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Potentially nilpotent and spectrally arbitrary even cycle sign patterns ☆

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Abstract

An $n \times n$ sign pattern $\mathscr{S}_n$ is potentially nilpotent if there is a real matrix having sign pattern $\mathscr{S}_n$ and characteristic polynomial x^n. A new family of sign patterns $\mathscr{C}_n$ with a cycle of every even length is introduced and shown to be potentially nilpotent by explicitly determining the entries of a nilpotent matrix with sign pattern $\mathscr{C}_n$. These nilpotent matrices are used together with a Jacobian argument to show that $\mathscr{C}_n$ is spectrally arbitrary, i.e., there is a real matrix having sign pattern $\mathscr{C}_n$ and characteristic polynomial $x^n + \sum_{i=1}^{n}(-1)^i \mu_i x^{n-i}$ for any real μ_i. Some results and a conjecture on minimality of these spectrally arbitrary sign patterns are given.

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Keywords: Characteristic polynomial; Cycle product; Digraph; Nilpotent matrix; Potentially nilpotent; Sign pattern; Spectrally arbitrary; Spectrum

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1. Introduction

A real $n \times n$ matrix $Y_n = [y_{ij}]$ has an associated digraph $D(Y_n)$ with vertices $1, 2, \ldots, n$ and an arc (i, j) from vertex i to vertex j if and only if $y_{ij} \neq 0$. If $y_{ii} \neq 0$, then the simple cycle (i, i) of length 1 in $D(Y_n)$ is called a loop at vertex i, and its associated cycle product of size 1 is y_{ii}. A simple cycle of length $k \geqslant 2$ (called a k-cycle) in $D(Y_n)$ is a sequence of arcs $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)$ with k distinct vertices, and its associated cycle product of size k is $(-1)^{k+1} y_{v_1 v_2} y_{v_2 v_3} \ldots y_{v_{k-1} v_k} y_{v_k v_1}$. A composite cycle of length k is a set of vertex disjoint simple cycles with lengths summing to k. Its associated cycle product of size k is $(-1)^m$ times the product of all matrix entries corresponding to these vertex disjoint cycles, where m is the number of such cycles of even length. If the characteristic polynomial of Y_n, $\det(xI - Y_n)$, is given by

$$p_n(x) = x^n - \mu_1 x^{n-1} + \mu_2 x^{n-2} - \cdots + (-1)^n \mu_n, \tag{1.1}$$

then it is well known that the coefficient μ_i of $(-1)^i x^{n-i}$ for $1 \leqslant i \leqslant n$ in the characteristic polynomial of Y_n is the sum of all cycle products of size i (see, for example, [4]).

A fixed matrix $Y_n = [y_{ij}]$ has an associated sign pattern (matrix) $\mathscr{S}_n = [s_{ij}]$ with $s_{ij} = \operatorname{sgn}(y_{ij})$ for all i, j, where $\operatorname{sgn}(y_{ij}) = +, -, 0$ according as y_{ij} is positive, negative, zero, respectively. We denote by $Q(\mathscr{S}_n)$ the set of all real matrices with associated sign pattern $\mathscr{S}_n$, thus $Y_n \in Q(\mathscr{S}_n)$. Also $D(\mathscr{S}_n) = D(Y_n)$ and cycles in $D(\mathscr{S}_n)$ are defined as above for $D(Y_n)$. A sign pattern $\mathscr{S}_n$ is a *spectrally arbitrary pattern* (SAP) if given any real monic polynomial $p_n(x)$ of degree n, there exists a real matrix $Y_n \in Q(\mathscr{S}_n)$ with characteristic polynomial $p_n(x)$. A sign pattern $\mathscr{S}_n$ is *potentially nilpotent* if there exists a matrix $Y_n \in Q(\mathscr{S}_n)$ that is nilpotent, i.e., the characteristic polynomial of Y_n is x^n. If $\mathscr{S}_n$ is a SAP, then clearly $\mathscr{S}_n$ is potentially nilpotent, but the converse is not necessarily true. However, Drew et al. [3] developed a methodology (based on the implicit function theorem) of using a nilpotent matrix Y_n to determine a spectrally arbitrary pattern.

The first known family of spectrally arbitrary patterns (for all $n \geqslant 2$) was given in [7] and is based on constructions using a Soules matrix. If $\mathscr{S}_n$ is a SAP, but no proper subpattern of $\mathscr{S}_n$ is a SAP, then $\mathscr{S}_n$ is a *minimal* SAP. The first known families of minimal spectrally arbitrary patterns were given in [1] by using the methodology of [3]. This was also used by Cavers and Vander Meulen [2] to introduce other families of SAPs. More recently, all spectrally arbitrary patterns with an associated star graph were determined in [6]. The characteristic polynomial of a matrix with a star graph is relatively simple, and consequently the matrix entries can be explicitly computed for any given characteristic polynomial. Results of [8] were used in [6] to characterize all potentially nilpotent star patterns. Note that SAPs and potentially nilpotent patterns are studied up to equivalence, i.e., transposition, negation, and permutation and signature similarity.

Here we introduce a new family of particular sign patterns $\mathscr{C}_n$ that have a cycle of every even length (which we call *even cycle sign patterns*), and show that this family is spectrally arbitrary. For n even, we prove that if $D(\mathscr{C}_n)$ has n loops and the product of entries corresponding to each of the cycles of even length is negative, then $\mathscr{C}_n$ is a SAP. Although the characteristic polynomial of $M_n \in Q(\mathscr{C}_n)$ is complicated, we use algebraic and graph theoretic techniques to find nilpotent matrices with these sign patterns, and then use the methodology of [3] to demonstrate that the pattern is spectrally arbitrary. When $n = 2k + 1$, the results and proofs are obtained from those for $n = 2k$ by requiring that $D(\mathscr{C}_{2k+1})$ has a Hamilton cycle and only $2k$ loops. Even cycle sign patterns are motivated by the observation [2, Lemma 1.5] that if $\mathscr{S}_n$ allows any inertia, which must be true if $\mathscr{S}_n$ is a SAP, then $D(\mathscr{S}_n)$ contains a 2-cycle with $s_{kj} s_{jk} < 0$ for $k \neq j$.

In Section 2, we begin by finding nilpotent even cycle matrices $M_n \in Q(\mathscr{C}_n)$, firstly for n even and then for n odd. In Section 3, we consider a Jacobian and show that it is nonzero, which allows us to use the methodology of [3]. In Section 4, which contains our main results, we prove that $\mathscr{C}_n$ is a SAP (with approximately $5n/2$ nonzero entries) and identify a subpattern that is potentially nilpotent (but not a SAP). We conclude with some results and a conjecture on minimality.

2. Nilpotent even cycle matrices

Throughout we restrict consideration to $n \times n$ matrices with $n \geqslant 4$ and having the following structures depending on the parity of n. For $n = 2k$, let

$$M_{2k} = \begin{bmatrix} a_1 & 1 & & & & \\ b_2 & a_2 & 1 & & & \\ 0 & & a_3 & 1 & 0 & \\ b_4 & & & a_4 & \ddots & \\ \vdots & & 0 & & \ddots & 1 & \\ 0 & & & & a_{2k-1} & 1 \\ b_{2k} & & & & & a_{2k} \end{bmatrix} \tag{2.1}$$

and for $n = 2k + 1$, let

$$M_{2k+1} = \begin{bmatrix} a_1 & 1 & & & & & \\ b_2 & a_2 & 1 & & & & \\ 0 & & a_3 & 1 & & 0 & \\ b_4 & & & a_4 & \ddots & & \\ \vdots & & & & \ddots & 1 & \\ 0 & & 0 & & & a_{2k-1} & 1 \\ b_{2k} & & & & & & a_{2k} & 1 \\ b_{2k+1} & & & & & & & a_{2k+1} \end{bmatrix}, \tag{2.2}$$

where $b_n \neq 0$ and all other variables are arbitrary. Note that both $D(M_{2k})$ and $D(M_{2k+1})$ have a loop at each vertex j for which $a_j \neq 0$ and exactly one simple cycle of length $2i$ if $b_{2i} \neq 0$; the corresponding cycle products are a_j (for the loop at vertex j) and $-b_{2i}$ (for the simple cycle of length $2i$). The digraph $D(M_{2k+1})$ also has one simple cycle of length $2k + 1$ with corresponding cycle product b_{2k+1}. Since all the simple cycles of $D(M_n)$ have even length, except for loops and (if n is odd) a Hamilton cycle, we call the matrices M_n *even cycle matrices*.

With respect to M_n in (2.1) and (2.2), let $A_{n,q} = \{n - q + 1, n - q + 2, \ldots, n\}$ for $1 \leqslant q \leqslant n$. For $1 \leqslant w \leqslant q$, define

$$F(q, w) = \sum_{\substack{B \subseteq A_{n,q} \\ |B| = w}} \prod_{i \in B} a_i,$$

i.e., $F(q, w)$ is the sum of all products of w distinct entries a_z, where $z \in A_{n,q}$. Define $F(q, 0) = F(0, 0) = 1$ and define $F(q, w) = 0$ if $w > q$.

We first consider the case that n is even. The coefficients of the characteristic polynomial (1.1) of M_{2k} in (2.1) can be specified in terms of the functions $F(q, w)$ (see Lemma 2.1). If, for $n = 2k$, we let

$$a_{2i-1} = 1 \quad \text{and} \quad a_{2i} = -1 \quad \text{for } 1 \leqslant i \leqslant k, \tag{2.3}$$

then the functions $F(q, w)$ are easily computed (see Lemma 2.2) and this enables us to determine a nilpotent matrix M_{2k} (in Theorem 2.3).

