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On special types of nonholonomic contact elements

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ABSTRACT

Our starting point has been a recent clarification of the role of semiholonomic contact elements in the theory of submanifolds of Cartan geometries, Kolář and Vitolo (2010) [5]. We deduce some further properties of the iterated contact elements by using the general concept of contact (n, F)-element for a regular subcategory F of the category of nonholonomic r-jets. Special attention is paid to the incidence relation of contact F-elements of different dimensions.

The nonholonomic and semiholonomic jets, introduced by C. Ehresmann [2], play in interesting role in various branches of differential geometry, see [7] for a survey. In [5], R. Vitolo and the author studied the semiholonomic and nonholonomic contact elements and pointed out that the semiholonomic ones can be used, in a remarkable way, in the theory of submanifolds of Cartan geometries [1,8]. The main aim of the present paper is to clarify some geometric phenomena from [5]. That is why we introduce the general concept of special type of nonholonomic contact elements and deduce their basic properties.

We start with the simplest type of nonholonomic jets, namely the iterated $[r, s]$-jets. The results from Section 1 frequently play the role of lemmas in the next research. In Section 2, we describe how $[r, s]$-jets generate the iterated contact elements. Then we recall the Ehresmann's idea of the category $\tilde{J}^r$ of nonholonomic r-jets. In Section 3 we introduce the concept of regular subcategory $F \subset \tilde{J}^r$. To our present knowledge, this is the most appropriate approach to the concept of special type of nonholonomic jets, that we attacked from another point of view in [3]. The contact elements of type F are studied in Section 4. In particular, the relation (19) testifies that our general construction has a reasonable geometrical meaning. In the last section we clarify that the incidence relation among the holonomic contact elements can be extended to each type of the nonholonomic ones. Finally, we present an example showing that the incidence relation is preserved under the absolute contact differentiation over submanifolds of Cartan geometries, [5].

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [6].

1. Nonholonomic $[r, s]$-jets

Consider a fibered manifold $p : Y \to M$ and write $J^r Y$ for its r-th jet prolongation. The nonholonomic $[r, s]$-jet prolongation of Y is defined by

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$$J^{r,s}Y = J^s\big(J^rY \to M\big). \tag{1}$$

If Y is the product $M \times N \to M$, the elements of $J^{r,s}(M \times N \to M) =: J^{r,s}(M,N)$ are called nonholonomic $[r,s]$-jets of M into N. The canonical injection $J^{r+s}Y \hookrightarrow J^{r,s}Y$ is of the form

$$j_x^{r+s}\sigma \mapsto j_x^s\big(j^r\sigma\big). \tag{2}$$

In particular, $J^{r+s}(M,N) \hookrightarrow J^{r,s}(M,N)$. For $s = 0$, we have $J^{r,0}(M,N) = J^r(M,N)$.

The composition of nonholonomic $[r,s]$-jets is introduced as follows. Consider $X \in J_x^{r,s}(M,N)_y$, $X = j_x^s F$, $F : M \to J^r(M,N)$ and $Z \in J_y^{r,s}(N,Q)_z$, $Z = j_y^s G$, $G : N \to J^r(N,Q)$. Write $f = \beta \circ F : M \to N$, where β is the target jet projection. Then we can construct the composition $G(f(u)) \circ F(u)$ of holonomic r-jets, $u \in M$, and we define

$$Z \circ X := j_x^s\big(G\big(f(u)\big) \circ F(u)\big) \in J_x^{r,s}(M,Q)_z. \tag{3}$$

This composition is associative. Indeed, if $W \in J_z^{r,s}(Q,P)_w$, $W = j_z^s H$, $H : Q \to J^r(Q,P)$ and $g = \beta \circ G : N \to Q$, then the associativity of the composition of holonomic r-jets implies

$$\begin{aligned} Z \circ (Y \circ X) &= j_x^s\big(H\big(g\big(f(u)\big)\big) \circ \big(G\big(f(u)\big) \circ F(u)\big)\big) \\ &= j_x^s\big(\big(H\big(g\big(f(u)\big)\big) \circ G\big(f(u)\big)\big) \circ F(u)\big) = (Z \circ Y) \circ X. \end{aligned}$$

By (2), $E_{x,M}^{r+s} = j_x^{r+s}\,\mathrm{id}_M$ is the unit at $x \in M$.

