Generalizing the Recursion Relationship for the Partition Function

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The known recursion relationship for the partition function \( p(n) \) which represents the number of partitions of the positive integer \( n \) is exhibited as the limit as \( q \to \infty \) in one identity and as the case 1 substituted for \( q \) in a second formula that arise from a matrix problem over the field of \( q \) elements. Such identities can be further generalized. © 1997 Academic Press

1. INTRODUCTION

The partition function \( p(n) \) is determined by the following identity:

\[
1 + \sum_{n=1}^{\infty} p(n) t^n = \prod_{n=1}^{\infty} (1 - t^n)^{-1}.
\]

Differentiation of this followed by multiplication by \( t \) yields

\[
p(n) = \frac{1}{n} \sum_{r=1}^{n} \sigma(r) p(n - r),
\]

where \( \sigma(r) \) represents the sum of divisors of \( r \).

In this paper we write \( \lambda \vdash n \) to denote “\( \lambda \) is a partition of \( n \)” \( \lambda \vdash n \) to denote the largest part of \( \lambda \), and set \( n = |\lambda| \). For positive integer \( r \), let \( \psi_r(q) \) denote the polynomial \((1 - q)(1 - q^2)\cdots(1 - q^r)\) and set \( \psi_0(q) = 1 \). Given a partition \( \mu = (1^n 2^\cdot \cdot \cdot) \), we let \( \beta_\mu(q) = \prod_{r \geq 1} \psi_{\mu_r}(q) \), and define

\[
P_\mu(n, q) = \sum_{\mu \vdash n} \frac{1}{\beta_\mu(q^{-1})}.
\]

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Thus $\lim_{q \to \infty} P_1(n, q) = p(n)$. Likewise we define

$$H_1(n, q) = \sum_{d|n} \frac{1}{d} 1 - q^d,$$

thus $\lim_{q \to \infty} H_1(n, q) = \sum_{d|n} n/d = \sigma(n)$.

Two generalizations of formula (1) are given in this paper. One is the following:

$$P_1(n, q) = \frac{1}{n} \sum_{r=1}^n H_1(r, q) P_1(n-r, q),$$

which turns out to be formula (1) when $q \to \infty$. The other is

$$M_1(n, q) = \sum_{r=1}^n E_1(r, q) M_1(n-r, q),$$

where $M_1(n, q) = \sum_{i=0}^n q^{i^2}$, $E_1(n, q) = \sum_{d|n} dq^{d^2}$. It turns out to be formula (1) when $q$ is substituted by 1. More general functions $P_g(n, q)$, $H_g(n, q)$, $M_g(n, q)$ and $E_g(n, q)$ for $g \geq 1$ are introduced later, and satisfy similar formulae.

2. THE MATRIX PROBLEM

Let $g$ be a fixed non-negative integer, $q$ a prime power, $F_q$ the field of $q$ elements, $\overline{F}_q$ the algebraic closure of $F_q$. Let $n$ be any positive integer, $G_n$ be the general linear group of degree $n$ over $F_q$, and $C_n$ be a subset of $G_n$ containing one element from each conjugacy class of $G_n$. Let $M_g(n, q)$ denote the number of classes of ordered $g$-tuples of $n \times n$ matrices over $F_q$ up to simultaneous similarity. In this context the Molien-Burnside orbit counting formula becomes:

$$M_g(n, q) = \frac{1}{|G_n|} \sum_{\gamma \in G_n} |X_{\gamma}|^g = \sum_{\gamma \in C_n} |X_{\gamma}|^g,$$

where $X_{\gamma} = \{ x \in M_g(F_q) \mid \gamma x \gamma^{-1} = x \}$, $Z_{\gamma} = \{ x \in G_n \mid \gamma x \gamma^{-1} = x \}$.

