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Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains

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ABSTRACT

For a linear, strictly elliptic second order differential operator in divergence form with bounded, measurable coefficients on a Lipschitz domain Ω we show that solutions of the corresponding elliptic problem with Robin and thus in particular with Neumann boundary conditions are Hölder continuous up to the boundary for sufficiently L^p-regular right-hand sides. From this we deduce that the parabolic problem with Robin or Wentzell–Robin boundary conditions is well-posed on $C(\bar{\Omega})$.

1. Introduction

In this article we show that solutions of elliptic Robin boundary value problems on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ are Hölder regular up to the boundary if the right-hand side is smooth enough in an L^p-sense. In particular, this result applies to Neumann boundary conditions. From this we obtain well-posedness of the parabolic problem with Neumann, Robin, and Wentzell–Robin boundary conditions in the space $C(\bar{\Omega})$.

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To be more precise, let L be a strictly elliptic operator in divergence form

$$Lu = -\sum_{j=1}^{N} D_j\left(\sum_{i=1}^{N} a_{ij} D_i u + b_j u\right) + \sum_{i=1}^{N} c_i D_i u + du \tag{1.1}$$

with bounded, measurable coefficients. We consider elliptic problems that formally take the form

$$\begin{cases} Lu = f_0 - \sum_{j=1}^{N} D_j f_j, & \text{on } \Omega, \\ \dfrac{\partial u}{\partial \nu_L} + \beta u = g + \sum_{j=1}^{N} f_j \nu_j, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where we set

$$\frac{\partial u}{\partial \nu_L} := \sum_{j=1}^{N}\left(\sum_{i=1}^{N} a_{ij} D_i u + b_j u\right)\nu_j,$$

and where ν denotes the outer normal on $\partial\Omega$. We assume β to be bounded and measurable, but make no assumptions on the sign of β.

In Section 2 we explain what is meant by a weak solution of (1.2). Section 3 is devoted to L^p-regularity and Hölder regularity results for solutions of (1.2), which are summarized in Theorem 3.14. The main idea is to extend weak solutions by reflection at the boundary, to show that this extension again solves an elliptic problem, and then to apply interior regularity results. This strategy is known, see for example [23, Section 2.4.3] or [6, Remark 3.10], but it seems that until now it has not been exploited to this extent.

In particular, if $f_0 \in L^{p/2}(\Omega)$, $f_j \in L^p(\Omega)$, $j = 1, \dots, N$, and $g \in L^{p-1}(\partial\Omega)$ for $p > N$, then every solution u of (1.2) is Hölder continuous on Ω. Weaker versions of this result can be found in [6,25,11,1]. On the other hand, a stronger version of this result has been obtained in [19], but by considerably more difficult methods that also might be less flexible in certain situations.

Using the elliptic regularity result we attack parabolic problems in spaces of continuous functions in Section 4. More precisely, we consider the initial value problems

$$\begin{cases} \dot{u}(t,x) = -Lu(t,x), & t > 0,\ x \in \Omega, \\ \dfrac{\partial u}{\partial \nu_L}(t,z) + \beta u(t,z) = 0, & t \geqslant 0,\ z \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

i.e., Robin or Neumann boundary conditions, and

$$\begin{cases} \dot{u}(t,x) = -Lu(t,x), & t > 0,\ x \in \Omega, \\ -Lu(t,z) + \dfrac{\partial u(t,z)}{\partial \nu_L} + \beta u(t,z) = 0, & t > 0,\ z \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \bar{\Omega}, \end{cases} \tag{1.4}$$

i.e., Wentzell–Robin boundary conditions. The solution operators for these equations form strongly continuous semigroups in appropriate Hilbert spaces. Those semigroups have extensively been studied, see for example [2,9,10,3,5,8]. In special cases it is known that these semigroups extrapolate to strongly continuous semigroups also on $C(\bar{\Omega})$, see [6,17,15,12,25], i.e., that the parabolic problem is well-posed in $C(\bar{\Omega})$. We extend these results to the case of arbitrary strongly elliptic differential operators with bounded, measurable coefficients. These results seem to be new in the literature. To get an idea why it is important to have generation results also in this space we refer to [7,24] and the references therein.