Write $\beta_1 : J^{r,s}Y \to J^rY$ for the target jet projection. The target projection $\beta : J^rY \to Y$ is extended into a map $\beta_2 := J^s\beta : J^{r,s}Y \to J^sY$. In the product case, both β_1 and β_2 preserve the jet composition, i.e. $\beta_1(Z \circ X) = \beta_1 Z \circ \beta_1 X$ and $\beta_2(Z \circ X) = \beta_2 Z \circ \beta_2 X$ with the classical composition of holonomic jets on the right-hand sides.

Proposition 1. *$X \in J_x^{r,s}(M,N)_y$ is invertible, iff both $\beta_1 X \in J_x^r(M,N)_y$ and $\beta_2 X \in J_x^s(M,N)_y$ are invertible.*

Proof. We have $X = j_x^s F(u)$. Since $\beta_1 X = F(x)$ is invertible, we can locally construct $F^{-1}(u)$. Since $\beta_2 X = j_x^s f$ is invertible, there exists locally the inverse map $\tilde{f}$ of f. Then

$$\tilde{X} := j_y^s\big(F^{-1} \circ \tilde{f}\big) \in J_y^{r,s}(N,M)_x$$

satisfies $\tilde{X} \circ X = E_{x,M}^{r+s}$ and $X \circ \tilde{X} = E_{y,N}^{r+s}$. □

Definition 1. An element $X \in J_x^{r,s}(M,N)_y$ is called regular, if there exists $Z \in J_y^{r,s}(N,M)_x$ such that $Z \circ X = E_{x,M}^{r+s}$.

For $s = 0$, this reduces to the well known fact that regular r-jets of M into N coincide with r-jets of immersions. Clearly, if X_1 and X_2 are composable regular $[r,s]$-jets, then $X_2 \circ X_1$ is regular as well.

Proposition 2. *X is regular, iff both $\beta_1 X$ and $\beta_2 X$ are regular.*

Proof. It is a direct modification of the proof of Proposition 1. □

In the holonomic case, one defines

$$L_{m,n}^r = J_0^r\big(\mathbb{R}^m, \mathbb{R}^n\big)_0 \quad \text{and} \quad L^r = \bigcup_{m,n} L_{m,n}^r,$$

[6]. Then L^r is a category over $\mathbb{N} \times \mathbb{N}$ with respect to the composition of jets. We have a left action of $G_m^r \times G_n^r$ on $L_{m,n}^r$,

$$X \mapsto h \circ X \circ g^{-1}, \qquad g \in G_m^r, \quad h \in G_n^r,\ X \in L_{m,n}^r.$$

Clearly, L^r is a skeleton of the category J^r of r-jets in that sense that $J^r(M,N)$ coincides with the associated bundle

$$J^r(M,N) = \big(P^rM \times P^rN\big)\big[L_{m,n}^r\big], \tag{4}$$

$m = \dim M$, $n = \dim N$, and the composition in L^r induces the composition of r-jets by

$$\{v, w, Z\} \circ \{u, v, X\} = \{u, w, Z \circ X\}, \tag{5}$$

$u \in P_x^rM$, $v \in P_y^rN$, $w \in P_z^rQ$, $X \in L_{m,n}^r$, $Z \in L_{n,p}^r$, $p = \dim Q$.

