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Numerical solutions of the Burgers' equation by the least-squares quadratic B-spline finite element method

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Abstract

In this study, a least-squares quadratic B-spline finite element method for calculating the numerical solutions of the one-dimensional Burgers-like equations with appropriate boundary and initial conditions is presented. Three test problems have been studied to demonstrate the accuracy of the present method. Results obtained by the method have been compared with the exact solution of each problem and are found to be in good agreement with each other. A Fourier stability analysis of the method is also investigated.

MSC: 65N06

Keywords: Burgers' equation; Least-squares finite element; Quadratic B-spline

1. Introduction

The one-dimensional Burgers' equation

$$U_t + UU_x = vU_{xx}, \quad a < x < b, \; t > 0, \tag{1}$$

where $v > 0$ is the coefficient of kinematic and the subscripts x and t denote differentiation, plays a major role in the study of nonlinear waves since it is used as a mathematical model in turbulence problems, in the theory of shock waves and in continuous stochastic processes [7].

Eq. (1), which was first introduced by Bateman [3] and later treated by Burgers [5] and after whom such an equation widely referred to as Burgers' equation, is one of a few well-known nonlinear partial differential equations, which have been solved analytically for a restricted set of arbitrary

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initial conditions [7,11]. Benton and Platzman [4] surveyed exact solutions of the one-dimensional Burgers-like equations. In many cases, these solutions involve infinite series which may converge very slowly for small values of $v > 0$ [14]. However, various numerical techniques specially based on finite difference, finite element and boundary element methods have been applied to solve numerically Eq. (1) under the following boundary conditions:

$$\begin{aligned} U(a,t) &= f_1(t), \quad t > 0, \\ U(b,t) &= f_2(t), \quad t > 0, \end{aligned} \tag{2}$$

and the initial condition:

$$U(x,0) = g(x), \quad a < x < b, \tag{3}$$

where $f_1(t)$, $f_2(t)$ and $g(x)$ are the prescribed functions of the variables.

In order to solve the Burgers' equation numerically, Varoglu and Finn [22] used a new finite element method based on a weighted residual formulation, Caldwell and Smith [6] finite difference and cubic spline finite element methods, Evans and Abdullah [8] alternating group explicit methods, Kakuda and Tosaka [12] the generalized boundary element approach, Ali et al. [2] a cubic B-spline finite element method based on a collocation formulation, Nguyen and Rynen [17] a linear space-time finite element method based on a least-squares approach, Mittal and Singhal [15,16] a technique of finitely reproducing nonlinearities to get a system of nonlinear differential equations, which are solved by a Runge–Kutta–Chebyshev method. Gardner et al. [9] used a Petrov–Galerkin method by a quadratic B-spline spatial finite elements, and they also used a least-squares technique using linear space-time finite elements in [10]. Özis and Özdes [19] used a direct variational method to generate an approximation solution in the form of a sequence solution. Recently, Kutluay et al. [13] proposed the exact-explicit finite difference method to the Burgers-like problems to obtain numerical solutions of adequate accuracy. More recently, Abd-el-Malek and El-Mansi [1] have used the group-theoretic methods for calculating the solution of Burgers' equation with appropriate boundary and initial conditions.

In this paper, we have applied a least-squares quadratic B-spline finite element method based on the work of Nguyen and Reynen [18] to the Burgers' equation (1) with a set of boundary and initial conditions given by Eqs. (2) and (3) to obtain its numerical solutions. The second-order Burgers' equation is reduced to a pentadiagonal matrix system by applying the classical weighted residual method over the finite elements, which can be solved by a variant of Thomas algorithm, together with an iteration process at each time step. In order to demonstrate the accuracy of the present method and make a comparison of numerical solutions with exact ones, we have chosen three test problems given in the following section so that each of them has an exact solution.

2. Statement of problems

We consider the Burgers' equation (1) with the boundary conditions $U(a,t) = U(b,t) = 0$ and the initial condition $U(x,0) = g(x)$ for a finite interval $a \leqslant x \leqslant b$.

