenerate diffusions: a viscosity solution approach, in: W.M. McEneaney, C.G. Yin, Q. Zhang (Eds.), Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W.H. Fleming, Birkhäuser, Boston, 1999, pp. 191–208.
[4] M. Bardi, R. Jensen, A geometric characterization of viable sets for controlled degenerate diffusions, Set-Valued Anal. 10 (2002) 129–141.
[5] F. Baudouin, An Introduction to the Geometry of Stochastic Flows, Imperial College Press, 2004.
[6] R. Buckdahn, C. Cardaliaguet, M. Quincampoix, Representation formula and mean curvature motion, SIAM J. Math. Anal. 33 (4) (2002) 827–846.
[7] R. Buckdahn, S. Peng, M. Quincampoix, C. Rainer, Existence of stochastic control under state constraints, C. R. Acad. Sci. Paris Sér. I Math. 327 (1998) 17–22.
[8] G. da Prato, H. Frankowska, Stochastic viability for compact sets in terms of the distance function, Dynamic Systems Appl. 10 (2001) 177–184.
[9] G. da Prato, H. Frankowska, Invariance of stochastic control systems with deterministic arguments, J. Differential Equations 200 (2004) 18–52.
[10] H. Doss, Liens entre équations différentielles stochastiques et ordinaires, Ann. Inst. H. Poincaré, Calcul Probab. Statist. 23 (1977) 99–125.
[11] S. Gautier, L. Thibault, Viability for constrained stochastic differential equations, Differential Integral Equations 6 (1993) 1395–1414.
[12] R.Sh. Liptser, A.N. Shiryayev, Theory of Martingales, Mathematics and its Applications (Soviet Series), vol. 49, Kluwer Academic Publishers Group, 1989.
[13] T. Lyons, N. Victoir, Curvature on Wiener Space, Proc. Roy. Soc. London A 460 (2041) (2004) 169–198.
[14] A. Milian, A note on stochastic invariance for Itô equations, Bull. Polish Acad. Sci. 41 (1993) 139–150.
[15] M. Quincampoix, C. Rainer, Stochastic control and compatible subsets of constraints, Bull. Sci. Math. 129 (2005) 39–55.
[16] H.J. Sussmann, On the gap between deterministic and stochastic ordinary differential equations, Ann. Probab. 6 (1978) 19–41.