Let $\Phi$ denote the set of all irreducible monic polynomials in $t$ over $F_q$, other than $t$, and let $P$ denote the set of partitions of positive integers. For each $f \in \Phi$, let $d(f)$ denote the degree of $f$, $J(f)$ denote the companion matrix of $f$ (see Macdonald [4], page 140), and for $m \geq 1$ let $J_m(f)$ denote the Jordan block matrix consisting of $m^2$ block $d(f) \times d(f)$ matrices with $J(f)$ in each diagonal block. For any partition $\pi = (1^{n_1} 2^{n_2} \cdots) \in P$, let

$$J(f, \pi) = J_1(f)^{n_1} \oplus J_2(f)^{n_2} \oplus \cdots.$$
This is a diagonal block matrix with \( n_i \) copies of \( J_i(f) \) in the diagonal. Then any element of \( G_n \) has Jordan canonical form as follows:

\[
J(f_1, \pi_1) \oplus J(f_2, \pi_2) \oplus \cdots \oplus J(f_k, \pi_k),
\]

with \( \sum_{i=1}^k d(f_i) |\pi_i| = n \), where \( f_1, \ldots, f_k \) are distinct polynomials from \( \Phi \), and where \( \pi_1, \ldots, \pi_k \in \mathcal{P} \), \( k \) is some positive integer.

Let \( \mathbb{N} \) denote the set of positive integers. Given a partition \( \mu = (\mu_1, \mu_2, \ldots) \) with \( \mu_1 \geq \mu_2 \geq \cdots \), we define \( n(\mu) = \sum_{i \geq 1} (i-1) \mu_i \).

**Lemma 1.** Given a partition \( \mu = (1^{n_1}2^{n_2}\cdots k^{n_k}), k \in \mathbb{N} \), there holds:

\[
\sum_{j=1}^k \sum_{i=1}^k n_i n_j \min(i, j) = |\mu| + 2n(\mu).
\]

**Proof.** Let \( \mu' = (\mu_1', \mu_2', \ldots) \) be the conjugate partition of \( \mu \), then \( \mu_i' = \sum_{j \geq 1} n_j \). Moreover \( n(\mu) = \sum_{i \geq 1} (\binom{i}{2}) \) (Macdonald [4], page 3).

\[
2n(\mu) = 2 \left( \binom{n_1 + n_2 + \cdots + n_k}{2} + 2 \binom{n_2 + n_3 + \cdots + n_k}{2} + \cdots + 2 \binom{n_k}{2} \right)
= (n_1 + n_2 + \cdots + n_k)(n_1 + n_2 + \cdots + n_k - 1)
+ (n_2 + n_3 + \cdots + n_k)(n_2 + n_3 + \cdots + n_k - 1) + \cdots + n_k(n_k - 1)
= n_1(n_1 + n_2 + n_3 + \cdots + n_k) + n_2(n_1 + 2n_2 + 2n_3 + \cdots + 2n_k)
+ \cdots + n_k(n_1 + 2n_2 + 3n_3 + \cdots + kn_k)
= \sum_{i=1}^k \sum_{j=1}^k n_i n_j \min(i, j) - |\mu|.
\]

**Lemma 2.** For any \( f \in \Phi \) with \( d(f) = d \), \( \mu = (1^{n_1}2^{n_2}\cdots k^{n_k}) \in \mathcal{P} \), the following formulae hold:

\[
|Z_{J_1, f, \mu}| = q^{d|\mu| + 2n(\mu)} b^d q^{-d},
\]

\[
|X_{J_1, f, \mu}| = q^{d|\mu| + 2n(\mu)}.
\]

**Proof.** The first formula is given by Macdonald ([4], page 139). The second is proved as follows.

Let \( A = F_q[x] \), \( \hat{A} = A \otimes_{F_q} \hat{F}_q \). Given any \( m \times m \) matrix \( x \) over \( F_q \), we define an \( A \)-module structure on \( F_q^m \) by \( x \cdot v = xv \) for \( v \in F_q^m \). Let \( V_n \) denote this module, and define \( \hat{V}_n = V_n \otimes_{F_q} \hat{F}_q \). Obviously, \( X_{J_1, f, \mu} = \text{End}_A(V_{J_1, f, \mu}) \).