Quite similarly, we write

$$L^{r,s}_{m,n} = J^{r,s}_0(\mathbb{R}^m, \mathbb{R}^n)_0 \quad \text{and} \quad L^{r,s} = \bigcup_{m,n} L^{r,s}_{m,n}. \tag{6}$$

This is a category over $\mathbb{N} \times \mathbb{N}$. The jet composition defines a left action of $G^{r+s}_m \times G^{r+s}_n$ on $L^{r,s}_{m,n}$, $J^{r,s}(M,N)$ coincides with the associated bundle

$$J^{r,s}(M,N) = \left(P^{r+s}M \times P^{r+s}N\right)\left[L^{r,s}_{m,n}\right] \tag{7}$$

and (5) holds even in this case. The following assertion describes $L^{r,s}$ in terms of L^r.

Proposition 3. *We have $L^{r,s}_{m,n} = L^s_{m,n} \times T^s_m L^r_{m,n}$ with the composition*

$$Z \circ X = \left(Z_1 \circ X_1, T^s_m \varkappa^r_{m,n,p}(Z_2 \circ X_1, X_2)\right), \tag{8}$$

where $\varkappa^r_{m,n,p} : L^r_{n,p} \times L^r_{m,n} \to L^r_{m,p}$ is the composition in L^r, $X = (X_1, X_2) \in L^{r,s}_{m,n}$ and $Z = (Z_1, Z_2) \in L^{r,s}_{n,p}$.

Proof. Consider the canonical identification $J^r(\mathbb{R}^m, \mathbb{R}^n) = \mathbb{R}^m \times L^r_{m,n} \times \mathbb{R}^n$ defined by the translations on $\mathbb{R}^m$ and $\mathbb{R}^n$. Hence a section $F : \mathbb{R}^m \to J^r(\mathbb{R}^m, \mathbb{R}^n)$ is identified with a pair of maps $f_1 : \mathbb{R}^m \to \mathbb{R}^n$, $f_2 : \mathbb{R}^m \to L^r_{m,n}$, so that $X = j^s_0 F$ is identified with $(j^s_0 f_1, j^s_0 f_2) \in L^s_{m,n} \times T^s_m L^r_{m,n}$. Then (3) implies (8). □

In the case $r = s$, we can define the bundle of semiholonomic $[r,r]$-jets of M into N by

$$\bar{J}^{r,r}(M,N) = \left\{X \in J^{r,r}(M,N);\ \beta_1 X = \beta_2 X\right\}. \tag{9}$$

Clearly, $\bar{J}^{r,r}$ is a subcategory of $J^{r,r}$.

2. Iterated contact elements

Let $N \subset M$ be an n-submanifold. If $\psi : \mathbb{R}^n \to N$ is a local parametrization of N, $\psi(0) = x$, then $X = j^r_0 \psi$ is a regular (n,r)-velocity on M. The set

$$k^r_x N := X \circ G^r_n \tag{10}$$

depends on N only and is called the r-th contact element of N at x. The bundle of all contact (n,r)-elements on M is denoted by $K^r_n M$, [6]. So (10) can be expressed as $K^r_n M = \operatorname{reg} T^r_n M / G^r_n$. We write $k : \operatorname{reg} T^r_n M \to K^r_n M$ for the factor projection. If $f : M \to Q$ is an immersion, then $K^r_n f : K^r_n M \to K^r_n Q$ is defined by

$$K^r_n f\left(X \circ G^r_n\right) = T^r_n f(X) \circ G^r_n.$$

Hence K^r_n is a functor on the category of all immersions.

In the case of $p : Y \to M$, we write $\operatorname{tr} T^r_n Y \subset \operatorname{reg} T^r_n Y$ for the subset of all $X = j^r_0 \varphi$ such that $j^r_0(p \circ \varphi) \in \operatorname{reg} T^r_n M$. Further, we define $\operatorname{tr} K^r_n Y \subset K^r_n Y$ as the subset of all elements whose underlying n-plane is transversal to the fibers. In both cases, we obtain an open and dense subset. Clearly, $X \in \operatorname{tr} T^r_n Y$ if and only if $k(X) \in \operatorname{tr} K^r_n Y$. Further, we write $\gamma_1 : K^s_n(K^r_n M) \to K^r_n M$ for the bundle projection. Analogously to Section 1, we define

