Quasisymmetrically minimal uniform Cantor sets✩

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Abstract

The uniform Cantor set $E(n,c)$ of Hausdorff dimension 1, defined by a bounded sequence $n$ of positive integers and a gap sequence $c$, is shown to be minimal for 1-dimensional quasisymmetric maps.

Keywords: Uniform Cantor set; Quasisymmetric map; Quasisymmetrically minimal set

1. Introduction

Let $X$, $Y$ be metric spaces and $f : X \to Y$ be a topological homeomorphism. The map $f$ is called quasisymmetric if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{|f(x) - f(a)|}{|f(x) - f(b)|} \leq \eta \left( \frac{|x - a|}{|x - b|} \right)$$

(1)

for all triples $a$, $b$, $x$ of distinct points in $X$. When $X = Y = \mathbb{R}^n$, we also say that $f$ is an $n$-dimensional quasisymmetric map. We call a set $E \subset \mathbb{R}^n$ quasisymmetrically minimal, if

$$\dim_H f(E) \geq \dim_H E$$

(2)

for any $n$-dimensional quasisymmetric map $f$.

Since $n$-dimensional quasisymmetric maps are locally Hölder continuous, they send sets of Hausdorff dimension zero to sets of Hausdorff dimension zero. When $n \geq 2$ they also preserve sets of Hausdorff dimension $n$ (see [1, 5, 11]). But, according to Tukia [10], a set in $\mathbb{R}$ of Hausdorff dimension 1 may be not minimal for 1-dimensional quasisymmetric maps.

For a set $E \subset \mathbb{R}^n$ with $0 < \dim_H E < n$, Bishop [2] proved that for every $\varepsilon > 0$ there is an $n$-dimensional quasisymmetric map $f$ such that $\dim_H f(E) > n - \varepsilon$. On the other hand, by Kovalev [7], if $0 < \dim_H E < 1$ then for every $\varepsilon > 0$ there is an $n$-dimensional quasisymmetric map $f$ such that $\dim_H f(E) < \varepsilon$. Thus, no sets in $\mathbb{R}^n$ of $\dim_H \in (0, 1)$

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can be quasisymmetrically minimal. Tyson [9] proved that for every \( \alpha \in [1, n] \) there are quasisymmetrically minimal sets in \( \mathbb{R}^n \) of Hausdorff dimension \( \alpha \).

It is natural to ask: which sets in \( \mathbb{R}^n \) of dim \( H \in [1, n] \) are minimal for \( n \)-dimensional quasisymmetric maps? We shall study this question for \( n = 1 \). Since a set in \( \mathbb{R} \) with nonempty interior has topological dimension 1, its image under a homeomorphism has Hausdorff dimension \( \geq 1 \). Thus, such sets are minimal. In addition, Staples and Ward’s quasisymmetrically thick sets are minimal for 1-dimensional quasisymmetric maps (see [8]). Recently, Hakobyan [6] proved that middle interval Cantor sets of Hausdorff dimension 1 are all minimal. These are the only known examples of minimal sets in \( \mathbb{R} \) of Hausdorff dimension 1. In the present paper, we consider the uniform Cantor set \( E(n, c) \) of Hausdorff dimension 1, defined by a bounded sequence \( n \) of positive integers and a gap sequence \( c \). We shall prove that the set \( E(n, c) \) is minimal for 1-dimensional quasisymmetric maps.

2. Theorem

We begin with the definition of uniform Cantor sets. Let \( E_0 = [0, 1] \). Let \( n = \{n_k\}_{k=1}^{\infty} \) be a sequence of positive integers and \( c = \{c_k\}_{k=1}^{\infty} \) be a sequence of real numbers in \( (0, 1) \), such that \( n_k c_k < 1 \) for all \( k \). Suppose \( \{E_k\}_{k=0}^{\infty} \) is a nested sequence of closed sets in \( [0, 1] \) satisfying the following conditions:

(a) For every \( k \geq 1 \), \( E_k \) is a union of disjoint closed intervals of the same length.
(b) Every component interval \( I \) of \( E_{k-1} \) contains \( n_k + 1 \) component intervals of \( E_k \). These \( n_k + 1 \) intervals are of the same spacing \( c_k |I| \), the most left one and \( I \) have the same left endpoint, and the most right one and \( I \) have the same right endpoint.