Note that \( \text{End}_A(V_{J_1, f, \mu}) \) is a finite dimensional vector space over \( F_q \), which...
has the same dimension as $\text{End}_A(\tilde{P}_{J(f, \mu)})$ over $\tilde{F}_q$. This reduces the calculation to the corresponding calculation over the field $\tilde{F}_q$. Since $J(f, \mu) = J_1(f, \mu) \oplus J_2(f, \mu) \oplus \cdots$, it follows that $\tilde{P}_{J(f, \mu)} \cong \tilde{P}_{J_1(f, \mu)} \oplus \tilde{P}_{J_2(f, \mu)} \oplus \cdots$. Therefore,

$$\text{End}_A(\tilde{P}_{J(f, \mu)}) \cong \bigoplus_{i,j \geq 1} \text{Hom}_A(\tilde{P}_{J_1(f, \mu)}, \tilde{P}_{J_2(f, \mu)}) \cong \bigoplus_{i,j \geq 1} \text{Hom}_A(\tilde{P}_{J_1(f, \mu)}, \tilde{P}_{J_2(f, \mu)}) \cong \bigoplus_{i,j \geq 1} \text{Hom}_A(\tilde{P}_{J_1(f, \mu)}, \tilde{P}_{J_2(f, \mu)}) \cong \bigoplus_{i,j \geq 1} \text{Hom}_A(\tilde{P}_{J_1(f, \mu)}, \tilde{P}_{J_2(f, \mu)})$$

The classical theorem that any finite field is separable implies that for any irreducible monic polynomial $f(t)$ in $F_q[t]$ there is an invertible matrix $X$ with entries in $F_q$ such that $XJ(f(t))X^{-1}$ is diagonal with distinct diagonal entries $\lambda_1, \ldots, \lambda_d$. Here $f(t) = \prod_{i=1}^d (t - \lambda_i)$. Thus $J(f)$ is similar over $F_q$ to $J(\lambda_1, i) \oplus \cdots \oplus J(\lambda_d, i)$, where $J(\lambda, i)$ represents the Jordan normal form. Therefore,

$$\text{Hom}_A(\tilde{P}_{J_1(f, \mu)}, \tilde{P}_{J_2(f, \mu)}) \cong \bigoplus_{i,j \geq 1} \text{Hom}_A(\tilde{P}_{J_1(f, \mu)}, \tilde{P}_{J_2(f, \mu)}) \cong \bigoplus_{i,j \geq 1} \text{Hom}_A(\tilde{P}_{J_1(f, \mu)}, \tilde{P}_{J_2(f, \mu)}) \cong \bigoplus_{i,j \geq 1} \text{Hom}_A(\tilde{P}_{J_1(f, \mu)}, \tilde{P}_{J_2(f, \mu)})$$

An elementary calculation with matrices reveals that

$$\dim_{F_q} \text{Hom}_A(\tilde{P}_{J_1, i}, \tilde{P}_{J_2, j}) = \min(i, j).$$

Thus

$$\dim_{F_q} X_{J(f, \mu)} = \sum_{i \geq 1} \sum_{j \geq 1} n_in_j \dim_{F_q} \text{Hom}_A(\tilde{P}_{J_1(f, \mu)}, \tilde{P}_{J_2(f, \mu)}) = \sum_{i \geq 1} \sum_{j \geq 1} n_in_j \min(i, j)d = d||\mu| + 2n(\mu)|| \quad \text{by Lemma 1}.\] Finally, $|X_{J(f, \mu)}| = q^d||\mu| + 2n(\mu)||$. Note that these formulae depend only on the degree of $f$.\]

Now we let $P_\mu(0, q) = 1$. For $n \geq 1$, we define

$$P_\mu(n, q) = \sum_{\mu \geq n} \frac{q^{(\mu - 1)\mu + 2n(\mu)}}{h_\mu(q^{-1})}.$$
Theorem 3. The generating function of $M_\epsilon(n, q)$ ($n \geq 0$) can be factorized as follows:

$$1 + \sum_{n=1}^{\infty} M_\epsilon(n, q) t^n = \prod_{d=1}^{\infty} \left( 1 + \sum_{j=1}^{\infty} P(d, q^d) t^d \right)^{\phi(d)},$$

where $t$ is an indeterminate, and where $\phi(d)$ denotes the number of polynomials in $\Phi$ which have degree $d$. Note that exceptionally $\phi(1) = q - 1$.