With respect to $D(M_{2k})$, the cycle products of size i are obtained from i loops, or from a simple even cycle of length $j \leqslant i$ and $i - j$ loops that are disjoint from the j-cycle. These observations give the following expressions for the coefficients of the characteristic polynomial of M_{2k}.

Lemma 2.1. *When $k \geqslant 2$ and $n = 2k$, the characteristic polynomial (1.1) of M_n has for $1 \leqslant r \leqslant k$,*

$$\mu_{2r} = F(2k, 2r) - \sum_{i=1}^{r} b_{2i} F(2k - 2i, 2r - 2i) \tag{2.4}$$

and for $0 \leqslant r \leqslant k - 1$,

$$\mu_{2r+1} = F(2k, 2r + 1) - \sum_{i=1}^{r} b_{2i} F(2k - 2i, 2r + 1 - 2i). \tag{2.5}$$

The F functions in the above lemma are now computed by assigning values to the variables a_j as in (2.3).

Lemma 2.2. *Let $k \geqslant 2$ and $n = 2k$. If (2.3) holds and $2 \leqslant p \leqslant k$, then for $r = 0, 1, \ldots, p$,*

$$F(2p, 2r) = (-1)^r \binom{p}{r} \tag{2.6}$$

and

$$F(2p, 2r + 1) = 0. \tag{2.7}$$

Proof. If (2.3) holds, then each product of entries a_z with $z \in A_{n,2p}$ is ± 1. Letting $B_p = \{i : i \text{ is odd and } i \in A_{n,2p}\}$ and $C_p = \{i : i \text{ is even and } i \in A_{n,2p}\}$, $F(2p, w)$ is the number of sets with w elements formed by taking an even number of elements from C_p (and the rest from B_p), minus the number of sets with w elements formed by taking an odd number of elements from C_p (and the rest from B_p). Thus,

$$F(2p, 2r) = \sum_{i=0}^{r} \binom{p}{2i} \binom{p}{2r - 2i} - \sum_{i=0}^{r-1} \binom{p}{2i + 1} \binom{p}{2r - (2i + 1)}. \tag{2.8}$$

This can easily be seen by noting that each term in these summations is of the form $\binom{p}{j}\binom{p}{\ell}$, where j elements are chosen from C_p and ℓ elements are chosen from B_p to form a set of size $j + \ell$. Note that $j + \ell = 2r$ in (2.8).

The coefficient of t^{2r} in the binomial expansion of $(1 - t^2)^p$ is $(-1)^r \binom{p}{r}$. Similarly, the coefficient of t^{2r} in the product of the binomial expansions of $(1 - t)^p$ and $(1 + t)^p$ is

$$\sum_{i=0}^{2r} (-1)^i \binom{p}{i} \binom{p}{2r - i} = F(2p, 2r)$$

by (2.8). Since $(1 - t)^p (1 + t)^p = (1 - t^2)^p$, they must have equal coefficients of t^{2r}, thus $F(2p, 2r) = (-1)^r \binom{p}{r}$.

By a similar argument as used for (2.8),

$$F(2p, 2r+1) = \sum_{i=0}^{r} \binom{p}{2r+1-(2i+1)} \binom{p}{2i+1} - \sum_{i=0}^{r} \binom{p}{2i+1} \binom{p}{2r+1-(2i+1)},$$

which is equal to 0. □

Theorem 2.3. *Let $k \geqslant 2$ and $n = 2k$. If $a_{2i-1} = 1$ and $a_{2i} = -1$ for $1 \leqslant i \leqslant k$, then M_n is nilpotent if and only if $b_{2r} = -\binom{k}{r}$ for $1 \leqslant r \leqslant k$.*

Proof. In the characteristic polynomial of M_n, by (2.5) and (2.7) it follows that $\mu_{2r+1} = 0$ for $0 \leqslant r \leqslant k-1$. Then M_n is nilpotent if and only if $\mu_{2r} = 0$ for $1 \leqslant r \leqslant k$. By (2.4) and (2.6), this holds if and only if for $1 \leqslant r \leqslant k$

$$\begin{aligned} b_{2r} &= F(2k, 2r) - \sum_{i=1}^{r-1} b_{2i} F(2k-2i, 2r-2i) \\ &= (-1)^r \binom{k}{r} - \sum_{i=1}^{r-1} b_{2i} (-1)^{r-i} \binom{k-i}{r-i}. \end{aligned} \tag{2.9}$$

Now we show by induction on r that (2.9) gives $b_{2r} = -\binom{k}{r}$. For $r = 1$, Eq. (2.9) gives $b_2 = (-1)^1 \binom{k}{1} - 0 = -\binom{k}{1}$, so the statement is true. Now assume that $b_{2r} = -\binom{k}{r}$ for all $r \leqslant u-1$. From (2.9) and the induction hypothesis,

$$\begin{aligned} b_{2u} &= (-1)^u \binom{k}{0}\binom{k}{u} + \sum_{i=1}^{u-1} (-1)^{u-i} \binom{k}{i}\binom{k-i}{u-i} \\ &= \sum_{i=0}^{u-1} (-1)^{u-i} \binom{k}{i}\binom{k-i}{u-i} \\ &= \sum_{i=0}^{u-1} (-1)^{u-i} \binom{k}{u}\binom{u}{i} = \binom{k}{u}\left[(-1)^u \sum_{i=0}^{u} (-1)^i \binom{u}{i} - 1\right]. \end{aligned}$$

Note that $\sum_{i=0}^{u} (-1)^i \binom{u}{i} = 0$ since the binomial expansion $(1-t)^u = \sum_{i=0}^{u} \binom{u}{i} (-t)^i$ with $t = 1$ gives $0 = \sum_{i=0}^{u} (-1)^i \binom{u}{i}$. Therefore $b_{2u} = -\binom{k}{u}$ and the statement follows by induction. □

We now consider M_n when n is odd as in (2.2). For a specified main diagonal (that simplifies the characteristic polynomial), a necessary condition for M_n to be nilpotent is given in the following lemma, and a nilpotent matrix M_n is determined in Theorem 2.5.

Lemma 2.4. *Let $k \geqslant 2$ and $n = 2k+1$, and let M_n have exactly one $a_v = 0$ for $1 \leqslant v \leqslant n$ and $b_{2r} \neq 0$ for all $1 \leqslant r \leqslant k$. Suppose the nonzero a_i alternate in value $1, -1, 1, -1, \dots, 1, -1$. If M_n is nilpotent, then either $a_{n-2} = 0$ or $a_{n-1} = 0$.*

Proof. Considering the cycle products of size n, the constant term of the characteristic polynomial of M_n is

$$\sum_{r=1}^{k}\left(b_{2r}\prod_{j=2r+1}^{n} a_j\right) - b_n - \prod_{i=1}^{n} a_i. \tag{2.10}$$

Since a_n appears in each term except $-b_n$, if $a_n = 0$ then (2.10) is $-b_n \neq 0$, and M_n is not nilpotent.

Now suppose $a_v = 0$ for some fixed v with $1 \leqslant v \leqslant n-3$, and let

$$s = \begin{cases} v & \text{for } v \text{ even} \\ v+1 & \text{for } v \text{ odd.} \end{cases}$$

Here s is the length of the shortest cycle of length at least 2 in $D(M_n)$ on which vertex v lies, and s is even with $2 \leqslant s \leqslant n-3$ and $2 \leqslant n-s-1 \leqslant n-3$. The coefficient of x^{n-s-1} in the characteristic polynomial of M_n with $a_v = 0$ is given by the sum of all cycle products of size $s+1$, i.e., by

$$F(2k+1, s+1) - \sum_{i=1}^{\frac{s-2}{2}} b_{2i} F(2k+1-2i, s+1-2i) - b_s\sigma, \tag{2.11}$$

where

$$\sigma = \sum_{i=s+1}^{n} a_i.$$

There are an odd number, namely $n-s$, of loops that are disjoint from the cycle of length s. The sets $A_{n,q}$ corresponding to each of the above F functions, respectively, are

$$A_{2k+1,2k+1}, A_{2k+1,2k-1}, A_{2k+1,2k-3}, \dots, A_{2k+1,2k+1-(s-2)}.$$

By definition, each of these sets contains the index v such that $a_v = 0$. Each F function in (2.11) is of the form $F(2j+1, w)$ for $2k+1-(s-2) \leqslant 2j+1 \leqslant 2k+1$ and w odd with $3 \leqslant w \leqslant s+1$, where the associated set $A_{2k+1,2j+1}$ contains j indices z for which $a_z = 1$ and j indices z for which $a_z = -1$. Thus its value is equal to $F(2j, w)$ when (2.3) holds. Since $s+1, s-1, s-3, \dots, 3$ are all odd, by (2.7) all of the F functions in (2.11) are equal to 0 and (2.11) reduces to $-b_s\sigma$. The value of σ is clearly nonzero since it is the sum of an odd number of variables with values from $\{1, -1\}$, therefore (2.11) is nonzero and M_n is not nilpotent.