The set
\[
E = E(n, c) = \bigcap_{k=0}^{\infty} E_k
\]
is called a uniform Cantor set.

If we do not assume that, in the condition (a), the intervals of \( E_k \) are of equal length, and that, in the condition (b), those intervals lying in \( I \) are of equal spacing, the resulting set \( E \) is called a Moran set (see [12]).

The main result of this paper is the following theorem.

**Theorem 1.** Let \( E = E(n, c) \) be a uniform Cantor set. If the sequence \( n \) is bounded and if \( \dim_H E = 1 \) then \( \dim_H f(E) = 1 \) for all 1-dimensional quasisymmetric maps \( f \).

It would be interesting to know whether the result holds without assuming that \( n \) is a bounded sequence.

The proof of Theorem 1 goes as follows. In order to prove \( \dim_H f(E) \geq 1 \), it suffices to show that \( \dim_H f(E) \geq d \) for any \( d \in (0, 1) \). For this purpose, given \( d \in (0, 1) \), a probability measure \( \mu \) on \( f(E) \) will be defined so that the inequality
\[
\mu(J) \leq C |J|^d
\]
holds for any interval \( J \subset \mathbb{R} \), where \( C \) is a positive constant independent of \( J \). Then the mass distribution principle [4] yields \( \dim_H f(E) \geq d \). A complete proof, which will be given in the rest of this paper, is based on some ideas of Hakobyan [6] and some fine properties of the set \( E \) and its image \( f(E) \).

3. Preliminary

By the definition of quasisymmetric maps, an increasing homeomorphism \( f: \mathbb{R} \to \mathbb{R} \) is quasisymmetric if and only if
\[
M^{-1} \leq \frac{|f(I)|}{|f(J)|} \leq M
\]
for all pairs of adjacent intervals \( I, J \) of equal length, where \( M = \eta(1), \eta \) is as in (1). In this case we also say that \( f \) is \( M \)-quasisymmetric. The following property of \( M \)-quasisymmetric maps is very useful for us.
Lemma 3.1. Let \( f \) be an \( M \)-quasisymmetric map. Then
\[
(1 + M)^{-2} \left( \frac{|J|}{|I|} \right)^q \leq \frac{|f(J)|}{|f(I)|} \leq 4 \left( \frac{|J|}{|I|} \right)^p
\]
for all pairs \( J, I \) of intervals with \( J \subset I \), where
\[
0 < p = \log_2(1 + M^{-1}) \leq 1 \leq q = \log_2(1 + M).
\]

**Proof.** Similar to the proof of Lemma 1 in [13]. \( \square \)

Let \( E = E(n, c) \) be a uniform Cantor set. Denote by \( N_k \) the number of component intervals of \( E_k \) and by \( \delta_k \) their common length. From the definition we have
\[
N_k = \prod_{i=1}^{k} (n_i + 1) \quad \text{and} \quad \delta_k = \prod_{i=1}^{k} \frac{1 - n_i c_i}{n_i + 1}.
\]
Thus, the total length of \( E_k \) is
\[
N_k \delta_k = \prod_{i=1}^{k} (1 - n_i c_i).
\]
The Hausdorff dimension of the set \( E \) depends on \( \{N_k\} \) and \( \{\delta_k\} \) as follows.

**Lemma 3.2.** (See [3, Theorem 2].) If \( E = E(n, c) \) is a uniform Cantor set then
\[
\dim_H E = \liminf_{k \to \infty} \frac{\log N_k}{-\log \delta_k}.
\]

When the sequence \( n \) is bounded and \( \dim_H E = 1 \), the defining parameters of the set \( E \) are restricted as follows.

**Lemma 3.3.** Let \( E = E(n, c) \) be a uniform Cantor set. If \( n \) is bounded and \( \dim_H E = 1 \) then:

1) \( \lim_{k \to \infty} (N_k \delta_k)^{1/k} = 1 \).
2) \( \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} c_i^p = 0 \) for any \( 0 < p \leq 1 \).
3) \( \lim_{k \to \infty} \sharp \{i: 0 \leq i \leq k, \ c_i \geq \epsilon \} / k = 0 \) for any \( \epsilon \in (0, 1) \), where \( \sharp \) denotes the cardinality.