Proof.

$$1 + \sum_{n=1}^{\infty} M_\epsilon(n, q) t^n = 1 + \sum_{n=1}^{\infty} \sum_{\pi \in C_n} \frac{|X_{\pi}|}{|Z_{\pi}|} t^n$$

where the summation is over $f \in \Phi$, which are distinct,

$$= 1 + \sum_{n=1}^{\infty} \sum_{f_1, \ldots, f_k \in \Phi} \frac{|X_{f_1, \ldots, f_k}|}{|Z_{f_1, \ldots, f_k}|} t^n \prod_{i=1}^{k} (d(f_i) | \pi_i|)$$

where the summation is over $f \in \Phi$, which are distinct,

$$= \prod_{f \in \Phi} \left( 1 + \sum_{\pi \in \Phi} \frac{|X_{f, \pi}|}{|Z_{f, \pi}|} t^n \right)^{\phi(|f|)}$$

(by Lemma 2)

$$= \prod_{d=1}^{\infty} \left( 1 + \sum_{j=1}^{\infty} P(d, q^d) t^d \right)^{\phi(d)} .$$

Let $M = (M_1, \ldots, M_g)$ be an ordered $g$-tuple of $n \times n$ matrices over $F_q$. Recall that $M$ is said to be decomposable over $F_q$ if there exists an invertible $n \times n$ matrix $X$ with entries in $F_q$ such that

$$(X M_1 X^{-1}, \ldots, X M_g X^{-1}) = \left( \begin{array}{cc} A_1 & 0 \\ 0 & B_1 \end{array} \right), \ldots, \left( \begin{array}{cc} A_g & 0 \\ 0 & B_g \end{array} \right) ,$$
and $A_1, \ldots, A_s$ are square matrices with the same degree. Otherwise $M$ is said to be indecomposable over $F_q$. Now, let $I_g(n, q)$ denote the number of classes of indecomposable ordered $g$-tuples of $n \times n$ matrices over $F_q$ up to simultaneous similarity. The Krull–Schmidt Theorem states that every $g$-tuple of $n \times n$ matrices over a field can be written as a direct sum of indecomposable $g$-tuples in a unique way up to order. Thus

$$1 + \sum_{n=1}^{\infty} M_g(n, q) t^n = \prod_{i=1}^{\infty} (1 - t)^{-l_i(i, q)}.$$ 

Then, by Theorem 3,

$$\prod_{d=1}^{\infty} \left( 1 + \sum_{j=1}^{\infty} P_g(j, q^d) t^{dj} \right)^{\phi_d(q)} = \prod_{i=1}^{\infty} (1 - t)^{l_i(i, q)}.$$ 

This implies upon taking logarithms:

$$\sum_{d=1}^{\infty} \phi_d(q) \log \left( 1 + \sum_{j=1}^{\infty} P_g(j, q^d) t^{dj} \right) = \sum_{i=1}^{\infty} I_g(i, q) \log \frac{1}{1-t}.$$ 

Now, we define

$$\log m_g(q, t) = \log \left( 1 + \sum_{j=1}^{\infty} P_g(j, q) t^j \right).$$

Note that the constant term of $\log m_g(q, t)$ is 0. Therefore,

$$\log \left( 1 + \sum_{n=1}^{\infty} M_g(n, q) t^n \right) = \sum_{i=1}^{\infty} I_g(i, q) \log \frac{1}{1-t} = \sum_{d=1}^{\infty} \phi_d(q) \log \log m_g(q^d, t^d).$$

(2)