Problem (a). We first take Eq. (1) with the homogeneous boundary conditions

$$U(0,t)=0, \quad t>0,$$

$$U(1,t)=0, \quad t>0, \tag{4}$$

and the initial condition

$$U(x,0)=\sin(\pi x), \quad 0<x<1.$$

The (exact) Fourier series solution of this problem given by Cole [7] is

$$U(x,t)=2\pi v\,\frac{\sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2 vt)n\sin(n\pi x)}{a_0+\sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2 vt)\cos(n\pi x)}, \tag{5}$$

where the Fourier coefficients are

$$a_0=\int_0^1 \exp\{-(2\pi v)^{-1}[1-\cos(\pi x)]\}\,\mathrm{d}x,$$

$$a_n=2\int_0^1 \exp\{-(2\pi v)^{-1}[1-\cos(\pi x)]\}\cos(n\pi x)\,\mathrm{d}x \quad (n=1,2,3,\ldots).$$

Problem (b). We secondly consider Eq. (1) with the boundary conditions given by Eq. (4) and the initial condition for this problem is

$$U(x,0)=4x(1-x), \quad 0<x<1.$$

The exact solution of this problem is given by Eq. (5), but in this case the Fourier coefficients are

$$a_0=\int_0^1 \exp\{-x^2(3v)^{-1}(3-2x)\}\,\mathrm{d}x,$$

$$a_n=2\int_0^1 \exp\{-x^2(3v)^{-1}(3-2x)\}\cos(n\pi x)\,\mathrm{d}x \quad (n=1,2,3,\ldots).$$

Problem (c). As our last test problem, consider Eq. (1) with the boundary conditions

$$U(a,t)=0, \quad t>0,$$

$$U(b,t)=0, \quad t>0,$$

and the initial condition at time $t=1$ given by

$$U(x,1)=\frac{x}{1+\exp[\frac{1}{4v}(x^2-\frac{1}{4})]}.$$

This problem has an exact solution of form [17]

$$U(x,t)=\frac{x/t}{1+(t/t_0)^{1/2}\exp(x^2/4vt)}, \quad t\geqslant 1,$$

where $t_0=\exp(1/8v)$.

3. Method of solution

The finite interval $[a,b]$ is partitioned into N finite elements of equal length h by the nodes x_i $(i=0,1,2,\ldots,N)$ such that $a=x_0<x_1<x_2<\cdots<x_N=b$ and $h(\equiv \Delta x)=x_i-x_{i-1}=(b-a)/N$.

The quadratic B-splines ϕ_m $(m=-1,0,\ldots,N)$ which form a basis over the interval $[a,b]$ are defined by the relationships [20]

$$\phi_m(x)=\frac{1}{h^2}\begin{cases}(x_{m+2}-x)^2-3(x_{m+1}-x)^2+3(x_m-x)^2, & [x_{m-1},x_m],\\ (x_{m+2}-x)^2-3(x_{m+1}-x)^2, & [x_m,x_{m+1}],\\ (x_{m+2}-x)^2, & [x_{m+1},x_{m+2}],\\ 0, & \text{otherwise.}\end{cases} \tag{6}$$

Each finite element $[x_m,x_{m+1}]$ is covered by three quadratic B-splines since each quadratic B-spline covers three elements. In each element, using the local coordinate transformation

$$x=x_m+\eta\Delta x,\quad 0\leqslant\eta\leqslant 1,$$
$$t=\tau\Delta t,\quad 0\leqslant\tau\leqslant 1, \tag{7}$$

Eq. (6) leads to the quadratic B-spline shape functions having representations over the element $[x_m,x_{m+1}]$ as

$$\begin{matrix}\phi_{m-1}\\ \phi_m\\ \phi_{m+1}\end{matrix} = \begin{cases}(1-\eta)^2,\\ 1+2\eta-2\eta^2,\ 0\leqslant\eta\leqslant 1,\\ \eta^2,\end{cases} \tag{8}$$

where $\Delta t(\equiv k)$ is the time step.