Thus if M_n is nilpotent and exactly one $a_v = 0$, then $v = n-2$ or $n-1$. □

Theorem 2.5. *Let $k \geqslant 2$ and $n = 2k+1$. Suppose M_n has exactly one of a_{n-2} and a_{n-1} equal to 0, and the nonzero a_i alternate in value $1, -1, 1, -1, \dots, 1, -1$. Then M_n is nilpotent if and only if $b_{2r} = -\binom{k}{r}$ for $1 \leqslant r \leqslant k$ and $b_{2k+1} = 1$.*

Proof. Let M_{2k} have characteristic polynomial (1.1) with (2.3) holding. Consider the characteristic polynomial (1.1) of M_{2k+1} with μ_i replaced by $\tilde{\mu}_i$ and where the a_i are assigned as in the theorem statement. For $1 \leqslant p \leqslant 2k$, we claim that the sets of all cycle products of size p in $D(M_{2k+1})$ and $D(M_{2k})$ are identical. This is true because:

(i) In $D(M_{2k+1})$, the arc from vertex $2k+1$ to vertex 1 belongs to no cycle of length $\leqslant 2k$ (and thus b_{2k+1} does not occur in these cycle products).
(ii) Every cycle product of size p in $D(M_{2k})$ and $D(M_{2k+1})$ corresponds to either
 (a) p loops (and each digraph has k loops with cycle product equal to 1 and k loops with cycle product equal to -1), or
 (b) one cycle of length $2r$ for $1 \leqslant r \leqslant k$ and $p-2r$ loops (and in each digraph, the cycles of length $2r$ have the same cycle product and each is disjoint from $2k-2r$ loops, $k-r$ of which have cycle product 1 and $k-r$ of which have cycle product -1).

Thus it follows that $\mu_i = \tilde{\mu}_i$ for $1 \leqslant i \leqslant 2k$ (in terms of the b_{2r}). By Theorem 2.3, $\mu_i = \tilde{\mu}_i = 0$ for $1 \leqslant i \leqslant 2k$ if and only if $b_{2r} = -\binom{k}{r}$ for $1 \leqslant r \leqslant k$. The constant term $\tilde{\mu}_n$ is the sum of cycle products of size $2k+1$ as in (2.10). With the a_i as assigned, the only such cycle products are obtained from the $(2k+1)$-cycle and from the composite cycle consisting of the loop at vertex $2k+1$ and the $2k$-cycle. These cycle products have values equal to b_{2k+1} and $-a_{2k+1}b_{2k}$, respectively. Thus, $\tilde{\mu}_n = 0$ if and only if

$$0 = -b_{2k+1} + a_{2k+1}b_{2k}, \tag{2.12}$$

that is, if and only if $b_{2k+1} = (-1)\left[-\binom{k}{k}\right] = 1.$ □

Remark 2.6. Theorems 2.3 and 2.5 give nilpotent matrices M_n for specific assignments of the a_j when $n \geqslant 4$. However, the proofs of both theorems can be adapted to show nilpotence for other values of the main diagonal. When $k \geqslant 2, n = 2k, a_{2i} \in \{1, -1\}$ and $a_{2i-1} = -a_{2i}$ for $1 \leqslant i \leqslant k$, then M_n is nilpotent if and only if $b_{2r} = -\binom{k}{r}$ for $1 \leqslant r \leqslant k$. When $k \geqslant 2$, $n = 2k+1$, $a_{2i} \in \{1, -1\}$ and $a_{2i-1} = -a_{2i}$ for $1 \leqslant i \leqslant k-1$, if $\{a_{2k-1}, a_{2k}\} = \{1, 0\}$ and $a_{2k+1} = -1$, then M_n is nilpotent if and only if $b_{2r} = -\binom{k}{r}$ for $1 \leqslant r \leqslant k$ and $b_{2k+1} = 1$. However, in this latter case if $\{a_{2k-1}, a_{2k}\} = \{-1, 0\}$ and $a_{2k+1} = 1$, then $b_{2r} = -\binom{k}{r}$ for $1 \leqslant r \leqslant k$ and (2.12) gives $b_{2k+1} = -1$ for nilpotence. Thus, from M_n we can find $2^{\lceil n/2 \rceil}$ nonequivalent potentially nilpotent sign patterns.

3. Nonzero Jacobian

With a view to obtaining spectrally arbitrary patterns from M_n, we now consider a particular Jacobian evaluated at values a_i, b_i obtained for a nilpotent matrix M_n. This enables us to use the methodology of [3, Observation 10]; see also [1, Lemma 2.1]. We first consider the case that n is even, i.e., $n = 2k$ for $k \geqslant 2$. In M_n, set

$$a_{2i} = -1 \quad \text{for } 1 \leqslant i \leqslant k. \tag{3.1}$$

Certain entries of the matrix D_n, which is defined below in (3.6), are determined by differentiating the coefficients of the characteristic polynomial μ_i with respect to the n remaining variables a_{2i-1} and b_{2i}; see Lemmas 3.1, 3.2, 3.4, 3.6 and 3.7.

For $1 \leqslant j \leqslant k$ and for some odd ℓ with $n - 2j + 1 \leqslant \ell \leqslant n$, define

$$\widetilde{A}_{n,2j-1} \equiv A_{n,2j} \backslash \{\ell\}.$$

For $1 \leqslant w \leqslant 2j - 1$, define

$$\widetilde{F}(2j-1, w) = \sum_{\substack{B \subseteq \widetilde{A}_{n,2j-1} \\ |B| = w}} \prod_{i \in B} a_i,$$

i.e., $\widetilde{F}(2j-1, w)$ is equal to the sum of all products of w distinct entries a_z, where $z \in \widetilde{A}_{n,2j-1}$. Define $\widetilde{F}(2j-1, 0) = 1$ and define $\widetilde{F}(2j-1, w) = 0$ if $w > 2j - 1$.

Lemma 3.1. *If* (2.3) *holds*, $0 \leqslant p \leqslant k - 1$ *and* $0 \leqslant r \leqslant p$, *then*

$$\widetilde{F}(2p+1, 2r) = (-1)^r \binom{p}{r}, \tag{3.2}$$

and

$$\widetilde{F}(2p+1, 2r+1) = (-1)^{r+1} \binom{p}{r}. \tag{3.3}$$

Proof. If (2.3) holds, then each product of entries a_z with $z \in \widetilde{A}_{n,2p+1}$ is ± 1. Letting $\widetilde{B}_p = \{i : i \text{ is odd and } i \in \widetilde{A}_{n,2p+1}\}$ and $\widetilde{C}_{p+1} = \{i : i \text{ is even and } i \in \widetilde{A}_{n,2p+1}\}$, $\widetilde{F}(2p+1, w)$ is the number of sets with w elements formed by taking an even number of elements from $\widetilde{C}_{p+1}$ (and the rest from $\widetilde{B}_p$) minus the number of sets with w elements formed by taking an odd number of elements from $\widetilde{C}_{p+1}$ (and the rest from $\widetilde{B}_p$). Note that $|\widetilde{B}_p| = p$ and $|\widetilde{C}_{p+1}| = p + 1$. As in the proof of Lemma 2.2, the expression for $\widetilde{F}(2p+1, w)$ is

$$\widetilde{F}(2p+1, 2r) = \sum_{i=0}^{r} \binom{p}{2i} \binom{p+1}{2r-2i} - \sum_{i=0}^{r-1} \binom{p}{2i+1} \binom{p+1}{2r-(2i+1)}.$$

Consider the binomial expansions

$$(1+t)^p = \sum_{i=0}^{p} \binom{p}{i} t^i \quad \text{and} \quad (1-t)^{p+1} = \sum_{i=0}^{p+1} (-1)^i \binom{p+1}{i} t^i.$$

The coefficient of t^{2r} in $(1+t)^p(1-t)^{p+1}$ is

$$\sum_{i=0}^{2r} (-1)^i \binom{p}{i} \binom{p+1}{2r-i} = \sum_{i=0}^{r} \binom{p}{2i} \binom{p+1}{2r-2i} - \sum_{i=0}^{r-1} \binom{p}{2i+1} \binom{p+1}{2r-(2i+1)},$$

which is $\widetilde{F}(2p+1, 2r)$. Also,

$$(1+t)^p(1-t)^{p+1} = (1-t)(1-t^2)^p = (1-t^2)^p - t(1-t^2)^p.$$

The coefficient of t^{2r} in $(1-t^2)^p - t(1-t^2)^p$ is equal to the coefficient of t^{2r} in $(1-t^2)^p$ since $-t(1-t^2)^p$ has no even powers of t. Therefore, since

$$(1-t^2)^p = \sum_{i=0}^{p} (-1)^i \binom{p}{i} t^{2i},$$

(3.2) follows.

For (3.3), observe that when F is defined with respect to $A_{n,2p+2}$, $\widetilde{F}$ is defined with respect to $\widetilde{A}_{n,2p+1} \equiv A_{n,2p+2}\backslash\{\ell\}$, and ℓ is odd with $n-2p-1 \leqslant \ell \leqslant n$,

$$F(2p+2, 2r+1) = \widetilde{F}(2p+1, 2r+1) + a_\ell \widetilde{F}(2p+1, 2r).$$

By (2.7), $F(2p+2, 2r+1) = 0$, so $-a_\ell \widetilde{F}(2p+1, 2r) = \widetilde{F}(2p+1, 2r+1)$. This implies that $\widetilde{F}(2p+1, 2r+1) = (-1)^{r+1}\binom{p}{r}$ by (3.2), since $a_\ell = 1$. □

Lemma 3.2. *Let $k \geqslant 2$ and $n = 2k$. If (3.1) holds, then the derivatives of the coefficients in the characteristic polynomial (1.1) of M_n are, for $1 \leqslant j \leqslant k$, given by*

$$\frac{\partial \mu_{2r}}{\partial a_{2j-1}} = \widetilde{F}(2k-1, 2r-1) - \sum_{i=1}^{r-1} b_{2i} \widetilde{F}(2k-1-2i, 2r-1-2i)$$

when $2 \leqslant 2r \leqslant 2j$ and

$$\frac{\partial \mu_{2r+1}}{\partial a_{2j-1}} = \widetilde{F}(2k-1, 2r) - \sum_{i=1}^{r} b_{2i} \widetilde{F}(2k-1-2i, 2r-2i)$$

when $1 \leqslant 2r+1 \leqslant 2j-1$.