For simplicity we consider second order linear equations only. We will work with bounded, real-valued coefficients and pure Robin boundary conditions, i.e., we do not allow for Dirichlet or mixed boundary conditions. We will not investigate whether the operators generate semigroups on spaces of Hölder continuous functions. In the generation results for Wentzell–Robin boundary conditions we will in addition assume that the coefficients b_j, $j = 1, \dots, N$, are Lipschitz continuous.

2. Preliminaries

Throughout the whole article Ω denotes a bounded Lipschitz domain in $\mathbb{R}^N$, i.e., Ω is an open, bounded set that is locally the epigraph of a Lipschitz regular function. When we work with Lebesgue spaces $L^p(\partial\Omega)$, we always equip $\partial\Omega$ with the natural surface measure, which coincides with the $(N-1)$-dimensional Hausdorff measure. Since Ω is a Lipschitz domain, there exists a bounded trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$, and we denote the trace of $u \in H^1(\Omega)$ by $u|_{\partial\Omega}$ or simply by u, if misunderstandings are not to be expected.

We consider a linear differential operator L in divergence form acting on functions on Ω, i.e., L is (formally) given by (1.1). We assume throughout that the coefficients a_{ij}, b_j, c_i, and d are bounded and measurable and that L is strictly elliptic, i.e., there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqslant \alpha|\xi|^2 \tag{2.1}$$

holds for all $\xi \in \mathbb{R}^N$ and almost every x in Ω. Moreover, we restrict ourselves to the case $N \geqslant 2$ since the results for $N = 1$ are easy, but cumbersome to include into the general statement.

For L as in (1.1) and $\beta \in L^\infty(\partial\Omega)$, we define the bilinear form $a_{L,\beta}$ via

$$\begin{aligned} a_{L,\beta}(u,v) := {} & \sum_{i,j=1}^{N} \int_\Omega a_{ij} D_i u D_j v \,\mathrm{d}\lambda + \sum_{j=1}^{N} \int_\Omega b_j u D_j v \,\mathrm{d}\lambda \\ & + \sum_{i=1}^{N} \int_\Omega c_i D_i u v \,\mathrm{d}\lambda + \int_\Omega d u v \,\mathrm{d}\lambda + \int_{\partial\Omega} \beta u v \,\mathrm{d}\sigma \end{aligned} \tag{2.2}$$

for u and v in $H^1(\Omega)$.

Given functions $f_j \in L^1(\Omega)$, $j = 1, \dots, N$, and $g \in L^1(\partial\Omega)$, we call a function u in $H^1(\Omega)$ that satisfies

$$a_{L,\beta}(u,v) = \int_\Omega f_0 v \,\mathrm{d}\lambda + \sum_{j=1}^{N} \int_\Omega f_j D_j v \,\mathrm{d}\lambda + \int_{\partial\Omega} g v \,\mathrm{d}\sigma \quad \text{for all } v \in \mathrm{C}^1(\bar{\Omega}) \tag{2.3}$$

a *weak solution of* (1.2). If all functions and the domain are sufficiently smooth, then in fact (2.3) is equivalent to (1.2) as can be seen from the divergence theorem.

If u satisfies (2.3) maybe not for all $v \in \mathrm{C}^1(\bar{\Omega})$, but at least for all smooth functions with compact support in Ω, i.e., for all $v \in \mathrm{C}_c^\infty(\Omega)$, we say that $u \in H^1(\Omega)$ solves the problem $Lu = f_0 - \sum_{j=1}^{N} D_j f_j$. Note that this condition does not depend on β.

In the proofs, we will frequently need Sobolev embedding theorems, which can be found for example in Grisvard's book [20, Theorems 1.5.1.3 and 1.4.4.1].

3. Elliptic problems

3.1. Neumann boundary conditions

In this section we consider (2.3) in the special case $\beta = 0$, i.e., elliptic problems with Neumann boundary conditions. We will see that for sufficiently regular right-hand sides, every solution admits a Hölder continuous representative.

Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain. By definition, for every $z \in \partial\Omega$ we can choose an orthogonal matrix $\mathcal{O}$, a radius $r > 0$, a Lipschitz continuous function $\psi : \mathbb{R}^{N-1} \to \mathbb{R}$, and

$$G := \bigl\{\bigl(y, \psi(y) + s\bigr):\ y \in B(0,r) \subset \mathbb{R}^{N-1},\ s \in (-r,r)\bigr\}$$

such that

$$\mathcal{O}(\Omega - z) \cap G = \bigl\{\bigl(y, \psi(y) + s\bigr):\ y \in B(0,r) \subset \mathbb{R}^{N-1},\ s \in (0,r)\bigr\}. \tag{3.1}$$

Convention 3.1. Since the assumptions of Section 2 are invariant under isometric transformations of $\mathbb{R}^N$, for local considerations we assume without loss of generality that $\mathcal{O} = I$ and $z = 0$.