For $i \geq 1$, we define $H_g(i, q)$ by

$$\log \left( 1 + \sum_{j=1}^{\infty} P_g(j, q) t^j \right) = \sum_{i=1}^{\infty} \frac{1}{i} H_g(i, q) t^i;$$

and define $E_g(i, q)$ by

$$\log \left( 1 + \sum_{j=1}^{\infty} M_g(j, q) t^j \right) = \sum_{i=1}^{\infty} \frac{1}{i} E_g(i, q) t^i.$$ 

Thus, identity (2) implies that

$$E_g(n, q) = \sum_{r \mid n} r \phi_r(q) H_g \left( \frac{1}{r}, q^r \right).$$
LEMMA 4.

\[ I_g(n, q) = \frac{1}{n} \sum_{d|n} \mu(d) E_g \left( \frac{n}{d}, q \right), \]

where \( \mu \) is the classical Möbius function.

Proof.

\[
\begin{align*}
\sum_{i=1}^{\infty} \frac{1}{i} E_g(i, q) t^i &= \sum_{i=1}^{\infty} I_g(i, q) \log \frac{1}{1-t^i} \\
&= \sum_{i=1}^{\infty} I_g(i, q) \sum_{j=1}^{\infty} \frac{1}{j} t^i \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} I_g(i, q) t^i.
\end{align*}
\]

By comparing the coefficients of \( t^n \) on both sides, we have

\[ I_g(n, q) = \sum_{d|n} \frac{1}{d} I_g \left( \frac{n}{d}, q \right). \]

It follows that

\[ E_g(n, q) = \sum_{d|n} n I_g \left( \frac{n}{d}, q \right) = \sum_{d|n} d I_g(d, q). \]

Möbius inversion of this shows that

\[ I_g(n, q) = \frac{1}{n} \sum_{d|n} \mu(d) E_g \left( \frac{n}{d}, q \right). \]

Let \( A_g(n, q) \) denote the number of classes of absolutely indecomposable ordered \( g \)-tuples of \( n \times n \) matrices over \( F_q \) up to simultaneous similarity. Recall that an ordered \( g \)-tuple is said to be absolutely indecomposable if it remains indecomposable when the field is extended to \( F_q \).

THEOREM 5. (Kac) \( I_g(n, q) \) and \( A_g(n, q) \) are linked by the following formulae:

\[
\begin{align*}
I_g(n, q) &= \sum_{d|n} \frac{1}{d} \sum_{r|d} \mu \left( \frac{d}{r} \right) A_g \left( \frac{n}{d}, q^r \right), \\
A_g(n, q) &= \sum_{d|n} \frac{1}{d} \sum_{r|d} \mu(r) I_g \left( \frac{n}{d}, q^r \right).
\end{align*}
\]
Proof. The first identity is given by Kac ([2], page 91), and can be found in Le Bruyn ([3] page 153). The second is the M"obius inverse of the first.

The following simple identity is needed below in dealing with double summations over divisors of integers:

$$\sum_{d \mid n} \sum_{r \mid n} f(d, r) = \sum_{r \mid n} \sum_{d \mid n/r} f(d, r) = \sum_{r \mid n} \sum_{d \mid n/r} \left( \frac{n}{r} \right).$$

The following is a known formula modified in the case $n = 1$:

$$\phi_n(q) = \frac{1}{n} \sum_{d \mid n} \mu(d)(q^{nd} - 1).$$

The M"obius inverse of this amounts to the following formula:

$$\sum_{d \mid n} \mu(d) \phi_{n,d}(q^n) = \mu(n)(q - 1).$$

Lemma 6.

$$A_q(n, q) = \frac{1}{n} \sum_{d \mid n} \mu(d) E_q \left( \frac{n}{d}, q^n \right).$$

Proof.