Proof. Consider (2.4), in which all $F(2p, w)$ are defined with respect to $A_{n,2p} = \{n-2p+1, n-2p+2, \ldots, n\}$ and (3.1) holds. For some fixed j with $1 \leqslant j \leqslant k$, factor a_{2j-1} from this expression to give

$$\mu_{2r} = a_{2j-1}\left(\widetilde{F}(2k-1, 2r-1) - \sum_{i=1}^{r-1} b_{2i} \widetilde{F}(2k-1-2i, 2r-1-2i)\right) + f(a_1, a_3, \ldots, a_{2j-3}, a_{2j+1}, \ldots, a_{2k-1}, b_2, b_4, \ldots, b_{2r}) \tag{3.4}$$

if $2r \leqslant 2j$, where each term $\widetilde{F}(2p-1, w)$ above is defined with respect to $\widetilde{A}_{n,2p-1} \equiv A_{n,2p}\backslash\{2j-1\}$. That is, all such (nontrivial) terms $F(2p, w)$ have $2j-1 \in A_{n,2p}$ and thus can be expressed as

$$F(2p, w) = a_{2j-1}\widetilde{F}(2p-1, w-1) + f_1(a_{2k-2p+1}, a_{2k-2p+3}, \ldots, a_{2j-3}, a_{2j+1}, \ldots, a_{2k-1}),$$

where f_1 is the sum of terms that do not include a_{2j-1} as a factor. Note that the product $a_{2j-1}b_{2i}$, for $1 \leqslant i \leqslant r-1$, occurs in (2.4) if and only if $2r \leqslant 2j$ (by the definition of $F(2p, w)$). Differentiating (3.4) with respect to a_{2j-1} gives

$$\frac{\partial \mu_{2r}}{\partial a_{2j-1}} = \widetilde{F}(2k-1, 2r-1) - \sum_{i=1}^{r-1} b_{2i} \widetilde{F}(2k-1-2i, 2r-1-2i)$$

as required.

Similarly, (2.5) can be written as

$$\mu_{2r+1} = a_{2j-1}\left(\widetilde{F}(2k-1,2r) - \sum_{i=1}^{r} b_{2i}\widetilde{F}(2k-1-2i,2r-2i)\right) + f(a_1,a_3,\ldots,a_{2j-3},a_{2j+1},\ldots,a_{2k-1},b_2,b_4,\ldots,b_{2r}), \tag{3.5}$$

if $2r+1 \leqslant 2j-1$. Note that the product $a_{2j-1}b_{2i}$, for $1 \leqslant i \leqslant r$, occurs in (2.5) if and only if $2r+1 \leqslant 2j-1$ (by the definition of $F(2p,w)$). Differentiating (3.5) with respect to a_{2j-1} gives

$$\frac{\partial\mu_{2r+1}}{\partial a_{2j-1}} = \widetilde{F}(2k-1,2r) - \sum_{i=1}^{r} b_{2i}\widetilde{F}(2k-1-2i,2r-2i)$$

as required. □

Lemma 3.3. *If $v \geqslant 0$ and $k \geqslant v+1$, then*

$$\sum_{u=0}^{v}(-1)^u\binom{v}{u}\frac{1}{k-u} = \frac{(-1)^v}{(k-v)\binom{k}{v}}.$$

The above identity (see [5, Identity 1.43]), which can be proven by induction, is used to prove the following lemma. For $k \geqslant 2$ and $n = 2k$, define the $n \times n$ matrix $D_n = [d_{ij}]$ where for $1 \leqslant i \leqslant n$,

$$d_{ij} = (-1)^i\frac{\partial\mu_i}{\partial a_{2j-1}} \quad\text{and}\quad d_{i,k+j} = (-1)^i\frac{\partial\mu_i}{\partial b_{2j}} \qquad \text{for } 1 \leqslant j \leqslant k. \tag{3.6}$$

Lemma 3.4. *Let $k \geqslant 2$, $n = 2k$, $1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant 2j$. If (3.1) holds and d_{ij} is evaluated with $b_{2r} = -\binom{k}{r}$ and $a_{2r-1} = 1$ for $1 \leqslant r \leqslant k$, then $d_{ij} = -1$.*

Proof. Fix j, $1 \leqslant j \leqslant k$ and fix i, $1 \leqslant i \leqslant 2j$. If i is odd ($i = 2g+1$), then by Lemma 3.2

$$\frac{\partial\mu_{2g+1}}{\partial a_{2j-1}} = \widetilde{F}(2k-1,2g) + \sum_{u=1}^{g}\binom{k}{u}\widetilde{F}(2k-1-2u,2g-2u).$$

Applying (3.2) gives

$$\begin{aligned}\frac{\partial\mu_{2g+1}}{\partial a_{2j-1}} &= (-1)^g\binom{k-1}{g} + \sum_{u=1}^{g}\binom{k}{u}(-1)^{g-u}\binom{k-1-u}{g-u}\\ &= \sum_{u=0}^{g}(-1)^{g-u}\binom{k}{u}\binom{k-1-u}{g-u}\\ &= \sum_{u=0}^{g}(-1)^{g-u}\frac{k}{k-u}\binom{k-1}{g}\binom{g}{u}\\ &= (-1)^g k\binom{k-1}{g}\sum_{u=0}^{g}(-1)^u\frac{1}{k-u}\binom{g}{u}.\end{aligned} \tag{3.7}$$

Using Lemma 3.3 with v as g gives

$$\frac{\partial\mu_{2g+1}}{\partial a_{2j-1}} = (-1)^g k \binom{k-1}{g} \frac{(-1)^g}{(k-g)\binom{k}{g}}$$
$$= \frac{k(k-1)!}{g!(k-1-g)!} \frac{g!(k-g)!}{(k-g)k!} = 1.$$

Recalling that $d_{ij} = -\frac{\partial\mu_i}{\partial a_{2j-1}}$ for i odd, $d_{2g+1,j} = -1$ as required.
If i is even ($i = 2g$), then using Lemma 3.2 and applying (3.3) gives

$$\frac{\partial\mu_{2g}}{\partial a_{2j-1}} = \widetilde{F}(2k-1, 2g-1) + \sum_{u=1}^{g-1} \binom{k}{u} \widetilde{F}(2k-1-2u, 2g-1-2u)$$
$$= (-1)^g \binom{k-1}{g-1} + \sum_{u=1}^{g-1} \binom{k}{u} (-1)^{g-u} \binom{k-1-u}{g-1-u}$$
$$= \sum_{u=0}^{g-1} (-1)^{g-u} \binom{k}{u} \binom{k-1-u}{g-1-u}$$
$$= -\sum_{u=0}^{g-1} (-1)^{g-1-u} \binom{k}{u} \binom{k-1-u}{g-1-u}.$$

This is the negative of (3.7) with g replaced by $g-1$. Recalling the definition in (3.6), a similar argument to the i odd case gives $d_{2g,j} = -1$ as required. □

The following identity established in the above proof is used in the proof of the next lemma.

Corollary 3.5. *For $k \geqslant 2$ and $0 \leqslant g \leqslant k-1$,*

$$\sum_{u=0}^{g} (-1)^{g-u} \binom{k}{u} \binom{k-1-u}{g-u} = 1.$$

Lemma 3.6. *Let $k \geqslant 2$, $n = 2k$ and $1 \leqslant j \leqslant k-1$. If (3.1) holds and $d_{2j+1,j}$ is evaluated with $b_{2r} = -\binom{k}{r}$ and $a_{2r-1} = 1$ for $1 \leqslant r \leqslant k$, then $d_{2j+1,j} \geqslant 1$.*

Proof. If the variables a_{2r-1} and b_{2r} for $1 \leqslant r \leqslant k$ are not assigned values, then μ_{2j+1} has terms containing the factors $a_{2j-1}b_h$ for all $h = 2, 4, \ldots, 2j-2$ since such a factor occurs in some cycle product of size $2j+1$. However, the factor $a_{2j-1}b_{2j}$ occurs nowhere in μ_{2j+1} since the cycle of length $2j$ and the loop at vertex $2j-1$ are not disjoint. Thus, analogous to (3.5), Eq. (2.5) can be written as

$$\mu_{2j+1} = a_{2j-1}\left(\widetilde{F}(2k-1, 2j) - \sum_{i=1}^{j-1} b_{2i}\widetilde{F}(2k-1-2i, 2j-2i)\right)$$
$$+ f(a_1, a_3, \ldots, a_{2j-3}, a_{2j+1}, \ldots, a_{2k-1}, b_2, b_4, \ldots, b_{2j}).$$