Define $T(y,s) := (y, \psi(y) + s)$ for $y \in \mathbb{R}^{N-1}$ and $s \in \mathbb{R}$. Then T is a bi-Lipschitz mapping from $B(0,r) \times (-r,r)$ to G with derivative

$$T'(y,s) = \begin{pmatrix} I & 0 \\ \nabla\psi(y) & 1 \end{pmatrix} \quad \text{and} \quad T'(y,s)^{-1} = \begin{pmatrix} I & 0 \\ -\nabla\psi(y) & 1 \end{pmatrix}$$

almost everywhere. Moreover, define the reflection $S : G \to G$ at the boundary $\partial\Omega$ by $S(T(y,s)) := T(y,-s)$. Then

$$S'\bigl(T(y,s)\bigr) = T'(y,-s) \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix} T'(y,s)^{-1} = \begin{pmatrix} I & 0 \\ 2\nabla\psi(y) & -1 \end{pmatrix}$$

almost everywhere. Note that $S(Sx) = x$, S' is bounded, $\det S'(x) = -1$ and $S'(x)^{-1} = S'(x)$. Moreover, $S'(y,s)$ does not depend on s, whence $S'(Sx) = S'(x)$.

Notation 3.2. We write U for $G \cap \Omega$ and V for $S(U) = G \setminus \bar{\Omega}$. For a function w on $D \subset G$, define w^* by $w^*(x) := w(Sx)$ on $S(D)$. For a function w on U, define $\tilde{w}$ (almost everywhere) on G by

$$\tilde{w}(x) := \begin{cases} w(x), & x \in U, \\ w^*(x), & x \in V. \end{cases}$$

In the following it will not matter that $\tilde{w}$ is not defined on the Lebesgue null set $\partial\Omega \cap G$ since we will apply this notation only to L^p-functions.

For the rest of the section we fix a linear, strictly elliptic differential operator L in divergence form and write a for the matrix (a_{ij}) and b and c for the vectors (b_j) and (c_i), respectively. Moreover, we define

$$\hat{a}(x) := \begin{cases} a(x), & x \in U, \\ S'(x)a^*(x)S'(x)^T, & x \in V, \end{cases} \qquad \hat{b}(x) := \begin{cases} b(x), & x \in U, \\ S'(x)b^*(x), & x \in V, \end{cases}$$

$$\hat{c}(x) := \begin{cases} c(x), & x \in U, \\ S'(x)c^*(x), & x \in V. \end{cases}$$

Lemma 3.3.

(i) *If w is in $H^1(D)$, then w^* is in $H^1(S(D))$, and $\nabla w^*(x) = \nabla w(Sx)S'(x)$ almost everywhere.*
(ii) *If w is in $H^1(U)$, then $w|_{\partial U} = w^*|_{\partial V}$ on $\partial\Omega \cap G$.*
(iii) *If w is in $H^1(U)$, then $\tilde{w}$ is in $H^1(G)$, and $\nabla\tilde{w} = \nabla w \mathbb{1}_U + \nabla w^* \mathbb{1}_V$.*
(iv) *For any $p \in [1, \infty]$, the extension operator $w \mapsto \tilde{w}$ is continuous from $L^p(U)$ to $L^p(G)$.*
(v) *The functions $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\tilde{d}$ are measurable and bounded on G.*

Proof. Assertion (i) follows from [26, Theorem 2.2.2]. Assertion (ii) is obvious if w is in addition continuous up to the boundary. Since U is a Lipschitz domain, those functions are dense in $H^1(U)$ and the claim follows by approximation. Let φ be a test function on G. The divergence theorem [14, Section 4.3] shows that

$$\int_G \tilde{w} D_i\varphi \,\mathrm{d}\lambda = \int_{\partial U} w\varphi\nu_i \,\mathrm{d}\sigma - \int_U D_i w\varphi \,\mathrm{d}\lambda + \int_{\partial V} w^*\varphi\nu_i \,\mathrm{d}\sigma - \int_V D_i w^*\varphi \,\mathrm{d}\lambda.$$