$$A_q(n, q) = \sum_{d \mid n} \frac{1}{d} \sum_{r \mid d} \mu(r) E_q \left( \frac{n}{d}, q^n \right)$$

$$= \sum_{d \mid n} \frac{1}{d} \sum_{r \mid d} \mu(r) \frac{d}{n} \sum_{s \mid n/d} \mu \left( \frac{n}{ds} \right) E_q(s, q^n) \quad \text{(by Lemma 4)}$$

$$= \frac{1}{n} \sum_{r \mid n} \sum_{d \mid r} \sum_{s \mid n/d} \mu(r) \mu \left( \frac{n}{ds} \right) E_q(s, q^n)$$

$$= \frac{1}{n} \sum_{r \mid n} \sum_{s \mid n/r} \sum_{d \mid n/rs} \mu(r) \mu \left( \frac{n}{rd} \right) E_q(s, q^n)$$

$$= \frac{1}{n} \sum_{r \mid n} \sum_{s \mid n/r} E_q(s, q^n) \sum_{d \mid n/rs} \mu \left( \frac{n}{rd} \right)$$

$$= \frac{1}{n} \sum_{r \mid n} \mu(r) E_q \left( \frac{n}{r}, q^n \right). \quad \Box$$
Theorem 7.

\[ A_g(n, q) = \frac{q - 1}{n} \sum_{d|n} \mu(d) H_e \left( \frac{n}{d}, q^d \right). \]

Proof.

\[ A_g(n, q) = \frac{1}{n} \sum_{d|n} \mu(d) E_
u \left( \frac{n}{d}, q^d \right) \quad \text{(by Lemma 6)} \]

\[ = \frac{1}{n} \sum_{d|n} \mu(d) \sum_{r|n/d} \phi_r(q^d) H_e \left( \frac{n}{d}, q^d \right) \]

\[ = \frac{1}{n} \sum_{r|n} \sum_{d|r} \mu(d) \frac{n}{dr} \phi_{r,d}(q^d) H_e (r, q^{nd}) \]

\[ = \frac{1}{n} \sum_{r|n} \sum_{d|r} \mu(d) \frac{r}{dr} \phi_{r,d}(q^d) H_e \left( \frac{n}{r}, q^r \right) \]

\[ = \frac{q - 1}{n} \sum_{r|n} \mu(r) H_e \left( \frac{n}{r}, q^r \right) \quad \text{(by (3)).} \]

Remark. For fixed \( n \) and \( g \) the functions \( M_g(n, q) \), \( I_g(n, q) \), \( A_g(n, q) \) of \( q \) are evidently rational functions of \( q \). As they take integer values for all integers \( q \) that are powers of primes, these functions are polynomial functions of \( q \) with rational coefficients. They have been calculated in various cases by Diane Maclagan and the author. The polynomials \( A_g(n, q) \) appear to have non-negative integer coefficients while \( I_g(n, q) \) do not. This can be viewed as extra support for Kac’s conjecture. More detail about Kac’s conjecture can be found in Le Bruyn [3].

3. RECURRENCE FORMULAE

In this section we prove the main results of this paper.

Proposition 8. \( P_g(n, q) \) and \( M_g(n, q) \) satisfy the following recursion formulae:

\[ P_g(n, q) = \frac{1}{n} \sum_{r=1}^{n} H_g(r, q) P_g(n-r, q). \]

\[ M_g(n, q) = \frac{1}{n} \sum_{r=1}^{n} E_g(r, q) M_g(n-r, q). \]
Proof. By the definition of $H_g(r, q)$, we have the following identity:

$$\log \left( \sum_{r=0}^{\infty} P_g(r, q)t^r \right) = \sum_{r=1}^{\infty} H_g(r, q)t^r/r.$$ 

Differentiate both sides respecting to $t$, we have

$$\frac{\sum_{r=1}^{\infty} rP_g(r, q)t^{r-1}}{\sum_{r=0}^{\infty} P_g(r, q)t^r} = \sum_{r=1}^{\infty} H_g(r, q)t^{r-1}.$$ 

Thus,

$$\sum_{r=1}^{\infty} rP_g(r, q)t^{r-1} = \left( \sum_{r=1}^{\infty} H_g(r, q)t^{r-1} \right) \left( \sum_{r=0}^{\infty} P_g(r, q)t^r \right).$$

By comparing the coefficients of $t^{r-1}$ on both sides, we get

$$nP_g(n, q) = \sum_{r=1}^{n} H_g(r, q) P_g(n-r, q).$$

Thus the first recursion formula has been established.