Differentiating with respect to a_{2j-1} gives

$$\frac{\partial\mu_{2j+1}}{\partial a_{2j-1}} = \widetilde{F}(2k-1, 2j) - \sum_{i=1}^{j-1} b_{2i}\widetilde{F}(2k-1-2i, 2j-2i).$$

Thus, if (3.1) holds, assigning $a_{2r-1} = 1$ and $b_{2r} = -\binom{k}{r}$ for $1 \leqslant r \leqslant k$, application of (3.2) gives

$$\begin{aligned}\frac{\partial\mu_{2j+1}}{\partial a_{2j-1}} &= (-1)^j\binom{k-1}{j} + \sum_{i=1}^{j-1}\binom{k}{i}(-1)^{j-i}\binom{k-1-i}{j-i}\\ &= \left[\sum_{i=0}^{j}(-1)^{j-i}\binom{k}{i}\binom{k-1-i}{j-i}\right] - (-1)^0\binom{k}{j}\binom{k-1-j}{0}.\end{aligned} \tag{3.8}$$

Applying Corollary 3.5 to (3.8) gives

$$\frac{\partial\mu_{2j+1}}{\partial a_{2j-1}} = 1 - \binom{k}{j}, \tag{3.9}$$

which implies that $d_{2j+1,j} = \binom{k}{j} - 1$. Since $\binom{k}{j} \geqslant 2$ if $1 \leqslant j \leqslant k-1$ and $k \geqslant 2$, it follows that $d_{2j+1,j} \geqslant 1$. □

We now consider columns $k+1, k+2, \ldots, 2k$ of D_n, the entries of which are defined in terms of derivatives with respect to b_{2j} for $j = 1, 2, \ldots, k$.

Lemma 3.7. *Let D_n be defined as in (3.6) with $k \geqslant 2$ and $n = 2k$.*

(i) *If $1 \leqslant j \leqslant k$ and $1 \leqslant i \leqslant 2j-1$, then $d_{i,k+j} = 0$.*
(ii) *If $1 \leqslant j \leqslant k$, then $d_{2j,k+j} = -1$.*
(iii) *If (2.3) holds, $1 \leqslant j \leqslant k-1$ and $i = 1, 3, \ldots, 2k-2j-1$, then $d_{2j+i,k+j} = 0$.*

Proof. Fix column $k+j$ for $1 \leqslant j \leqslant k$. The variable b_{2j} does not appear in $\mu_1, \mu_2, \ldots, \mu_{2j-1}$ since b_{2j} occurs only in a cycle product of size at least $2j$. Therefore $d_{i,k+j} = 0$ for $i = 1, 2, \ldots, 2j-1$, and (i) follows.

Fix column $k+j$ for $1 \leqslant j \leqslant k$. In μ_{2j}, the variable b_{2j} appears once with coefficient -1, therefore $\frac{\partial\mu_{2j}}{\partial b_{2j}} = d_{2j,k+j} = -1$, and (ii) follows.

Fix column $k+j$ for $1 \leqslant j \leqslant k-1$ and assume that (2.3) holds. Consider $\mu_{2j+1}, \mu_{2j+3}, \ldots, \mu_{2k-1}$ as given by (2.5). The coefficient of b_{2j} in μ_{2j+i} is $-F(2k-2j, i)$, $i = 1, 3, \ldots, 2k-2j-1$. This coefficient is 0 by (2.7) and (iii) follows. □

Theorem 3.8. *Let $k \geqslant 2$ and $n = 2k$. If D_n is evaluated when (2.3) holds and $b_{2r} = -\binom{k}{r}$, $1 \leqslant r \leqslant k$, then* $\det(D_n) \neq 0$.

Proof. By Lemmas 3.4, 3.6 and 3.7, if D_n is evaluated with a_i and b_{2r} as stated, then D_n has the form

$$\begin{bmatrix}
-1 & -1 & -1 & \dots & -1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\
-1 & -1 & -1 & & -1 & -1 & -1 & 0 & 0 & & 0 & 0 \\
c_1 & -1 & \ddots & \ddots & -1 & -1 & 0 & 0 & \ddots & \ddots & 0 & 0 \\
d_{41} & -1 & \ddots & \ddots & -1 & -1 & d_{4,k+1} & -1 & 0 & \ddots & 0 & 0 \\
d_{51} & c_2 & -1 & \ddots & -1 & -1 & 0 & 0 & 0 & \ddots & 0 & 0 \\
d_{61} & d_{62} & -1 & \ddots & -1 & -1 & d_{6,k+1} & d_{6,k+2} & -1 & & 0 & 0 \\
d_{71} & d_{72} & c_3 & & -1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\
d_{81} & d_{82} & d_{83} & & -1 & -1 & d_{8,k+1} & d_{8,k+2} & d_{8,k+3} & & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
d_{n-2,1} & d_{n-2,2} & d_{n-2,3} & \dots & -1 & -1 & d_{n-2,k+1} & d_{n-2,k+2} & d_{n-2,k+3} & \dots & -1 & 0 \\
d_{n-1,1} & d_{n-1,2} & d_{n-1,3} & \dots & c_{k-1} & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\
d_{n1} & d_{n2} & d_{n3} & \dots & d_{n,k-1} & -1 & d_{n,k+1} & d_{n,k+2} & d_{n,k+3} & \dots & d_{n,2k-1} & -1
\end{bmatrix},$$

where $c_i \geqslant 1$ for $1 \leqslant i \leqslant k-1$ by Lemma 3.6. Expansion along columns in the order $2k, 2k-1, \dots, k+1$, gives

$$\det(D_n) = (-1)^{\lceil k/2 \rceil} \det(G_k)$$

with

$$G_k = \begin{bmatrix}
-1 & -1 & -1 & -1 & \dots & -1 & -1 \\
c_1 & -1 & -1 & -1 & & -1 & -1 \\
d_{51} & c_2 & \ddots & \ddots & \ddots & -1 & -1 \\
d_{71} & d_{72} & c_3 & \ddots & \ddots & -1 & -1 \\
d_{91} & d_{92} & d_{93} & c_4 & \ddots & -1 & -1 \\
\vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
d_{n-1,1} & d_{n-1,2} & d_{n-1,3} & d_{n-1,4} & & c_{k-1} & -1
\end{bmatrix}.$$

Using elementary row operations (subtracting row 1 from row i for $2 \leqslant i \leqslant k$), $\det(G_k) = \det(\widehat{G}_k)$ where

$$\widehat{G}_k = \begin{bmatrix}
-1 & -1 & -1 & -1 & \dots & -1 & -1 \\
c_1+1 & 0 & 0 & 0 & & 0 & 0 \\
d_{51}+1 & c_2+1 & \ddots & \ddots & \ddots & 0 & 0 \\
d_{71}+1 & d_{72}+1 & c_3+1 & \ddots & \ddots & 0 & 0 \\
d_{91}+1 & d_{92}+1 & d_{93}+1 & c_4+1 & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
d_{n-1,1}+1 & d_{n-1,2}+1 & d_{n-1,3}+1 & d_{n-1,4}+1 & & c_{k-1}+1 & 0
\end{bmatrix}.$$

Since each $c_i \geqslant 1$, $\det(\widehat{G}_k) \neq 0$ and thus $\det(D_n) \neq 0$. □

We now consider the case that n is odd. For $k \geqslant 2$, let M_{2k+1} have characteristic polynomial $p_{2k+1}(x)$ as in (1.1) with μ_i replaced by $\tilde{\mu}_i$, and let $\{\hat{a}, \tilde{a}\} = \{a_{2k-1}, a_{2k}\}$. Define the $(2k+1) \times (2k+1)$ matrix $\widetilde{D}_{2k+1} = [\tilde{d}_{ij}]$ where for $1 \leqslant i \leqslant 2k+1$,

$$\tilde{d}_{ij} = (-1)^i \frac{\partial \tilde{\mu}_i}{\partial a_{2j-1}}, \quad 1 \leqslant j \leqslant k-1,$$

$$\tilde{d}_{i,k+j} = (-1)^i \frac{\partial \tilde{\mu}_i}{\partial b_{2j}}, \quad 1 \leqslant j \leqslant k,$$

$$\tilde{d}_{ik} = (-1)^i \frac{\partial \tilde{\mu}_i}{\partial \hat{a}} \quad \text{and} \quad \tilde{d}_{i,2k+1} = (-1)^i \frac{\partial \tilde{\mu}_i}{\partial b_{2k+1}}.$$

Let M_{2k} be as in (2.1) with characteristic polynomial $p_{2k}(x)$ as in (1.1), and let D_{2k} be the associated $2k \times 2k$ matrix of partial derivatives as in (3.6). The following result relates D_{2k} to $\widetilde{D}_{2k+1}$ as defined above.