We seek the approximation $U_N(x,t)$ to the solution $U(x,t)$, which uses these quadratic B-splines as trial functions [18]

$$U_N(\eta,\tau)=\sum_{j=m-1}^{m+1}\phi_j(\eta)(\sigma_j+\tau\Delta\sigma_j). \tag{9}$$

Applying the least-squares approach to Eq. (1) in time and space leads to the integral equation

$$\delta\int_0^t\int_a^b(U_t+UU_x-vU_{xx})^2\,\mathrm{d}x\,\mathrm{d}t=0. \tag{10}$$

Using the transformation (7), Eq. (10) becomes

$$\delta\int_0^1\int_0^1\left(U_\tau+\hat{U}\,\frac{\Delta t}{\Delta x}\,U_\eta-v\,\frac{\Delta t}{\Delta x^2}\,U_{\eta\eta}\right)^2\mathrm{d}\eta\,\mathrm{d}\tau=0, \tag{11}$$

where $\hat{U}$ is taken to be a constant over the element $[x_m, x_{m+1}]$ to simplify the integral. From the variational principle, Eq. (11) can be written as

$$\int_0^1 \int_0^1 [U_\tau + \alpha U_\eta - \beta U_{\eta\eta}]\delta[U_\tau + \alpha U_\eta - \beta U_{\eta\eta}]\,\mathrm{d}\eta\,\mathrm{d}\tau = 0, \tag{12}$$

where $\alpha = \hat{U}(\Delta t/\Delta x)$ and $\beta = v(\Delta t/\Delta x^2)$.

The nodal values and their first derivatives at the knot x_m are given in terms of the parameters σ_m by

$$U_m = U(x_m) = \sigma_{m-1} + \sigma_m, \tag{13}$$

$$U'_m = U'(x_m) = \frac{2}{\Delta x}(\sigma_m - \sigma_{m-1}), \tag{14}$$

where "$'$" denotes differentiation with respect to η.

In Eq. (12), the term $\delta[U_\tau + \alpha U_\eta - \beta U_{\eta\eta}]$ can be considered as a weighting function w. The variation of U given by Eq. (9) over the element $[x_m, x_{m+1}]$ is

$$\delta U^e = \sum \phi_j(\eta)\tau\Delta\sigma_j.$$

Hence, weighting functions w_i can be obtained as

$$w_i = \phi_i + \alpha\tau\phi'_i - \beta\tau\phi''_i, \quad i = m-1,\ m,\ m+1. \tag{15}$$

Substitution of Eq. (15) into Eq. (12), we obtain the least-squares space-time weak formulation

$$\int_0^1 \int_0^1 (U_\tau + \alpha U_\eta - \beta U_{\eta\eta})(\phi_i + \alpha\tau\phi'_i - \beta\tau\phi''_i)\,\mathrm{d}\eta\,\mathrm{d}\tau = 0, \tag{16}$$

which can also be regarded as a Petrov–Galerkin method with weighting functions w_i given by Eq. (15).

Substituting expression (9) in Eq. (16), with some manipulation, leads to

$$\sum_{j=m-1}^{m+1} \left\{ \int_0^1 \left[\phi_i\phi_j + \frac{\alpha}{2}(\phi_i\phi'_j + \phi'_i\phi_j) + \left(\frac{\alpha^2}{3} + \beta\right)\phi'_i\phi'_j - \frac{\alpha\beta}{3}(\phi'_i\phi''_j + \phi''_i\phi'_j) \right.\right.$$
$$\left.\left. + \frac{\beta^2}{2}\phi''_i\phi''_j \right] \mathrm{d}\eta - \frac{\beta}{2}(\phi_i\phi'_j + \phi'_i\phi_j)\Big|_0^1 \right\} \Delta\sigma_j + \sum_{j=m-1}^{m+1} \left\{ \int_0^1 \left[\alpha\phi_i\phi'_j + \left(\frac{\alpha^2}{2} + \beta\right)\phi'_i\phi'_j \right.\right.$$
$$\left.\left. - \frac{\alpha\beta}{2}(\phi'_i\phi''_j + \phi''_i\phi'_j) + \frac{\beta^2}{2}\phi''_i\phi''_j \right] \mathrm{d}\eta - \beta\phi_i\phi'_j\Big|_0^1 \right\} \sigma_j = 0,$$

which can also be written in matrix form as

$$\sum_{j=m-1}^{m+1}\left\{A^e+\frac{\alpha}{2}(B^e+(B^e)^{\mathrm{T}})+\left(\frac{\alpha^2}{3}+\beta\right)C^e-\frac{\alpha\beta}{3}(D^e+(D^e)^{\mathrm{T}})\right.$$
$$\left.+\frac{\beta^2}{3}E^e-\frac{\beta}{2}(\phi_i\phi_j'+\phi_i'\phi_j)\Big|_0^1\right\}\Delta\sigma^e+\sum_{j=m-1}^{m+1}\left\{\alpha B^e+\left(\frac{\alpha^2}{2}+\beta\right)C^e\right.$$
$$\left.-\frac{\alpha\beta}{2}(D^e+(D^e)^{\mathrm{T}})+\frac{\beta^2}{2}E^e-\beta\phi_i\phi_j'\Big|_0^1\right\}\sigma^e=0. \quad (17)$$