$$\gamma_2 : \operatorname{tr} K^s_n\left(K^r_n M \to M\right) \to K^s_n M.$$

Having in mind the coming Section 3, we introduce

$$K^{r,s}_n M := \operatorname{tr} K^s_n\left(K^r_n M \to M\right).$$

Further, we define $G^{r,s}_n = \operatorname{inv} J^{r,s}_0(\mathbb{R}^n, \mathbb{R}^n)_0 \subset G^{r+s}_n$. (Using (8), one verifies directly $G^{r,s}_n = W^s_n G^r_n$ in the notation from [6].)

Proposition 4. *We have $K^{r,s}_n M = \operatorname{reg} T^{r,s}_n M / G^{r,s}_n$.*

Proof. By Proposition 2, we have $\operatorname{reg} T^{r,s}_n M \approx \operatorname{tr} T^s_n(\operatorname{reg} T^r_n M)$. Every $X \in K^s_n(K^r_n M)$ is of the form

$$k\left(j^s_0 k\left(\varphi(u)\right)\right), \qquad \varphi : \mathbb{R}^n \to \operatorname{reg} T^r_n M,$$

so that $j^s_0 \varphi \in \operatorname{reg} T^s_n(\operatorname{reg} T^r_n M)$. We have $X \in \operatorname{tr} K^s_n(K^r_n M)$ if and only if $j^s_0 \varphi \in \operatorname{tr} T^s_n(\operatorname{reg} T^r_n M)$.

Assume $X \in \operatorname{tr} K^s_n(K^r_n M)$ and consider another $\psi(v) : \mathbb{R}^n \to \operatorname{reg} T^r_n M$ such that

$$k\left(j^s_0\left(k\psi(v)\right)\right) = k\left(j^s_0\left(k\varphi(u)\right)\right). \tag{11}$$

By locality, we may assume $M = \mathbb{R}^m$. Then $\operatorname{reg} T_n^r \mathbb{R}^m = \mathbb{R}^m \times \operatorname{reg} L_{n,m}^r$, so that $\varphi = (\varphi_1, \varphi_2)$, $\psi = (\psi_1, \psi_2)$, $\varphi_1, \psi_1 : \mathbb{R}^n \to \mathbb{R}^m$, $\varphi_2, \psi_2 : \mathbb{R}^n \to \operatorname{reg} L_{n,m}^r$. Since $k(j_0^s \varphi_1) = k(j_0^s \psi_1)$, there exists an origin preserving diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$j_0^s \psi_1 = j_0^s (\varphi_1 \circ f). \tag{12}$$

Then (11) is equivalent to $j_0^s \psi_2 = j_0^s (\varphi_2 \circ f)$. Hence there is a map $g : \mathbb{R}^n \to G_n^r$ such that

$$j_0^s \big(\psi_2(u)\big) = j_0^s \big(\varphi_2\big(f(u)\big) \circ g(u)\big). \tag{13}$$

We have $(j_0^s f, j_0^s g) \in G_n^s \times T_n^s G_n^r = G_n^{r,s}$ and (12) and (13) correspond to the action of $G_n^{r,s}$ on $\operatorname{reg} T_n^{r,s} \mathbb{R}^m$ determined by (8). $\square$

In the semiholonomic case, we define

$$\bar{K}_n^{r,r} M = \big\{ X \in \operatorname{tr} K_n^r \big(K_n^r M\big);\ \gamma_1(X) = \gamma_2(X) \big\}.$$

Then Proposition 4 implies

$$\bar{K}_n^{r,r} M = \operatorname{reg} \bar{T}_n^{r,r} M / \bar{G}_n^{r,r}, \tag{14}$$

where $\bar{T}_n^{r,r} M = \bar{J}_0^{r,r}(\mathbb{R}^n, M)$ and $\bar{G}_n^{r,r} = \operatorname{inv} \bar{J}_0^{r,r}(\mathbb{R}^n, \mathbb{R}^n)_0$.