Theorem 3.9. *Let $k \geqslant 2$. Suppose D_{2k} is evaluated when (2.3) holds and $b_{2r} = -\binom{k}{r}$ for $1 \leqslant r \leqslant k$, and $\widetilde{D}_{2k+1}$ is evaluated when $0 = \tilde{a} \in \{a_{2k-1}, a_{2k}\}$, all other a_i (including $\hat{a}$) alternate in value $1, -1, 1, -1, \ldots, 1, -1$, $b_{2r} = -\binom{k}{r}$ for $1 \leqslant r \leqslant k$ and $b_{2k+1} = 1$. Then*

$$\widetilde{D}_{2k+1} = \begin{bmatrix} & & & & & 0 \\ & & D_{2k} & & & 0 \\ & & & & & \vdots \\ & & & & & 0 \\ 0 & 0 & \ldots & 0 & -1 & -1 \end{bmatrix} \quad \text{and} \quad \det(\widetilde{D}_{2k+1}) \neq 0.$$

Proof. To show that $\tilde{d}_{ij} = d_{ij}$ for $1 \leqslant i \leqslant 2k$ and $1 \leqslant j \leqslant 2k$ when they are evaluated with a_i and b_{2r} as stated, consider the sums of cycle products of size m that give coefficients μ_m and $\tilde{\mu}_m$ for $1 \leqslant m \leqslant 2k$ of $p_{2k}(x)$ and $p_{2k+1}(x)$, respectively. Recall that each such cycle product corresponds to a simple cycle, a composite cycle of loops, or a composite cycle of some even length cycle and some loops. Note that $-\mu_1 = -\sum_{j=1}^{2k} a_j$ and $-\tilde{\mu}_1 = -\sum_{j=1}^{2k+1} a_j$, and thus $\tilde{d}_{1j} = d_{1j} = -1$ for $1 \leqslant j \leqslant 2k$.

With the equality established when the relevant cycle products are size $m = 1$, let $m \geqslant 2$. The cycle products corresponding to a simple cycle are of even size and are given by $-b_m$, which appears in both μ_m and $\tilde{\mu}_m$. The sum of the cycle products corresponding to a composite cycle of loops is given by $F(2k, m)$ in μ_m (see the first terms in (2.4) and (2.5)) and analogously by $F(2k+1, m)$ in $\tilde{\mu}_m$. The sum of the cycle products that correspond to a composite cycle of some even length cycle and some loops is given by $-b_{2r}F(2k-2r, m-2r)$ in μ_m (see the summations in (2.4) and (2.5)) and analogously by $-b_{2r}F(2k+1-2r, m-2r)$ in $\tilde{\mu}_m$ for $2 \leqslant 2r < m \leqslant 2k$. We now argue that with a_i as specified, $F(2k, m)$ and $F(2k+1, m)$ are equal for $2 \leqslant m \leqslant 2k$ and that $F(2k-2r, m-2r)$ and $F(2k+1-2r, m-2r)$ are equal for fixed r, m and k such that $2 \leqslant 2r < m \leqslant 2k$. By the definition of cycle products and the F function, each such F function in μ_m is defined with respect to $A_{2k,2k-2r} = \{2r+1, 2r+2, \ldots, 2k\} \equiv A_{\text{even}}$. Similarly, each such F function in $\tilde{\mu}_m$ is defined with respect to $A_{2k+1,2k+1-2r} = \{2r+1, 2r+2, \ldots, 2k-2, 2k-1, 2k, 2k+1\} \equiv A_{\text{odd}}$. Define A to be the intersection of these sets, that is $A \equiv A_{\text{even}} \cap A_{\text{odd}} = \{2r+1, 2r+2, \ldots, 2k-2\}$. Note that $A_{\text{even}} = A \cup \{2k-1, 2k\}$, $A_{\text{odd}} = A \cup \{2k-1, 2k, 2k+1\}$ and that A is empty when $2r = 2k-2$. Since terms in $p_{2k+1}(x)$ with $\tilde{a}$ as a factor are equal to 0, all cycle products in μ_m are the same as those in $\tilde{\mu}_m$ with $\hat{a}$ replaced by a_{2k-1} and with a_{2k+1} replaced by a_{2k}. Therefore, after differentiating μ_m and $\tilde{\mu}_m$ with respect to one of $a_1, a_3, \ldots, a_{2k-3}, b_2, b_4, \ldots, b_{2k}$ and assigning values to all variables as in the theorem statement, it follows that $\tilde{d}_{ij} = d_{ij}$ for $1 \leqslant i \leqslant 2k$ and $j = 1, 2, \ldots, k-1, k+1, \ldots, 2k$. Note that this assignment has a_{2k-1} and a_{2k} in μ_m taking the values 1 and -1, respectively, $\hat{a}$ and a_{2k+1} in $\tilde{\mu}_m$ taking the values 1 and -1, respectively, and each of the variables $a_1, a_2, \ldots, a_{2k-3}, b_2, b_4, \ldots, b_{2k}$ taking the same value in μ_m and $\tilde{\mu}_m$. Similarly, differentiating μ_m with respect to a_{2k-1} and evaluating the variables gives the same value as differentiating $\tilde{\mu}_m$ with respect to $\hat{a}$ and evaluating the variables. Thus $\tilde{d}_{ik} = d_{ik}$ for $1 \leqslant i \leqslant 2k$, and it follows that

$$\tilde{d}_{ij} = d_{ij} \quad \text{for } 1 \leqslant i \leqslant 2k, 1 \leqslant j \leqslant 2k. \tag{3.10}$$

The corresponding cycle product b_{2k+1} of the $(2k+1)$-cycle in $D(M_{2k+1})$ does not appear in any of the coefficients $\tilde{\mu}_m$ for $1 \leqslant m \leqslant 2k$ in $p_{2k+1}(x)$. Therefore,

$$\tilde{d}_{i,2k+1} = 0 \quad \text{for } 1 \leqslant i \leqslant 2k. \tag{3.11}$$

In $D(M_{2k+1})$ the only cycle products of size $2k+1$ are b_{2k+1} from the $(2k+1)$-cycle and $-a_{2k+1}b_{2k}$ from the composite cycle of the loop at vertex $2k+1$ and the $2k$-cycle, so $-\tilde{\mu}_{2k+1} = -(b_{2k+1} - a_{2k+1}b_{2k})$. Assigning a_{2k+1} the value -1 gives $-\tilde{\mu}_{2k+1} = -b_{2k+1} - b_{2k}$, which implies that

$$\tilde{d}_{2k+1,2k+1} = -1, \quad \tilde{d}_{2k+1,2k} = -1 \quad \text{and} \quad \tilde{d}_{2k+1,j} = 0 \quad \text{for } 1 \leqslant j \leqslant 2k-1. \tag{3.12}$$

Thus, by (3.10), (3.11) and (3.12) the matrix $\widetilde{D}_{2k+1}$ has the stated form. By expanding $\det(\widetilde{D}_{2k+1})$ along column $2k+1$ and using Theorem 3.9, it follows that $\det(\widetilde{D}_{2k+1}) \neq 0$. □

Remark 3.10. Similar results as in Theorems 3.8 and 3.9 hold for other assignments of the a_i as in Remark 2.6. When $k \geqslant 2$ and $n = 2k$, define D_n with respect to the partial derivatives of the positive a_i and b_{2j}, cf. (3.6). Then D_n has the same form as in the proof of Theorem 3.8, and $\det(D_n) \neq 0$. For $k \geqslant 2$ and $n = 2k+1$, with a_i assigned as in Remark 2.6, the method used in Theorem 3.9 holds showing that $\det(\widetilde{D}_{2k+1}) = -\det(D_{2k}) \neq 0$.

4. Spectrally arbitrary even cycle patterns and minimality

Define the $n \times n$ sign pattern $\mathscr{C}_n$ as follows. For $k \geqslant 2$ and $n = 2k$, let

$$\mathscr{C}_{2k} = \begin{bmatrix} + & + & & & & & & \\ - & - & + & & & & & \\ 0 & & + & + & & 0 & & \\ - & & & - & + & & & \\ 0 & & & & + & \ddots & & \\ \vdots & & 0 & & & \ddots & + & \\ 0 & & & & & & + & + \\ - & & & & & & & - \end{bmatrix}$$

and for $n = 2k+1$, let

$$\mathscr{C}_{2k+1} = \begin{bmatrix} + & + & & & & & & \\ - & - & + & & & & & \\ 0 & & + & + & & & 0 & \\ - & & & - & \ddots & & & \\ \vdots & & & & \ddots & + & & \\ 0 & & 0 & & & + & + & \\ - & & & & & & 0 & + \\ + & & & & & & & - \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} + & + & & & & & & \\ - & - & + & & & & & \\ 0 & & + & + & & & 0 & \\ - & & & - & \ddots & & & \\ \vdots & & & & \ddots & + & & \\ 0 & & 0 & & & 0 & + & \\ - & & & & & & + & + \\ + & & & & & & & - \end{bmatrix}.$$

The nilpotent matrices M_n defined in Theorems 2.3 and 2.5 have $M_n \in Q(\mathscr{C}_n)$. Note that $D(\mathscr{C}_n)$ has a cycle of every even length with the product of entries corresponding to each even cycle being negative, and thus we call $\mathscr{C}_n$ an *even cycle sign pattern*. Using the results of Theorems 2.3 and 3.8 (for n even) and Theorems 2.5 and 3.9 (for n odd) with the methodology of [3, Observation 10], the following result is immediate.

Theorem 4.1. *For $n \geqslant 4$, $\mathscr{C}_n$ is spectrally arbitrary. Moreover, any superpattern of $\mathscr{C}_n$ is spectrally arbitrary.*

To investigate if $\mathscr{C}_n = [c_{ij}]$ is a minimal SAP for n even, we proceed as follows. Let $\mathscr{R}_n = [r_{ij}]$ be an irreducible subpattern of $\mathscr{C}_n$ (thus r_{n1} is negative, $r_{i,i+1}$ is positive for $1 \leqslant i \leqslant n-1$ and the other r_{ij} can be equal to c_{ij} or 0). If $Y_n \in Q(\mathscr{R}_n)$ for $k \geqslant 2$ and $n = 2k$, then without loss of generality the superdiagonal entries of Y_n can be normalized to 1 by a positive diagonal similarity transformation. Thus we can restrict attention to $M_n \in Q(\mathscr{R}_n)$ as given in (2.1). Our goal is to show that if $M_n \in Q(\mathscr{R}_n)$ is nilpotent for $k \geqslant 2$ and $n = 2k$, then $a_{2i} = -a_{2i-1}$ and $b_{2i} < 0$ for $1 \leqslant i \leqslant k$ (see Corollary 4.5). From this, some properties of the structure of irreducible potentially nilpotent subpatterns of $\mathscr{C}_n$ are given in Theorem 4.6 and a conjecture on the minimality of $\mathscr{C}_n$ is stated.