In the above equation, $\sigma^e=(\sigma_{m-1},\sigma_m,\sigma_{m+1})$ are the element parameters and A^e, B^e, C^e, D^e and E^e are the element matrices which are to be determined by Eq. (8) as follows:

$$A_{ij}^e=\int_0^1\phi_i\phi_j\,\mathrm{d}\eta=\frac{1}{30}\begin{bmatrix}6&13&1\\13&54&13\\1&13&6\end{bmatrix},\quad B_{ij}^e=\int_0^1\phi_i\phi_j'\,\mathrm{d}\eta=\frac{1}{6}\begin{bmatrix}-3&2&1\\-8&0&8\\-1&-2&3\end{bmatrix},$$

$$C_{ij}^e=\int_0^1\phi_i'\phi_j'\,\mathrm{d}\eta=\frac{2}{3}\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix},\quad D_{ij}^e=\int_0^1\phi_i'\phi_j''\,\mathrm{d}\eta=2\begin{bmatrix}-1&2&-1\\0&0&0\\1&-2&1\end{bmatrix},$$

$$E_{ij}^e=\int_0^1\phi_i''\phi_j''\,\mathrm{d}\eta=4\begin{bmatrix}1&-2&1\\-2&4&-2\\1&-2&1\end{bmatrix}.$$

Combining together the contributions from all elements, Eq. (17) leads to the system of equations

$$\left[A+\frac{\alpha}{2}(B+B^{\mathrm{T}})+\left(\frac{\alpha^2}{3}+\beta\right)C-\frac{\alpha\beta}{3}(D+D^{\mathrm{T}})+\frac{\beta^2}{3}E-\frac{\beta}{2}(\phi_i\phi_j'+\phi_i'\phi_j)\Big|_0^1\right]\Delta\sigma$$
$$+\left[\alpha B+\left(\frac{\alpha^2}{2}+\beta\right)C-\frac{\alpha\beta}{2}(D+D^{\mathrm{T}})+\frac{\beta^2}{2}E-\beta\phi_i\phi_j'\Big|_0^1\right]\sigma=0, \quad (18)$$

where $\sigma=(\sigma_{-1},\sigma_0,\dots,\sigma_N)^{\mathrm{T}}$, $\alpha=(\Delta t/\Delta x)(\sigma_{j-1}+\sigma_j)$ and A, B, C, D and E are assembling matrices which are derived from the element matrices A^e, B^e, C^e, D^e and E^e, respectively.

Identifying $\sigma=\sigma^n$, $\Delta\sigma=\sigma^{n+1}-\sigma^n$ and using in Eq. (18), we obtain the $(N+2)\times(N+2)$ pentadiagonal matrix system

$$\left[A+\frac{\alpha}{2}(B+B^{\mathrm{T}})+\left(\frac{\alpha^2}{3}+\beta\right)C-\frac{\alpha\beta}{3}(D+D^{\mathrm{T}})+\frac{\beta^2}{3}E-\frac{\beta}{2}(\phi_i\phi_j'+\phi_i'\phi_j)\Big|_0^1\right]\sigma^{n+1}$$
$$=\left[A+\frac{\alpha}{2}(B^{\mathrm{T}}-B)-\frac{\alpha^2}{6}C+\frac{\alpha\beta}{6}(D+D^{\mathrm{T}})-\frac{\beta^2}{6}E+\frac{\beta}{2}(\phi_i\phi_j'-\phi_i'\phi_j)\Big|_0^1\right]\sigma^n. \quad (19)$$

Using the boundary conditions (2) in system (19) leads to an $N \times N$ matrix system which can be solved by the Thomas algorithm. Two or three inner iterations are applied to $\sigma^{n\star} = \sigma^n + \frac{1}{2}(\sigma^n - \sigma^{n-1})$ at each time in order to improve the accuracy.