3. The general concept of r-jet category

The bundle of nonholonomic r-jets of M into N is defined by the iteration

$$\tilde{J}^r(M, N) = J^1\big(\tilde{J}^{r-1}(M, N) \to M\big), \qquad \tilde{J}^1(M, N) = J^1(M, N).$$

The composition $Z \circ X \in \tilde{J}_x^r(M, Q)_z$ of $X = j_x^1 F \in \tilde{J}_x^r(M, N)_y$ and $Z = j_y^1 G \in \tilde{J}_y^r(N, Q)_z$ is defined by (3) with $s = 1$ and with the composition of nonholonomic $(r-1)$-jets on the right-hand side. Thus we obtain the category $\tilde{J}^r$ of nonholonomic r-jets, [2,4]. In particular, $J^{1,1} = \tilde{J}^2$. Modifying (2), we find a canonical inclusion $J^{r,s} \hookrightarrow \tilde{J}^{r+s}$. Clearly, we have

$$\tilde{J}^r(M, N \times Q) = \tilde{J}^r(M, N) \times_M \tilde{J}^r(M, Q).$$

Analogously to Section 1, we write

$$\tilde{L}_{m,n}^r = \tilde{J}_0^r\big(\mathbb{R}^m, \mathbb{R}^n\big)_0 \quad \text{and} \quad \tilde{L}^r = \bigcup_{m,n} \tilde{L}_{m,n}^r.$$

Then $\tilde{L}^r$ is the skeleton of $\tilde{J}^r$ in the fiber sense, i.e.

$$\tilde{J}^r(M, N) = \big(P^r M \times P^r N\big)\big[\tilde{L}_{m,n}^r\big]$$

and the composition in $\tilde{L}^r$ determines the composition of nonholonomic r-jets analogously to (5).

There exist r canonical projections

$$\varphi_i^r : \tilde{J}^r(M, N) \to J^1(M, N), \quad i = 1, \ldots, r.$$

By induction, we have $r-1$ projections $\varphi_k^{r-1} : \tilde{J}^{r-1}(M, N) \to J^1(M, N)$, $k = 1, \ldots, r-1$. Consider the target jet projection $\beta_{r-1} : \tilde{J}^r(M, N) \to \tilde{J}^{r-1}(M, N)$. Then we set

$$\varphi_k^r = \varphi_k^{r-1} \circ \beta_{r-1} \quad \text{and} \quad \varphi_r^r\big(j_x^1 F\big) = j_x^1(\beta \circ F) \in J^1(M, N).$$

A direct modification of Definition 1 yields the concept of regular nonholonomic r-jet. Using Propositions 1, 2 and the induction, we deduce

Proposition 5. *$X \in \tilde{J}^r(M, N)$ is regular or invertible, iff all 1-jets $\varphi_i^r(X) \in J^1(M, N)$, $i = 1, \ldots, r$, are regular or invertible, respectively.*

The following definition represents a reasonable approach to the concept of special type of nonholonomic r-jets.

Definition 2. A regular subcategory $F \subset \tilde{J}^r$ is a rule transforming every pair (M, N) of manifolds into a fibered submanifold $F(M, N) \subset \tilde{J}^r(M, N)$ such that

(i) $J^r(M, N) \subset F(M, N)$ is a fibered submanifold,

(ii) if $X \in F(M,N)_y$ and $Z \in F_y(N,Q)$, then $Z \circ X \in F(M,Q)$,
(iii) if $X \in F_x(M,N)_y$ is regular in $\tilde{J}^r$, then there exists $Z \in F_y(N,M)_x$ such that $Z \circ X = E^r_{x,M}$,
(iv) $F(M, N \times Q) = F(M,N) \times_M F(M,Q)$.

We also say that F is a nonholonomic r-jet category.

(iii) implies directly that if $X \in F(M,N)$ is invertible in $\tilde{J}^r$, then $X^{-1} \in F(N,M)$.