For $v \geqslant 1$, define $P_{2v}(x_1, x_2, \ldots, x_n)$ equal to the sum of all products of nonnegative even powers of x_i where the sum of the powers is $2v$. That is,

$$P_{2v}(x_1, x_2, \ldots, x_n) = \sum {x_1}^{2i_1} {x_2}^{2i_2} \cdots {x_n}^{2i_n}, \quad \text{where} \quad i_1 + i_2 + \cdots + i_n = v,$$

and define $P_0(x_1, x_2, \ldots, x_n) = 1$.

Lemma 4.2. *For real c_i, s and t and integer $r \geqslant 0$, if*

$$H_{2v}(c_1, c_2, \ldots, c_r; s, t) = \begin{cases} \frac{P_{2v}(c_1,c_2,\ldots,c_r,s) - P_{2v}(c_1,c_2,\ldots,c_r,t)}{s^2-t^2} & if\ |s| \neq |t|, \\ \lim_{|s|\to|t|} \frac{P_{2v}(c_1,c_2,\ldots,c_r,s) - P_{2v}(c_1,c_2,\ldots,c_r,t)}{s^2-t^2} & if\ |s| = |t|, \end{cases}$$

then

$$H_{2v}(c_1, c_2, \ldots, c_r; s, t) = P_{2v-2}(c_1, c_2, \ldots, c_r, s, t).$$

Proof. When $r = 0$,

$$H_{2v}(s, t) = \begin{cases} \frac{s^{2v} - t^{2v}}{s^2 - t^2} & \text{if } |s| \neq |t|, \\ \lim_{|s|\to|t|} \frac{s^{2v}-t^{2v}}{s^2-t^2} & \text{if } |s| = |t|. \end{cases}$$

Since

$$s^{2v} - t^{2v} = (s^2 - t^2)\sum_{i=0}^{v-1} s^{2i} t^{2v-2i-2},$$

$$H_{2v}(s,t) = \begin{cases} \sum_{i=0}^{v-1} s^{2i} t^{2v-2i-2} = P_{2v-2}(s,t) & \text{if } |s| \neq |t|, \\ \lim_{|s|\to|t|} \sum_{i=0}^{v-1} s^{2i} t^{2v-2i-2} = P_{2v-2}(t,t) & \text{if } |s| = |t|. \end{cases}$$

When $r \geqslant 1$, since

$$P_{2v}(c_1, c_2, \ldots, c_r, s) = \sum_{i=0}^{v} s^{2i} P_{2v-2i}(c_1, c_2, \ldots, c_r),$$

$$\begin{aligned} &\frac{P_{2v}(c_1, c_2, \ldots, c_r, s) - P_{2v}(c_1, c_2, \ldots, c_r, t)}{s^2 - t^2} \\ &\quad = \sum_{i=1}^{v} \frac{s^{2i} - t^{2i}}{s^2 - t^2} P_{2v-2i}(c_1, c_2, \ldots, c_r) \\ &\quad = \sum_{i=1}^{v} P_{2i-2}(s,t) P_{2v-2i}(c_1, c_2, \ldots, c_r), \text{ by the } r = 0 \text{ case above,} \\ &\quad = P_{2v-2}(c_1, c_2, \ldots, c_r, s, t). \end{aligned}$$

The above argument also shows the limiting case. □

For $k \geqslant 2$ and $n = 2k$, let $M_n \in Q(\mathscr{R}_n)$ be nilpotent. From (2.1), the characteristic polynomial $\det(xI - M_n)$ is given by

$$p_n(x) = x^n = -\sum_{i=0}^{n/2} b_{2i} \prod_{j=2i+1}^{n} (x - a_j), \quad \text{where } b_0 \equiv -1.$$

Remark 4.3. Consider $p_n(a_n) = a_n^n = -b_n$, and $p_n(a_{n-1}) = a_{n-1}^n = -b_n$. Therefore, for some $c_0 > 0$ (since $b_n \neq 0$), $a_n = -c_0$, $a_{n-1} = c_0$ and $b_n = -c_0^n = -P_n(c_0)$. Thus, if M_n is nilpotent, it follows that a_n and a_{n-1}, which are of opposite sign since $M_n \in Q(\mathscr{R}_n)$, must be nonzero and of the same magnitude.

As motivation for the proof of the next lemma, consider

$$p_n(a_{n-2}) = -b_{n-2}(a_{n-2} - c_0)(a_{n-2} + c_0) - b_n = a_{n-2}^n.$$

Thus

$$-b_{n-2}(a_{n-2}^2 - c_0^2) + c_0^n = a_{n-2}^n$$

and if $|a_{n-2}| \neq |c_0|$, then by Lemma 4.2

$$-b_{n-2} = \frac{a_{n-2}^n - c_0^n}{a_{n-2}^2 - c_0^2} = P_{n-2}(c_0, a_{n-2}).$$

If $|a_{n-2}| = |c_0|$, then the same result follows by the limiting case of Lemma 4.2. Simliarly, considering $p_n(a_{n-3})$ leads to the equation

$$-b_{n-2} = P_{n-2}(c_0, a_{n-3}).$$

This implies that

$$P_{n-2}(c_0, a_{n-2}) = P_{n-2}(c_0, a_{n-3}). \tag{4.1}$$

If $g(x) = P_n(c_0, x)$, then $g(x)$ is even and strictly increasing on $(0, \infty)$. Since (4.1) implies that $g(a_{n-2}) = g(a_{n-3})$, it follows that $|a_{n-2}| = |a_{n-3}|$. That is, for some $c_1 \geqslant 0$, $a_{n-2} = -c_1$, $a_{n-3} = c_1$ and $b_{n-2} = -P_{n-2}(c_0, c_1)$.

Lemma 4.4. *Let $k \geqslant 2$ and M_{2k} be nilpotent with $M_{2k} \in Q(\mathscr{R}_{2k})$. For a fixed p with $1 \leqslant p \leqslant k-1$, suppose $c_0 > 0$, $c_{k-j} \geqslant 0$ for $p+1 \leqslant j \leqslant k-1$, $-a_{2i} = a_{2i-1} = c_{k-i}$ and $-b_{2i} = P_{2i}(c_0, c_1, \ldots, c_{k-i})$ for $p+1 \leqslant i \leqslant k$. Then*

$$-b_{2p} \prod_{j=p+1}^{q} (a_{2p}^2 - c_{k-j}^2) + \sum_{i=p+1}^{q} P_{2i}(c_0, c_1, \ldots, c_{k-i}) \prod_{j=i+1}^{q} (a_{2p}^2 - c_{k-j}^2)$$
$$= P_{2q}(c_0, c_1, \ldots, c_{k-q-1}, a_{2p})$$

for $p \leqslant q \leqslant k-1$.

Proof. Since M_{2k} is nilpotent,

$$p_{2k}(a_{2p}) = a_{2p}^{2k} = -\sum_{i=0}^{k} b_{2i} \prod_{j=2i+1}^{2k} (a_{2p} - a_j)$$

and it follows from Remark 4.3 that

$$-\sum_{i=p}^{k-1} b_{2i} \prod_{j=2i+1}^{2k} (a_{2p} - a_j) = a_{2p}^{2k} - c_0^{2k}.$$

Thus, by the assumptions on b_{2i}, a_{2i} and a_{2i-1},

$$-b_{2p} \prod_{j=p+1}^{k} (a_{2p}^2 - c_{k-j}^2) + \sum_{i=p+1}^{k-1} P_{2i}(c_0, c_1, \ldots, c_{k-i}) \prod_{j=i+1}^{k} (a_{2p}^2 - c_{k-j}^2) = a_{2p}^{2k} - c_0^{2k}$$

and if $|a_{2p}| \neq |c_0|$, then

$$-b_{2p} \prod_{j=p+1}^{k-1} (a_{2p}^2 - c_{k-j}^2) + \sum_{i=p+1}^{k-1} P_{2i}(c_0, c_1, \ldots, c_{k-i}) \prod_{j=i+1}^{k-1} (a_{2p}^2 - c_{k-j}^2) = \frac{a_{2p}^{2k} - c_0^{2k}}{a_{2p}^2 - c_0^2}$$
$$= P_{2k-2}(c_0, a_{2p}) \text{ by Lemma 4.2.}$$

If $|a_{2p}| = |c_0|$, then the same result follows by the limiting case of Lemma 4.2. This establishes the statement for $q = k-1$. Now assume the statement holds for $q = v$ with $p+1 \leqslant v \leqslant k-1$; i.e.,

$$-b_{2p} \prod_{j=p+1}^{v} (a_{2p}^2 - c_{k-j}^2) + \sum_{i=p+1}^{v} P_{2i}(c_0, c_1, \ldots, c_{k-i}) \prod_{j=i+1}^{v} (a_{2p}^2 - c_{k-j}^2)$$
$$= P_{2v}(c_0, c_1, \ldots, c_{k-v-1}, a_{2p}).$$