The boundary integrals cancel due to (ii) since the boundaries ∂V and ∂U have opposite orientations. This proves (iii). Assertion (iv) follows from [14, Section 3.4.3], and assertion (v) is obvious. □

Lemma 3.4. *There exists a constant $\hat{\alpha} > 0$ such that $\xi^T\hat{a}(x)\xi \geqslant \hat{\alpha}|\xi|^2$ for all $\xi \in \mathbb{R}^N$ and almost every $x \in G$.*

Proof. Let $w \in \mathbb{R}^{N-1}$ be an arbitrary row vector and define $W := \left(\begin{smallmatrix} I & 0 \\ w & -1 \end{smallmatrix}\right)$. Given a positive definite matrix $M := \left(\begin{smallmatrix} A & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{N\times N}$, the matrix

$$WMW^T = \begin{pmatrix} A & Aw^T - b \\ wA - c & wAw^T - wb - cw^T + d \end{pmatrix}$$

is positive definite as well. In fact, it suffices to check that the leading principal minors are positive. Since M is positive definite, all minors of M are positive. Hence the first $N-1$ leading principal minors of WMW^T are positive and, moreover, $\det M > 0$. Thus $\det(WMW^T) > 0$ by the multiplicativity of the determinant since $\det W = \det W^T = -1$, which proves the claim.

By what we have shown, the least eigenvalue $\lambda_1(WMW^T)$ of WMW^T is positive whenever M is positive definite. Since λ_1 depends continuously on the entries of the matrix this shows that $\lambda_1(WMW^T) \geqslant \delta$ for some $\delta > 0$ as M ranges over a compact subset of the set of all positive definite matrices, and w ranges over a compact subset of $\mathbb{R}^{N-1}$.

Recall that a matrix $A \in \mathbb{R}^{N\times N}$ satisfies $\xi^T A\xi \geqslant \alpha|\xi|^2$, $\alpha > 0$, for all $\xi \in \mathbb{R}^N$ if and only if $\lambda_1((A + A^T)/2) \geqslant \alpha$. Thus by assumption (2.1)

$$\frac{1}{2}\big(a(x) + a(x)^T\big) \in K_1 := \big\{M \in \mathbb{R}^{N\times N}\colon\ M = M^T,\ \lambda_1(M) \geqslant \alpha,\ \|M\| \leqslant c\big\}$$

for some constant c and for almost all $x \in U$. The set K_1 is a compact subset of the positive definite matrices. Let $K_2 \subset \mathbb{R}^{N-1}$ be a closed ball whose radius is large enough such that $2\nabla\psi(y) \in K_2$ for almost all y.

Using the first part of this proof, we see that there is $\delta > 0$ such that

$$\lambda_1\left(\frac{1}{2}S'(x)\big(a(Sx) + a(Sx)^T\big)S'(x)^T\right) \geqslant \delta$$

for almost every $x \in U$. Thus $\xi^T\hat{a}(x)\xi \geqslant \delta|\xi|^2$ almost everywhere on V, from which the claim follows with $\hat{\alpha} := \min\{\alpha, \delta\}$. □

Lemma 3.5. *Let $\hat{L}$ denote the differential operator on G for the coefficients $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\tilde{d}$. Assume that there exists $p > N$ such that $f_0 \in L^{p/2}(\Omega)$, $f_j \in L^p(\Omega)$, $j = 1, \dots, N$, and $g \in L^{p-1}(\partial\Omega)$. Let $u \in H^1(\Omega)$ satisfy* (2.3) *(recall that we allow only for $\beta = 0$ in this section). Then there exist $s > N$ and functions $h_0 \in L^{s/2}(G)$ and $h_j \in L^s(G)$, $j = 1, \dots, N$, that satisfy*

$$a_{\hat{L},0}(\tilde{u}, v) = \int_G h_0 v \,\mathrm{d}\lambda + \sum_{i=1}^N \int_G h_i D_i v \,\mathrm{d}\lambda \tag{3.2}$$

for every function $v \in C_c^\infty(G)$.