Utilising relations $U_N(x_i, 0) = U(x_i, 0)$, $(i = 0, 1, \ldots, N)$ together with an extra condition, which can be obtained as $U'(x_0, 0) = U_N'(x_0, 0)$ since the first derivative of the approximate initial condition shall agree with that of the exact initial condition, initial vector σ^0 can be determined from the matrix equation

$$\begin{bmatrix} -1 & 1 & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & & \ddots & \\ & & & 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{-1}^0 \\ \sigma_0^0 \\ \vdots \\ \sigma_{N-1}^0 \\ \sigma_N^0 \end{bmatrix} = \begin{bmatrix} U'(x_0, 0) \\ U(x_0, 0) \\ \vdots \\ U(x_{N-1}, 0) \\ U(x_N, 0) \end{bmatrix},$$

which can be solved using a variant of the Thomas algorithm. Hence, the approximate solution function $U(x,t)$ can be recovered from σ^n using Eqs. (13) and (14) if required.

3.1. Stability analysis

A typical member of Eq. (19) in terms of the nodal parameters σ_m^n is

$$\begin{aligned} &\alpha_1\sigma_{m-2}^{n+1} + \alpha_2\sigma_{m-1}^{n+1} + \alpha_3\sigma_m^{n+1} + \alpha_2\sigma_{m+1}^{n+1} + \alpha_1\sigma_{m+2}^{n+1} \\ &\quad = \alpha_4\sigma_{m-2}^n + \alpha_5\sigma_{m-1}^n + \alpha_6\sigma_m^n + \alpha_7\sigma_{m+1}^n + \alpha_8\sigma_{m+2}^n, \end{aligned} \tag{20}$$

where

$$\alpha_1 = \frac{1}{30} - \frac{2}{3}\left(\frac{\alpha^2}{3} + \beta\right) + \frac{4\beta^2}{3}, \quad \alpha_2 = \frac{26}{30} - \frac{4}{3}\left(\frac{\alpha^2}{3} + \beta\right) - \frac{16\beta^2}{3},$$

$$\alpha_3 = \frac{66}{30} + 4\left(\frac{\alpha^2}{3} + \beta\right) + 8\beta^2, \quad \alpha_4 = \frac{1}{30} + \frac{\alpha}{6} + \frac{\alpha^2}{9} - \frac{2\beta^2}{3},$$

$$\alpha_5 = \frac{26}{30} + \frac{5\alpha}{3} + \frac{2\alpha^2}{9} + \frac{8\beta^2}{3}, \quad \alpha_6 = \frac{66}{30} - \frac{2\alpha^2}{3} - 4\beta^2,$$

$$\alpha_7 = \frac{26}{30} - \frac{5\alpha}{3} + \frac{2\alpha^2}{9} + \frac{8\beta^2}{3}, \quad \alpha_8 = \frac{1}{30} - \frac{\alpha}{6} + \frac{\alpha^2}{9} - \frac{2\beta^2}{3}.$$

For stability analysis it is convenient to use the Fourier method (see, e.g., [21]). Substituting the Fourier mode $\sigma_m^n = \xi^n e^{i\theta mh}$, $(i = \sqrt{-1})$ into scheme (20) gives the growth factor ξ of the form

$$\xi = \frac{a + ib}{c},$$

where

$$a = (\alpha_4 + \alpha_8)\cos(2\theta h) + (\alpha_5 + \alpha_7)\cos(\theta h) + \alpha_6,$$

$$b = (\alpha_8 - \alpha_4)\sin(2\theta h) + (\alpha_7 - \alpha_5)\sin(\theta h)$$

Table 1
Comparison of results at $t_f = 0.1$ for $v = 1$, $k = 0.00001$ and various mesh sizes