Therefore

$$
\begin{aligned}
&-b_{2p}\prod_{j=p+1}^{v}(a_{2p}^2-c_{k-j}^2)+\sum_{i=p+1}^{v-1}P_{2i}(c_0,c_1,\ldots,c_{k-i})\prod_{j=i+1}^{v}(a_{2p}^2-c_{k-j}^2)\\
&\quad = P_{2v}(c_0,c_1,\ldots,c_{k-v-1},a_{2p})-P_{2v}(c_0,c_1,\ldots,c_{k-v-1},c_{k-v})
\end{aligned}
$$

and thus

$$
\begin{aligned}
&-b_{2p}\prod_{j=p+1}^{v-1}(a_{2p}^2-c_{k-j}^2)+\sum_{i=p+1}^{v-1}P_{2i}(c_0,c_1,\ldots,c_{k-i})\prod_{j=i+1}^{v-1}(a_{2p}^2-c_{k-j}^2)\\
&\quad = \frac{P_{2v}(c_0,c_1,\ldots,c_{k-v-1},a_{2p})-P_{2v}(c_0,c_1,\ldots,c_{k-v-1},c_{k-v})}{a_{2p}^2-c_{k-v}^2}\quad \text{if } |a_{2p}|\neq|c_{k-v}|\\
&\quad = P_{2v-2}(c_0,c_1,\ldots,c_{k-v-1},c_{k-v},a_{2p}) \text{ by Lemma 4.2.}
\end{aligned}
$$

If $|a_{2p}| = |c_{k-v}|$, then the same result follows by the limiting case of Lemma 4.2. Thus, the statement is true for $q = v - 1$, and the result follows by downward induction on q. □

The following result shows that if M_n is nilpotent and $n = 2k$, then $a_{2i} + a_{2i-1} = 0$ and $b_{2i} < 0$ for $1 \leqslant i \leqslant k$.

Corollary 4.5. *Let $k \geqslant 2$ and M_{2k} be nilpotent with $M_{2k} \in Q(\mathscr{R}_{2k})$. For a fixed p with $1 \leqslant p \leqslant k-1$, suppose $c_0 > 0$, $c_{k-j} \geqslant 0$ for $p+1 \leqslant j \leqslant k-1$, $-a_{2i} = a_{2i-1} = c_{k-i}$ and $-b_{2i} = P_{2i}(c_0, c_1, \ldots, c_{k-i})$ for $p+1 \leqslant i \leqslant k$. Then $a_{2p} = -a_{2p-1}$ and $-b_{2p} = P_{2p}(c_0, c_1, \ldots, c_{k-p})$.*

Proof. By Lemma 4.4 when $q = p$,

$$-b_{2p} = P_{2p}(c_0, c_1, \ldots, c_{k-p-1}, a_{2p}).$$

By considering $p_n(a_{2p-1})$, an identical argument to that in Lemma 4.4 can be used to show

$$-b_{2p} = P_{2p}(c_0, c_1, \ldots, c_{k-p-1}, a_{2p-1}).$$

Since $P_{2p}(c_0, c_1, \ldots, c_{k-p-1}, a_{2p})$ and $P_{2p}(c_0, c_1, \ldots, c_{k-p-1}, a_{2p-1})$ are equal and are even polynomials in a_{2p} and a_{2p-1} that are strictly increasing on $(0, \infty)$, it follows that $|a_{2p}| = |a_{2p-1}| = c_{k-p}$ for some $c_{k-p} \geqslant 0$. Therefore,

$$-b_{2p} = P_{2p}(c_0, c_1, \ldots, c_{k-p-1}, c_{k-p}),$$

and the sign pattern implies that $a_{2p} = -c_{k-p}$ and $a_{2p-1} = c_{k-p}$. □

Suppose that $k \geqslant 2$, $n = 2k$ and $M_n \in Q(\mathscr{R}_n)$ is nilpotent. Then $b_{2k} < 0$ and by Remark 4.3, $a_{2k-1} = -a_{2k} > 0$. By Corollary 4.5 with $p = k-1$, $a_{2k-3} = -a_{2k-2} \geqslant 0$ and $-b_{2k-2} = P_{2k-2}(c_0, c_1) > 0$, and considering $p = k-2, k-3, \ldots, 1$, the following result is obtained.

Theorem 4.6. *Let $k \geqslant 2$ and $n = 2k$. If $\mathscr{R}_n$ is an irreducible subpattern of $\mathscr{C}_n$ that is potentially nilpotent, then $D(\mathscr{R}_n)$ has a simple cycle of each even length, a loop at vertices $2k-1$ and $2k$, and has, for $1 \leqslant i \leqslant k-1$, a loop at vertex $2i$ if and only if it has a loop at vertex $2i-1$.*

The following result identifies a family of subpatterns $\mathscr{R}_{2k}$ from Theorem 4.6 that is potentially nilpotent but not spectrally arbitrary, and a similar family $\mathscr{R}_{2k+1}$.

Theorem 4.7. *For $k \geqslant 2$ and $n = 2k$ or $n = 2k + 1$, if $\mathscr{R}_n$ has $r_{ii} = 0$ for $1 \leqslant i \leqslant 2k - 2$ and $r_{ij} = c_{ij}$ otherwise, then $\mathscr{R}_n$ is potentially nilpotent, but not spectrally arbitrary.*

Proof. Let $n = 2k$ and suppose $M_n \in Q(\mathscr{R}_n)$, that is, M_n has $a_1 = a_2 = \cdots = a_{n-2} = 0, a_{n-1} > 0$, $a_n < 0$ and $b_{2j} < 0$ for $1 \leqslant j \leqslant k$. Consider the characteristic polynomial of M_n as in (1.1) with coefficients μ_i as given by (2.4) and (2.5). For i even, $\mu_i = -a_{n-1}a_nb_{i-2} - b_i$, and for i odd, $\mu_i = -(a_{n-1} + a_n)b_{i-1}$, with $b_0 \equiv -1$. Assigning $a_{n-1} = 1$ and $a_n = -1$ gives $\mu_i = 0$ for i odd, and gives $\mu_i = 0$ for i even if and only if $b_{2i} = -1$ for $1 \leqslant i \leqslant k$.

Similarly, let $n = 2k + 1$ and suppose $M_n \in Q(\mathscr{R}_n)$, that is, M_n has $a_1 = a_2 = \cdots = a_{n-3} = 0$, $\{a_{n-2}, a_{n-1}\} = \{0, \hat{a}\}$ with $\hat{a} > 0$, $a_n < 0$, $b_{2j} < 0$ for $1 \leqslant j \leqslant k$ and $b_{2k+1} > 0$. Let the characteristic polynomial of M_n be given by (1.1) with μ_i replaced by $\tilde{\mu}_i$. The coefficients $\tilde{\mu}_i$ are given by $\tilde{\mu}_i = -\hat{a}a_nb_{i-2} - b_i$ for i even, and $\tilde{\mu}_i = -(\hat{a} + a_n)b_{i-1}$ for i odd, where $1 \leqslant i \leqslant 2k$ and $b_0 \equiv -1$. Assigning $\hat{a} = 1$ and $a_{2k+1} = -1$ gives $\tilde{\mu}_i = 0$ for i odd, and gives $\tilde{\mu}_i = 0$ for i even if and only if $b_{2i} = -1$ for $1 \leqslant i \leqslant k$. The constant term μ_{2k+1}, given by (2.12), is then 0 if and only if $b_{2k+1} = 1$.

Thus, for $n \geqslant 4$, $M_n \in Q(\mathscr{R}_n)$ as specified above is nilpotent and therefore $\mathscr{R}_n$ is potentially nilpotent. The pattern $\mathscr{R}_n$ is clearly not a SAP since $\mu_1 = 0$ implies that $\mu_3 = 0$ and $\tilde{\mu}_1 = 0$ implies that $\tilde{\mu}_3 = 0$. □

Note that if n is odd and $a_{2k+1} = 0$, then (2.12) gives $b_{2k+1} = 0$, and if $\hat{a} = 0$, then $\tilde{\mu}_{2k} = 0$ implies $b_{2k} = 0$, which from (2.12) implies that $b_{2k+1} = 0$. In the even case, if $a_{2k-1} = 0$ or $a_{2k} = 0$, then $b_{2k} = 0$ (from Remark 4.3). But $b_n \neq 0$ for an irreducible matrix, so we conclude that $\mathscr{R}_n$ in the theorem above has no irreducible proper subpattern that is potentially nilpotent.

Theorems 4.1, 4.6 and 4.7 imply that $\mathscr{C}_4$ is a minimal SAP. However, for $k \geqslant 3$ and $n = 2k$, it is not known whether or not $\mathscr{C}_n$ is minimal. In particular, it is unknown whether or not a subpattern $\mathscr{R}_n$ of $\mathscr{C}_n$ with $r_{2i,2i} = r_{2i-1,2i-1} = 0$ for some $1 \leqslant i \leqslant k - 1$ is a SAP.

Conjecture 4.8. *If $k \geqslant 3$ and $n = 2k$, then $\mathscr{C}_n$ is a minimal SAP.*

For $n = 5$, a careful analysis shows that $\mathscr{C}_5$ is a minimal SAP, but minimality in the general case for n odd remains to be explored.

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