Proof. By definition of a solution of (2.3) we have that

$$\sum_{i,j=1}^N \int_U \hat{a}_{ij} D_i \tilde{u} D_j v \,\mathrm{d}\lambda + \sum_{j=1}^N \int_U \hat{b}_j \tilde{u} D_j v \,\mathrm{d}\lambda + \sum_{i=1}^N \int_U \hat{c}_i D_i \tilde{u} v \,\mathrm{d}\lambda + \int_U \tilde{d}\tilde{u} v \,\mathrm{d}\lambda$$
$$= \int_U f_0 v \,\mathrm{d}\lambda + \sum_{j=1}^N \int_U f_j D_j v \,\mathrm{d}\lambda + \int_{\partial U} g v \,\mathrm{d}\sigma \tag{3.3}$$

holds for every $v \in C_c^\infty(G)$.

Using part (i) of Lemma 3.3 and the change of variables formula [14, Section 3.4.3] (replacing x by Sx) we obtain

$$\begin{aligned}
&\int_V (\nabla\tilde{u})\hat{a}(\nabla v)^T \,\mathrm{d}\lambda + \int_V \tilde{u}(\nabla v)\hat{b} \,\mathrm{d}\lambda + \int_V (\nabla\tilde{u})\hat{c} v \,\mathrm{d}\lambda + \int_V \tilde{d}\tilde{u} v \,\mathrm{d}\lambda \\
&\quad = \int_V \nabla u(Sx) S'(x) S'(x) a(Sx) S'(x)^T \big(\nabla v(x)\big)^T \,\mathrm{d}x \\
&\qquad + \int_V u(Sx) \nabla v(x) S'(x) b(Sx) \,\mathrm{d}x + \int_V \nabla u(Sx) S'(x) S'(x) c(Sx) v(x) \,\mathrm{d}x \\
&\qquad + \int_V d(Sx) u(Sx) v(x) \,\mathrm{d}x \\
&\quad = \int_U \nabla u(x) a(x) S'(x)^T \big(\nabla v(Sx)\big)^T \,\mathrm{d}x + \int_U u(x) \nabla v(Sx) S'(x) b(x) \,\mathrm{d}x \\
&\qquad + \int_U \nabla u(x) c(x) v(Sx) \,\mathrm{d}x + \int_U d(x) u(x) v(Sx) \,\mathrm{d}x \\
&\quad = \int_U \nabla u(x) a(x) \big(\nabla v^*(x)\big)^T \,\mathrm{d}x + \int_U u(x) \nabla v^*(x) b(x) \,\mathrm{d}x \\
&\qquad + \int_U \nabla u(x) c(x) v^*(x) \,\mathrm{d}x + \int_U d(x) u(x) v^*(x) \,\mathrm{d}x
\end{aligned}$$

$$= \int_U f_0 v^* \,\mathrm{d}\lambda + \sum_{j=1}^N \int_U f_j D_j v^* \,\mathrm{d}\lambda + \int_{\partial U} g v^* \,\mathrm{d}\sigma$$

$$= \int_U f_0(x) v(Sx)\,\mathrm{d}x + \sum_{j=1}^N \int_U f_j(x) \sum_{i=1}^N D_i v(Sx)\big(S'(x)\big)_{ij}\,\mathrm{d}x + \int_{\partial U} g v^* \,\mathrm{d}\sigma$$

$$= \int_V f_0^* v \,\mathrm{d}\lambda + \sum_{i=1}^N \int_V \sum_{j=1}^N \big(S'\big)_{ij} f_j^* D_i v \,\mathrm{d}\lambda + \int_{\partial U} g v \,\mathrm{d}\sigma \tag{3.4}$$

for every $v \in C_c^\infty(\Omega)$, where we have used in addition that u satisfies (2.3).

Adding Eqs. (3.3) and (3.4) we obtain

$$a_{\hat{L},0}(\tilde{u}, v) = \int_G \tilde{f}_0 v \,\mathrm{d}\lambda + \sum_{j=1}^N \int_G \hat{f}_j D_j v \,\mathrm{d}\lambda + 2\int_{\partial U} g v \,\mathrm{d}\sigma \tag{3.5}$$

for $\hat{f}_j \in L^p(G)$ defined by

$$\hat{f}_j(x) := \begin{cases} f_j(x), & x \in U, \\ \sum_{i=1}^N (S'(x))_{ji} f_i^*(x), & x \in V. \end{cases}$$