**Proof.** 1) As \( \dim_H E = 1 \), we get from the dimension formula (7)
\[
\liminf_{k \to \infty} \frac{\log N_k}{-\log \delta_k} = \lim_{k \to \infty} \frac{\log N_k}{-\log \delta_k} = \lim_{k \to \infty} \frac{\log N_k}{\log N_k - \log N_k \delta_k} = 1,
\]
and hence
\[
\lim_{k \to \infty} \frac{\log N_k \delta_k}{\log N_k} = 0.
\]
Let \( N = 1 + \sup_k n_k < \infty \). One has \( N_k \leq N^k \). It follows that
\[
\lim_{k \to \infty} \frac{\log N_k \delta_k}{k \log N} = 0,
\]
giving the conclusion 1) of the lemma.

2) Since
\[
(N_k \delta_k)^{1/k} = \left( \prod_{i=1}^{k} (1 - n_i c_i) \right)^{1/k} \leq \frac{1}{k} \sum_{i=1}^{k} (1 - n_i c_i) = 1 - \frac{1}{k} \sum_{i=1}^{k} n_i c_i \leq 1,
\]
by the conclusion 1), we have \( \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} n_i c_i = 0 \). Then it follows from the boundedness of \( n \) that 
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} c_i = 0.
\]

for any \( 0 < p \leq 1 \). This proves the conclusion 2).

3) Fixed \( \varepsilon \in (0, 1) \), we have from the conclusion 2)
\[
\sharp \{ i: 1 \leq i \leq k, c_i \geq \varepsilon \} / k \leq 1
\]
\[
\frac{1}{k} \sum_{i=1}^{k} c_i \to 0
\]
as \( k \) tends to \( \infty \). This proves the conclusion 3). \( \square \)

4. The proof of Theorem 1

Let \( E = E(n, c) = \bigcap_{k=1}^{\infty} E_k \) be a uniform Cantor set satisfying the conditions of Theorem 1. Let \( f: \mathbb{R} \to \mathbb{R} \) be an \( M \)-quasisymmetric map and \( d \in (0, 1) \). Without loss of generality assume that \( f([0, 1]) = [0, 1] \). Then \( f(E) = \bigcap_{k=1}^{\infty} f(E_k) \) is a Moran set. The images of component intervals of \( E_k \) are component intervals of \( f(E_k) \).

We define a probability measure \( \mu \) on \( f(E) \) as follows: Let \( \mu([0, 1]) = 1 \). For every \( k \geq 1 \) and for every component interval \( J \) of \( f(E_{k-1}) \), let \( J_{k0}, J_{k1}, \ldots, J_{knk} \) denote the \( nk+1 \) component intervals of \( f(E_k) \) lying in \( J \). Define
\[
\mu(J_{ki}) = \frac{|J_{ki}|}{\|J\|_d} \mu(J), \quad i = 0, 1, \ldots, nk,
\]
where
\[
\|J\|_d = \sum_{i=0}^{nk} |J_{ki}|^d.
\]

We are going to show that the measure \( \mu \) satisfies
\[
\mu(J) \leq C |J|^d
\]
for any interval \( J \subset [0, 1] \), where \( C \) is a positive constant independent of \( J \). We do this in two steps.

Step 1. Suppose that \( J \) is a component interval of \( f(E_k) \). For every \( i, 0 \leq i \leq k \), let \( J_i \) be the component interval of \( f(E_i) \) such that
\[
J = J_k \subset J_{k-1} \subset \cdots \subset J_1 \subset J_0 = [0, 1].
\]

Then, by the definition of the measure \( \mu \),
\[
\frac{\mu(J)}{|J|^d} = \frac{1}{\|J\|_d} \frac{|J_{k-1}|^d}{\|J_{k-1}\|_d} \cdots \frac{|J_1|^d}{\|J_1\|_d} = \frac{|J_{k-1}|^d}{\|J_{k-1}\|_d} \cdots \frac{|J_1|^d}{\|J_1\|_d} \frac{|J_0|^d}{\|J_0\|_d}.
\]

Let
\[
|r_i| = \frac{|J_i|^d}{|J_i|^d}, \quad i = 0, 1, \ldots, k-1.
\]