	Numerical					Exact
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.0125$	$h = 0.00625$	
0.1	0.11132	0.11051	0.11003	0.10978	0.10965	0.10954
0.2	0.21293	0.21141	0.21060	0.21019	0.20998	0.20979
0.3	0.29572	0.29386	0.29288	0.29238	0.29213	0.29190
0.4	0.35203	0.35004	0.34899	0.34845	0.34818	0.34792
0.5	0.37575	0.37375	0.37267	0.37212	0.37185	0.37158
0.6	0.36317	0.36122	0.36015	0.35960	0.35932	0.35905
0.7	0.31380	0.31198	0.31096	0.31044	0.31017	0.30991
0.8	0.23110	0.22958	0.22872	0.22827	0.22805	0.22782
0.9	0.12259	0.12177	0.12125	0.12097	0.12083	0.12069
$\|e\|_1$	0.012165	0.006941	0.003651	0.001858	0.000928	

and

$$c = 2\alpha_1 \cos(2\theta h) + 2\alpha_2 \cos(\theta h) + \alpha_3.$$

For the stability, ξ must be satisfied the inequality $|\xi| \leqslant 1$ which is mathematically equivalent to $|a+\mathrm{i}b|^2 - |c|^2 \leqslant 0$. By virtue of any algebraic package programme, it is seen that $|a+\mathrm{i}b|^2 - |c|^2 \leqslant 0$ for $\alpha > 0$ and $\beta > 0$. So, scheme (20) is unconditionally stable.

4. Numerical result and conclusions

All calculations were performed in double precision arithmetic on a Pentium 4 processor using Microsoft Fortran Compiler. The least-squares quadratic B-spline solution of the Burgers' equation leads to pentadiagonal matrix system, which is solved easily by using the Thomas algorithm.

In order to show how good the numerical solutions of the above problems in comparison with the exact ones, we shall use the weighted 1-norm $\|e\|_1$ defined by

$$\|e\|_1 = \frac{1}{N}\sum_{i=1}^{N-1}\left|\frac{U(x_i,t_j) - U_{i,j}}{U(x_i,t_j)}\right|, \quad e = [e_1, e_2, \ldots, e_{N-1}]^{\mathrm{T}}.$$

Tables 1 and 2 display the numerical solutions of problem (a) for different values of mesh sizes and viscosity coefficients, respectively. It is observed that the numerical solutions are seen to be satisfactorily in agreement with the exact ones, and exhibit the expected convergence as the grid size h is refined.

The numerical solutions of problem (b) obtained by the present method have been compared with the exact solution in Table 3 for $v = 1.0,\ 0.1,\ 0.01$ with $h = 0.0125$ and $k = 0.0001$ at times from $t_f = 0.4$ to 3.0. Again, good agreement with exact values is evident, as is convergence.

The numerical solutions of problem (c) are given in Table 4 for $v = 0.5$, $[a, b] = [0, 8]$ with $h = 0.05$ and $k = 0.0001$ at $t_f = 1.5,\ 3.0$ and 4.5. The agreement between our numerical results and the exact solution is satisfactorily good. Since both solutions hit each other after $x = 5.0$, they are not given in Table 4.

Table 2
Comparison of results at different times for $v = 1.0$, 0.1 and 0.01 with $h = 0.0125$ and $k = 0.0001$

		$v = 1.0$		$v = 0.1$		$v = 0.01$	
x	t_f	Numerical	Exact	Numerical	Exact	Numerical	Exact
0.25	0.4	0.01359	0.01357	0.31215	0.30889	0.34819	0.34191
	0.6	0.00189	0.00189	0.24360	0.24074	0.27536	0.26896
	0.8	0.00026	0.00026	0.19815	0.19568	0.22752	0.22148
	1.0	0.00004	0.00004	0.16473	0.16256	0.19375	0.18819
	3.0	0.00000	0.00000	0.02771	0.02720	0.07754	0.07511
0.50	0.4	0.01927	0.01924	0.57293	0.56963	0.66543	0.66071
	0.6	0.00268	0.00267	0.45088	0.44721	0.53525	0.52942
	0.8	0.00037	0.00037	0.36286	0.35924	0.44526	0.43914
	1.0	0.00005	0.00005	0.29532	0.29192	0.38047	0.37442
	3.0	0.00000	0.00000	0.04097	0.04021	0.15362	0.15018
0.75	0.4	0.01365	0.01363	0.63038	0.62544	0.91201	0.91026
	0.6	0.00189	0.00189	0.49268	0.48721	0.77132	0.76724
	0.8	0.00026	0.00026	0.37912	0.37392	0.65254	0.64740
	1.0	0.00004	0.00004	0.29204	0.28747	0.56157	0.55605
	3.0	0.00000	0.00000	0.03038	0.02977	0.22874	0.22481