As $\log(\sum_{r=\geq 0} M_g(r, q)t^r) = \sum_{r=1} E_g(r, q)t^r/r$, the second identity can be proved similarly. 

**Proposition 9.** The functions $H_1(n, q)$, $P_1(n, q)$, $E_1(n, q)$ and $M_1(n, q)$ have the forms specified in the introduction.

**Proof.** Let $g=1$. By the Jordan Normal Form Theorem, the classes of $n \times n$ of matrices over $\mathbb{F}_q$ under conjugation are in one-one correspondence with Jordan normal forms $J(\lambda, n)$, where $\lambda \in \mathbb{F}_q$. As a consequence, $A_1(n, q) = q$ for all $n \geq 1$.

The Möbius inverse of Theorem 7 with $g=1$ now amounts to

$$H_1(n, q) = \sum_{d \mid n} \mu(n/d) q^d q^{d-1} = \sum_{d \mid n} \frac{1}{d-1} q^{n-d}.$$ 

By the definition of $P_1(n, q)$ in Section 2, $P_1(n, q)$ has the form as required.

Möbius inversion of Lemma 6 shows that $E_1(n, q) = \sum_{d \mid n} d A_1(d, q) q^{d-1}$, so $E_1(n, q) = \sum_{d \mid n} dq^{n-d}$.
Let $R_i(n, q) = \sum_{\lambda \vdash n} q^{\lambda(i)}$. Note that $\sum_{\lambda \vdash n} q^{\lambda(i)} = \sum_{\lambda \vdash n} q^{\lambda'}$, where $\lambda'$ is the partition conjugate to $\lambda$. Also note that $l(\lambda')$ is equal to the number of parts of $\lambda$. We claim that $M_i(n, q) = R_i(n, q)$. In fact,

$$1 + \sum_{n=1}^{\infty} R_i(n, q) t^n = 1 + \sum_{\lambda \vdash n} q^{\lambda(i)} t^n$$

$$= 1 + \sum_{\lambda \vdash n, \lambda' \vdash n} q^{\lambda(i)} t^{\lambda(i)}$$

$$= 1 + \sum_{\lambda \vdash n, \lambda' \vdash n} (q t^i)^{n_1} (q t^i)^{n_2} (q t^i)^{n_3} \ldots$$

$$= \prod_{i=1}^{\infty} \left(1 + \sum_{j=1}^{\infty} (q t^i)^{j}\right)$$

$$= \prod_{i=1}^{\infty} \frac{1}{1 - q t^i}.$$

Thus, taking logarithms on both sides implies

$$\log \left(1 + \sum_{n=1}^{\infty} R_i(n, q) t^n\right) = \sum_{i=1}^{\infty} \log \frac{1}{1 - q t^i}$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} (q t^i)^{j}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{d|n} \frac{d}{n} q^{n/d}\right) t^n$$

$$= \sum_{n=1}^{\infty} E_i(n, q) t^n/n.$$

Therefore, $\log(1 + \sum_{n=1}^{\infty} R_i(n, q) t^n) = \log(1 + \sum_{n=1}^{\infty} M_i(n, q) t^n)$. It follows that $M_i(n, q) = R_i(n, q) = \sum_{\lambda \vdash n} q^{\lambda(i)}$, for all $n \geq 1$.

**Corollary 10.**

$$1 + \sum_{n=1}^{\infty} M_i(n, q) t^n = \prod_{n=1}^{\infty} \frac{1}{1 - q t^n}.$$

**Remark.** Le Bruyn mentioned analogous characteristic 0 results due to H. Kraft and D. Peterson in [3]. More precisely, if $C$ is an algebraically closed field of characteristic 0, and $R_{\text{iso}}(S_1, n|\lambda)$ denotes the isomorphism classes of $n$-dimensional representations of $C[x]$ whose root-multiplicity-partition is conjugate to $\lambda$ (see Le Bruyn [3], page 144), then $R_{\text{iso}}(S_1, n|\lambda)$
is in a natural way an affine space of dimension equal to \( l(\lambda) \). This can be translated into the following identity:

\[
1 + \sum_{n=1}^{\infty} \sum_{\lambda \vdash n} s^{-\dim R^{n}(S, n(\lambda))} t^n = \prod_{n=1}^{\infty} \frac{1}{1 - t^n}.
\]