Consider an integer s. One verifies easily that the rule

$$(M,N) \mapsto J^s\big(F(M,N) \to M\big) \subset \tilde{J}^{r+s}(M,N)$$

defines a nonholonomic $(r+s)$-jet category, that is called the s-th jet prolongation of F.

We define

$$G^F_n = \operatorname{inv} F_0\big(\mathbb{R}^n, \mathbb{R}^n\big)_0.$$

This is a Lie subgroup of $\tilde{G}^r_n = \operatorname{inv} \tilde{J}^r_0(\mathbb{R}^n,\mathbb{R}^n)_0$. Analogously to Section 1, we write

$$L^F_{m,n} = F_0\big(\mathbb{R}^m, \mathbb{R}^n\big)_0, \qquad L^F = \bigcup_{m,n} L^F_{m,n}.$$

Then L^F is a skeleton of the category F in the fiber sense. In the case of J^sF, we find, analogously to Section 1,

$$L^{J^sF}_{m,n} = L^s_{m,n} \times T^s_m L^F_{m,n} \tag{15}$$

and the composition in L^{J^sF} is described by (8) with $\varkappa^r_{m,n,p}$ replaced by the composition map $L^F_{n,p} \times L^F_{m,n} \to L^F_{m,p}$.

We introduce the functor of (n,F)-velocities by

$$T^F_n M = F_0\big(\mathbb{R}^n, M\big), \qquad T^F_n f(X) = \big(j^r_x f\big) \circ X,$$

$f : M \to N$, $X \in F_0(\mathbb{R}^n, M)_x$. This is a bundle functor on $\mathcal{M}f$. Every $W \in L^F_{m,n}$ determines a natural transformation $T^F_n \to T^F_m$ by

$$X \mapsto X \circ W, \qquad X \in T^F_n M.$$

By (iv), every functor T^F_n preserves products.

Remark 1. In [3], we studied a concept equivalent to F under the name “total r-jet functor” and we characterized it in terms of Weil algebras.

An important example of F is the category $\bar{J}^r$ of semiholonomic r-jets. For $r = 2$, we have $\bar{J}^2 = \bar{J}^{1,1}$; for the general case we refer to [5] or [2, p. 361].

4. Contact (n, F)-elements

We define

$$K^F_n M = \operatorname{reg} T^F_n M / G^F_n.$$

Every $X \in (\operatorname{reg} T^F_n M)_x$ induces r underlying regular elements of $T^1_n M$. Each of them determines an n-plane in $T_x M$. Take an $(m-n)$-plane $Q \subset T_x M$ transversal to all of them and choose a local coordinate system $\mathbb{R}^n \times \mathbb{R}^{m-n}$ on M such that $Q = T_x \mathbb{R}^{m-n}$. Then $T^F_n M$ is locally identified with $T^F_n \mathbb{R}^n \times T^F_n \mathbb{R}^{m-n}$ and $X = (X_1, X_2)$. By Proposition 5, X_1 is regular. Using translations on $\mathbb{R}^n$, we find $\operatorname{reg} T^F_n \mathbb{R}^n = \mathbb{R}^n \times G^F_n$. Hence locally $\operatorname{reg} T^F_n M = \mathbb{R}^n \times G^F_n \times T^F_n \mathbb{R}^{m-n}$. This introduces a manifold structure on the factor space $\operatorname{reg} T^F_n M / G^F_n$.

Definition 3. $K^F_n M$ is called the bundle of contact (n,F)-elements on M.

Proposition 6. $\operatorname{reg} T^F_n M$ *is a principal bundle over* $K^F_n M$ *with structure group* G^F_n.

Proof. It remains to discuss X, $W \in \operatorname{reg} T^F_n M$ and U, $V \in G^F_n$ satisfying $X \circ U = W$, $X \circ V = W$. Since X is regular, there exists $Z \in F(M, \mathbb{R}^n)_0$ such that $Z \circ X = E^r_{0,\mathbb{R}^n}$. Hence $U = Z \circ W = V$. □

For every immersion $f : M \to Q$, we define $K^F_n f : K^F_n M \to K^F_n Q$ by

$$K_n^F f\big(X \circ G_n^F\big) = T_n^F f(X) \circ G_n^F.$$

Then K_n^F is a functor on the category of all immersions.