Table 3
Comparison of results at different times for $v = 1.0$, 0.1 and 0.01 with $h = 0.0125$ and $k = 0.0001$

		$v = 1.0$		$v = 0.1$		$v = 0.01$	
x	t_f	Numerical	Exact	Numerical	Exact	Numerical	Exact
0.25	0.4	0.01403	0.01400	0.32091	0.31752	0.36911	0.36226
	0.6	0.00195	0.00195	0.24910	0.24614	0.28905	0.28204
	0.8	0.00027	0.00027	0.20211	0.19956	0.23703	0.23045
	1.0	0.00004	0.00004	0.16782	0.16560	0.20069	0.19469
	3.0	0.00000	0.00000	0.02828	0.02776	0.07865	0.07613
0.50	0.4	0.01988	0.01985	0.58788	0.58454	0.68818	0.68368
	0.6	0.00276	0.00276	0.46174	0.45798	0.55425	0.54832
	0.8	0.00038	0.00038	0.37111	0.36740	0.46011	0.45371
	1.0	0.00005	0.00005	0.30183	0.29834	0.39206	0.38568
	3.0	0.00000	0.00000	0.04185	0.04107	0.15576	0.15218
0.75	0.4	0.01409	0.01407	0.65054	0.64562	0.92194	0.92050
	0.6	0.00195	0.00195	0.50825	0.50268	0.78676	0.78299
	0.8	0.00027	0.00027	0.39068	0.38534	0.66777	0.66272
	1.0	0.00004	0.00004	0.30057	0.29586	0.57491	0.56932
	3.0	0.00000	0.00000	0.03106	0.03044	0.23183	0.22774

Table 4
Comparison of results at different times for $v = 0.5$ and $[a, b] = [0, 8]$ with $h = 0.05$ and $k = 0.0001$

	$t_f = 1.5$		$t_f = 3.0$		$t_f = 4.5$	
x	Numerical	Exact	Numerical	Exact	Numerical	Exact
0.5	0.15398	0.15327	0.06468	0.06426	0.03825	0.03799
1.0	0.26634	0.26577	0.11942	0.11880	0.07231	0.07187
1.5	0.30451	0.30412	0.15576	0.15509	0.09847	0.09793
2.0	0.26190	0.26142	0.16832	0.16762	0.11399	0.11339
2.5	0.17268	0.17217	0.15699	0.15630	0.11761	0.11698
3.0	0.08839	0.08807	0.12803	0.12738	0.11011	0.10949
3.5	0.03594	0.03582	0.09185	0.09134	0.09425	0.09369
4.0	0.01189	0.01186	0.05834	0.05798	0.07409	0.07361
4.5	0.00325	0.00325	0.03305	0.03284	0.05367	0.05330
5.0	0.00074	0.00074	0.01684	0.01674	0.03597	0.03572

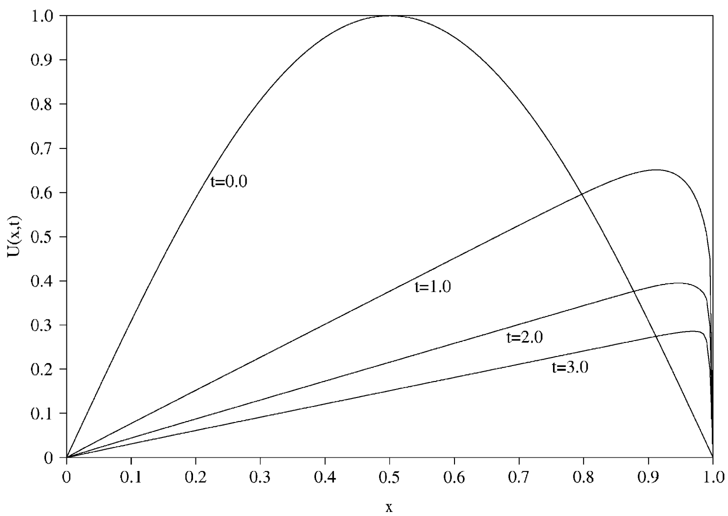

Fig. 1. Solutions at different times for $v = 0.001$, $h = 0.005$, $k = 0.125$.