The rest of this section is devoted to the algebraic interpretation of Corollary 10. Let \( F \) be any field, \( M_n(F) \) be the set of all \( n \times n \) matrices over \( F \), \( GL_n(F) \) be the general linear group. \( GL_n(F) \) acts on \( M_n(F) \) by conjugation, the orbit space is denoted by \( M_n(F)/GL_n(F) \). Let \( f \) be a monic polynomial over \( F \), say \( f(t) = t^n - a_{m-1}t^{m-1} - \cdots - a_1 t - a_0 \), recall that the companion matrix \( J(f) \) of \( f \) is defined by:

\[
J(f) =
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
a_0 & a_1 & a_2 & \cdots & a_{m-1}
\end{bmatrix}
\]

By Theorem 24 (Birkhoff–Mac Lane [1], page 332), we know that any \( n \times n \) matrix \( M \) over \( F \) is similar over \( F \) to one and only one direct sum of companion matrices \( J(f_1) \oplus J(f_2) \oplus \cdots \oplus J(f_k) \), such that \( f_{i+1} \mid f_i \) for all \( i \leq k - 1 \), where \( f_i \) (\( i \geq 1 \)) are monic polynomials over \( F \). Note that \( \sum_{i=1}^{k} \deg f_i = n \), thus \( \lambda = (\deg f_1, \deg f_2, \ldots, \deg f_k) \) forms a partition of \( n \). We call \( \lambda \) the rational partition afforded by \( M \).

**Theorem 11.** Let \( Q_\lambda \) denote the set of all similarity classes whose rational partitions are \( \lambda \), then

\[
M_n(F)/GL_n(F) = \bigcup_{\lambda \vdash n} Q_\lambda,
\]

where the union is a disjoint union. \( Q_\lambda \) is in a natural way an affine space of dimensional equal to \( l(\lambda) \).

**Proof.** The first statement follows from the Rational Canonical Form Theorem.

Suppose \( \lambda = (\lambda_1, \ldots, \lambda_k) \). Let \( \mu_i = \lambda_i - \lambda_{i+1} \) for \( i \leq k - 1 \), and \( \mu_k = \lambda_k \). By the previous discussion, every \( M \in Q_\lambda \) is similar uniquely over \( F \) to \( J(f_1) \oplus J(f_2) \oplus \cdots \oplus J(f_k) \), where \( f_i \) is some monic polynomial of degree \( \lambda_i \) over \( F \), such that \( f_{i+1} \mid f_i \) for all \( i \leq k - 1 \). Now, let \( g_i = f_i/f_{i+1} \) for \( i \leq k - 1 \), and \( g_k = f_k \), then \( g_i \) is a monic polynomial of degree \( \mu_i \) over \( F \) for all \( i \geq 1 \). Note that \( g_i \) (\( 1 \leq i \leq k \)) are uniquely determined by \( M \), and vice versa. Thus
the elements in $\mathbb{Q}_1$ are in one-one correspondence with the elements in the following set:

$$S = \{ (g_1, \ldots, g_k) \mid g_i \in \mathbb{F}[t] \text{ monic, } \deg g_i = \mu_i, \ 1 \leq i \leq k \}.$$ 

If we define $S_i = \{ g \mid g \in \mathbb{F}[t] \text{ monic, } \deg g = \mu_i \}$, then $S = \bigcap_{i=1}^{k} S_i$, where the product is the Cartesian product. It is trivial to prove that $S_i$ is in a natural way isomorphic to the affine space $\mathbb{F}^{\mu_i}$. Note that $\mu_1 + \cdots + \mu_k = \lambda_1 = l(\lambda)$, thus $Q_1$ is in a natural way isomorphic to the affine space $\mathbb{F}^{\lambda(\lambda)}$. Here “natural” means compatible with change of field.

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