The contact elements determined by the s-th jet prolongation $J^s F$ are closely related with the construction of iterated contact elements. A direct modification of the proof of Proposition 4 yields the following assertion.

Proposition 7. *We have* $K_n^{J^s F} M = \operatorname{tr} K_n^s(K_n^F M \to M)$.

The iteration of K_n^1 leads to the iterated Grassmann bundles. In [5], we defined

$$\tilde K_n^r M = K_n^1\big(\tilde K_n^{r-1} M\big), \qquad \operatorname{tr} \tilde K_n^r M = \operatorname{tr} K_n^1\big(\operatorname{tr} \tilde K_n^{r-1} M \to M\big). \tag{16}$$

The repeated application of Proposition 7 yields

$$K_n^{\tilde J^r} M = \operatorname{tr} \tilde K_n^r M. \tag{17}$$

In [5], we presented a direct definition of the bundle $\bar K_n^r M$ of semiholonomic contact (n,r)-elements on M. In the case $r = 2$, we have $\bar K_n^{1,1} M = \bar K_n^2 M$. Formula (3) from [5] implies

$$\bar K_n^r M = K_n^{\bar J^r} M, \tag{18}$$

so that both approaches coincide.

Consider another regular subcategory $H \subset \tilde J^r$. Writing $F \subset H$ we always assume $F(M,N) \subset H(M,N)$ is a fibered submanifold for every M, N. Hence we have

$$T_n^F M \subset T_n^H M, \qquad G_n^F \subset G_n^H.$$

Then the rule

$$X \circ G_n^F \mapsto X \circ G_n^H, \qquad X \in T_n^F M$$

defines an injection $K_n^F M \to K_n^H M$. In particular, $J^r \subset F \subset \tilde J^r$ yields a remarkable formula

$$K_n^r M \subset K_n^F M \subset \operatorname{tr} \tilde K_n^r M. \tag{19}$$

5. The incidence relation for contact elements

In the classical situation, a p-submanifold $P \subset M$ is said to have r-th order contact with an n-submanifold $N \subset M$ at a common point x, $p \leqslant n$, if there exists a p-submanifold $\bar P \subset N$ such that $k_x^r P = k_x^r \bar P$. This can be formalized in the following way.

Definition 4. We say that two contact elements $Q = X \circ G_p^r \in (K_p^r M)_x$ and $S = Z \circ G_n^r \in (K_n^r M)_x$ are incident, and we write $Q \,\varepsilon\, S$, if there exists $W \in \operatorname{reg} L_{p,n}^r$ such that $Z \circ W = X$.

This concept can be directly extended to arbitrary F. For $Q = X \circ G_p^F \in (K_p^F M)_x$ and $S = Z \circ G_n^F \in (K_n^F M)_x$, $Q \,\varepsilon\, S$ means that there exists $W \in \operatorname{reg} L_{p,n}^F$ such that $Z \circ W = X$. One verifies directly that ε is a transitive relation.

Example. Consider a Cartan space $\mathcal{S}(M) = (P, \Gamma, E, s)$, [5]. The absolute contact (n,r)-differentiation introduced in [5] is a map

$$\Gamma_n^r : K_n^r M \to \bigcup_{x \in M} \big(\bar K_n^r(E_x)\big)_{s(x)}.$$

Analyzing the constructions from [5], we deduce: If $Q \in (K_p^r M)_x$ and $S \in (K_n^r M)_x$ are incident, then the absolute contact differentials $\Gamma_p^r(Q) \in (\bar K_p^r(E_x))_{s(x)}$ and $\Gamma_n^r(S) \in (\bar K_n^r(E_x))_{s(x)}$ are incident in the semiholonomic sense.

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