It is known that the exact solutions for $v < 0.01$ fail because of the slow convergence of the infinite series (see, e.g., [14]). Therefore, these results are not compared to the exact solutions. Nevertheless, in order to show how good the numerical predictions of the problems (a) and (b) for $v < 0.01$ exhibit the correct physical behaviour of problem we give the graphs in Figs. 1 and 2, which show the numerical solutions at different times for $v = 0.001$ with $k = 0.125$ and $h = 0.005$.

Fig. 3 illustrates the numerical and exact solutions of problem (c) at different values of t for $v = 0.005$ with $h = 0.012$ and $k = 0.05$. Both solutions of the problem are drawn on the same diagram, but curves cannot be distinguishable since they are very close to each other.

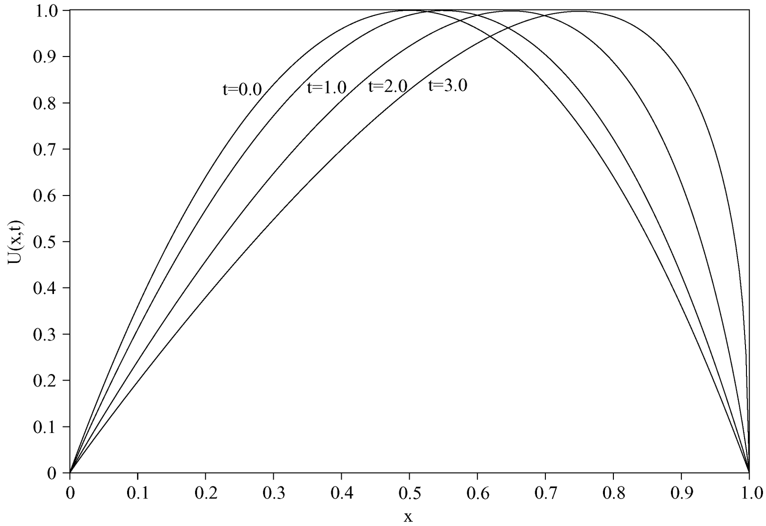

Fig. 2. Solutions at different times for $v = 0.001$, $h = 0.005$, $k = 0.125$.

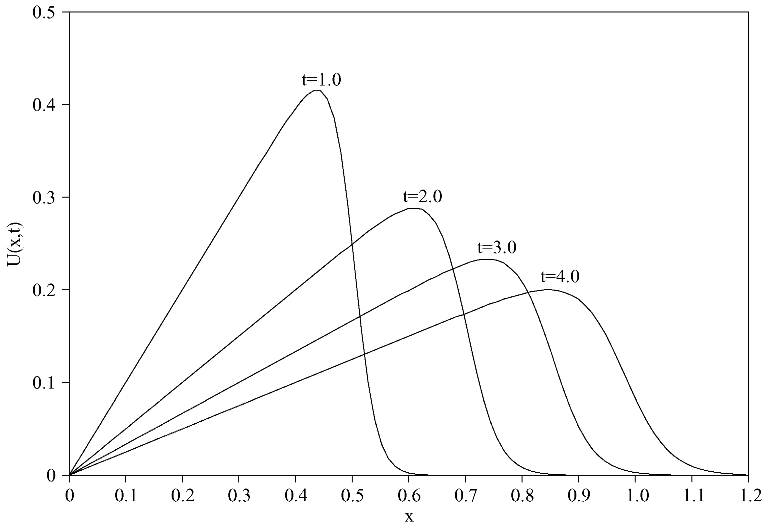

Fig. 3. Solutions at different times for $v = 0.005$, $h = 0.012$, $k = 0.05$.

In this paper, a numerical method based on the least-squares finite element method using quadratic B-splines as trial functions has been presented to find numerical solutions of Burgers-like equations with a set of boundary and initial conditions. It is known that the use of higher order of B-splines in the numerical methods provides least errors theoretically. However, the use of higher-order B-spline functions increases computational complexity. For instance, if the cubic B-splines were used in place of the quadratic ones, the second-order Burgers' equation considered in this study would reduce to a septadiagonal matrix system which is rather expensive computationally.

We conclude that a least-squares quadratic B-spline finite element method is capable of solving Burgers-like equations since it produces reasonably good results, even for small values of viscosity.

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