The above equality can be rewritten as
\[
\frac{\mu(J)}{|J|^d} = \left( \prod_{i=1}^{k} r_i^{1-1} \right).
\]

To prove (9), it suffices to show
\[
\lim_{k \to \infty} \prod_{i=1}^{k} r_i^{1-1} = \infty.
\]
Given an \( i \), \( 1 \leq i \leq k \), we are going to estimate \( r_{i-1} \). Let \( J_{i-1} \) be the component interval of \( f(E_{i-1}) \) in the sequence (10). Recall that \( J_i \subset J_{i-1} \) is a component interval of \( f(E_i) \). Let \( J_{i1}, J_{i2}, \ldots, J_{in_i} \) be other \( n_i \) component intervals of \( f(E_i) \) lying in \( J_{i-1} \). Let \( G_{i1}, G_{i2}, \ldots, G_{in_i} \) be the \( n_i \) gaps between these \( n_i + 1 \) intervals. Put

\[
I_{i-1} = f^{-1}(J_{i-1}), \quad I_i = f^{-1}(J_i) \quad \text{and} \quad I_{ij} = f^{-1}(J_{ij}), \quad j = 1, \ldots, n_i.
\]

Then \( I_1, I_{11}, I_{12}, \ldots, I_{in_1} \) are component intervals of \( E_i \) lying in the component interval \( I_{i-1} \) of \( E_{i-1} \). Since \( f \) is \( M \)-quasisymmetric, it follows from Lemma 3.1 and the construction of \( E \) that

\[
\frac{|G_{ij}|}{|J_{i-1}|} \leq 4c_i^p, \quad j = 1, \ldots, n_i,
\]

and that

\[
\frac{|J_{ij}|}{|J_{i-1}|} \geq \frac{1}{(1 + M)^{2i}} \left( \frac{|I_{ij}|}{|I_{i-1}|} \right)^q = \frac{1}{(1 + M)^{2i}} \left( \frac{1 - n_i c_i}{1 + n_i} \right)^q \geq \frac{(1 - n_i c_i)^q}{(1 + M)^{2i}N^q}.
\]

Here and below \( N = 1 + \sup_k n_k, \) \( p, q \) are numbers defined as in (4). The inequality (14) yields

\[
\frac{|J_{i}| + |J_{i1}| + \cdots + |J_{in_i}|}{|J_{i-1}|} = \frac{|J_{i-1}| - |G_{i1}| - \cdots - |G_{in_i}|}{|J_{i-1}|} \geq 1 - 4n_i c_i^p.
\]

The inequality (15) gives

\[
\frac{|J_{i}|}{|J_{i-1}|} = \frac{|J_{i}|}{|J_{i1}|} \geq \frac{1}{2N}, \quad j = 1, 2, \ldots, n_i.
\]

It then follows from the left hand inequality of (3) that

\[
A \leq \frac{|J_{ij}|}{|J_i|} = \frac{|f(I_{ij})|}{|f(I_i)|} \leq 1, \quad j = 1, 2, \ldots, n_k.
\]

where \( A = (1 + M)^{2}(2N)^{-q} \). Therefore,

\[
\frac{|J_{i}|^d + |J_{i1}|^d + \cdots + |J_{in_i}|^d}{(|J_i| + |J_{i1}| + \cdots + |J_{in_i}|)^d} = \frac{1 + x_1^d + \cdots + x_{n_i}^d}{(1 + x_1 + \cdots + x_{n_i})^d} \geq (1 + A)^{1-d},
\]

where \( x_j = |J_{ij}|/|J_i| \in [A, 1] \).

Write simply \( \alpha_2 = (1 + A)^{1-d} \). For \( i \in S(k, p) \) we get from (16) and (19)

\[
\frac{r_{i-1}}{|J_{i-1}|} = \frac{|J_i|^d + |J_{i1}|^d + \cdots + |J_{in_i}|^d}{|J_{i-1}|^d} \geq \alpha_2(1 - 4n_i c_i^p)^d.
\]

Since \( a = 1 - \frac{4N}{4N^{\frac{1}{p}}}, \) one has

\[
1 - 4mx \geq (1 - x)^{4m+1}
\]
for all $x \in (0, a)$ and all positive integers $m \leq N$. Note that $n_i < N$ and $c_i^p \in (0, a)$ for $i \in S(k, p)$. The last inequality together with (20) yields

$$r_{i-1} \geq \alpha_2 (1 - c_i^p)^{(4n_i+1)d}.$$  

(21)

Now we are in a position to prove (13). Using the estimate (17) for $i \notin S(k, p)$ and the estimate (21) for $i \in S(k, p)$, we get from (6)

$$\prod_{i=1}^{k} r_{i-1} \geq \prod_{i \notin S(k, p)} \alpha_1 (1 - n_i c_i)^{dq} \prod_{i \in S(k, p)} \alpha_2 (1 - c_i^p)^{(4n_i+1)d} \geq \xi_k \eta_k,$$

(22)

where

$$\xi_k = \alpha_1^{-\frac{k-\sharp S(k, p)}{N_k \delta_k}} (N_k \delta_k)^{dq} \alpha_2^{-\sharp S(k, p)}$$

and

$$\eta_k = \prod_{i \in S(k, p)} (1 - c_i^p)^{(4n_i+1)d}.$$  

It is clear that

$$\lim_{k \to \infty} \xi_k^{1/k} = \alpha_2$$

(23)

due to the conclusion 1) of Lemma 3.3 and the equality (18). On the other hand, since $\log(1 - x) \geq -2x$ for $0 < x < 1$, the conclusion 2) of Lemma 3.3 together with the equality (18) yields

$$\frac{1}{k} \log \eta_k = \frac{1}{k} \log \prod_{i \in S(k, p)} (1 - c_j^p)^{(4n_i+1)d} \geq \frac{4Nd}{k} \sum_{i \in S(k, p)} \log (1 - c_i^p)$$

$$\geq -\frac{8Nd}{k} \sum_{i \in S(k, p)} c_i^p \geq -\frac{8Nd}{k} \sum_{i=1}^{k} c_i^p \to 0$$

as $k$ tends to $\infty$. This implies

$$\lim_{k \to \infty} \eta_k^{1/k} = 1.$$  

(24)

It follows from (22)–(24) that

$$\liminf_{k \to \infty} \left( \prod_{i=1}^{k} r_{i-1} \right)^{1/k} \geq \alpha_2.$$  

As $\alpha_2 > 1$, the equality (13) then follows.

Step 2. It remains to prove (9) for any interval $J \subset [0, 1]$. For such a $J$ let $k$ be the unique positive integer such that

$$\delta_k \leq |f^{-1}(J)| < \delta_{k-1},$$

where, as mentioned, $\delta_k$ denotes the length of component intervals of $E_k$. Then the set $f^{-1}(J)$ meets at most two component intervals of $E_{k-1}$ and hence at most $2n_k + 2$ component intervals of $E_k$. Equivalently, the set $J$ meets at most $2n_k + 2$ component intervals of $f(E_k)$.

Let $J_1, J_2, \ldots, J_l$, $l \leq 2n_k + 2$, be those component intervals of $f(E_k)$ meeting $J$. Using the conclusion of Step 1, we get

$$\mu(J) \leq \mu(J_1) + \mu(J_2) + \cdots + \mu(J_l) \leq C \sum_{j=1}^{l} |J_j|^d.$$  

(25)

In addition, since $\delta_k \leq |f^{-1}(J)|$, we see easily that

$$f^{-1}(J_i) \subset 3f^{-1}(J), \quad i = 1, 2, \ldots, l,$$

where $3f^{-1}(J)$ is the interval of length $3|f^{-1}(J)|$ concentric with $f^{-1}(J)$. So we have

$$|J_i| \leq |f(3f^{-1}(J))| \leq K|J|, \quad i = 1, 2, \ldots, l,$$
where $K > 0$ is a constant depending on $M$ only. This together with (25) gives
\[
\mu(J) \leq C K^d |J|^d \leq 2 NC K^d |J|^d.
\]
This proves (9).

Finally, according to the mass distribution principle, it follows from (9) that $\dim_H f(E) \geq d$, and hence $\dim_H f(E) = 1$ due to the arbitrariness of $d \in (0, 1)$. This completes the proof of Theorem 1.

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