Since g is in $L^{p-1}(\partial U)$ and the trace operator is bounded from $W^{1,r}(G)$ to $L^{(N-1)r/(N-r)}(\partial U)$ for every $r \in (1, N)$ the mapping

$$C_c^\infty(G) \to \mathbb{R}, \qquad v \mapsto 2\int_{\partial U} g v \,\mathrm{d}\sigma$$

extends to a continuous linear functional on $W_0^{1,r_0}(G)$ for $r_0 := \frac{(p-1)N}{(p-2)N+1}$. Thus there exist functions $(k_j)_{j=0}^N$ in $L^{r_0'}(G)$ such that

$$2\int_{\partial U} g v \,\mathrm{d}\sigma = \int_G k_0 v \,\mathrm{d}\lambda + \sum_{j=1}^N \int_G k_j D_j v \,\mathrm{d}\lambda$$

holds for all $v \in C_c^\infty(G)$, see [26, Theorem 4.3.3]. Note that by assumption $r_0' = \frac{(p-1)N}{N-1} > N$. Hence (3.2) follows from (3.5) by setting $h_0 := \tilde{f}_0 + k_0$ and $h_j := \hat{f}_j + k_j$ for $j = 1, \ldots, N$. $\square$

Proposition 3.6. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and $p > N$. There exist $\gamma > 0$ and a constant c with the following property. If $f_0 \in L^{p/2}(\Omega)$, $f_j \in L^p(\Omega)$, $j = 1, \ldots, N$, and $g \in L^{p-1}(\partial\Omega)$, then every solution u of* (2.3) (*recall that at the moment we allow only for $\beta = 0$*) *is in $C^{0,\gamma}(\Omega)$ and satisfies*

$$\|u\|_{C^{0,\gamma}(\Omega)} \leqslant c\left(\|u\|_{L^2(\Omega)} + \|f_0\|_{L^{p/2}(\Omega)} + \sum_{j=1}^N \|f_j\|_{L^p(\Omega)} + \|g\|_{L^{p-1}(\partial\Omega)}\right). \tag{3.6}$$

Proof. Fix z in $\partial\Omega$ and consider a neighborhood G of z as in (3.1). By Lemma 3.5 there exist $s > N$ and functions $h_0 \in L^{s/2}(G)$ and $h_j \in L^s(G)$, $j = 1, \ldots, N$, such that the extension $\tilde{u} \in H^1(G)$ of u solves the problem

$$\hat{L}\tilde{u} = h_0 - \sum_{j=1}^{N} D_j h_j.$$

By Lemmata 3.3 and 3.4, the differential operator $\hat{L}$ on G satisfies the assumptions of Section 2. Thus it follows from interior regularity results [18, Theorem 8.24] that for every relatively compact subset G_0 of G there exists $\gamma_0 > 0$ such that $\tilde{u}$ is in $C^{0,\gamma_0}(G_0)$ and satisfies an estimate of the kind (3.6). Thus u is in $C^{0,\gamma_0}(G_0 \cap \Omega)$ and satisfies an appropriate estimate.

Since $\partial\Omega$ is compact, we can cover $\partial\Omega$ by finitely many such sets. Thus u is Hölder continuous in an interior neighborhood of $\partial\Omega$ and its Hölder norm can be controlled as in (3.6). Finally, we use the result about interior regularity once again to control u in the remaining part of Ω. □

3.2. Robin boundary conditions

In this section we apply Proposition 3.6 to obtain similar results also for Robin boundary conditions, i.e., for solutions of (2.3) if β does not necessarily equal zero. As a stepping stone, we investigate the L^p-regularity of solutions also in cases where the data is less regular than in Proposition 3.6. Thus even for $\beta = 0$ the results of this section extend those of the previous one.

As before, let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain and L be a linear, strictly elliptic differential operator on Ω. Moreover, let β be an arbitrary function in $L^\infty(\partial\Omega)$.

For $\omega \in \mathbb{R}$ we introduce the form $a^{\omega}_{L,\beta}$ defined by

$$a^{\omega}_{L,\beta}(u, v) := a_{L,\beta}(u, v) + \omega \int_{\Omega} uv \, \mathrm{d}\lambda \tag{3.7}$$

for u and v in $H^1(\Omega)$ and investigate the class of functions $u \in H^1(\Omega)$ that satisfy

$$a^{\omega}_{L,\beta}(u, v) = \int_{\Omega} f_0 v \, \mathrm{d}\lambda + \sum_{j=1}^{N} \int_{\Omega} f_j D_j v \, \mathrm{d}\lambda + \int_{\partial\Omega} g v \, \mathrm{d}\sigma \quad \text{for all } v \in C^1(\bar{\Omega}). \tag{3.8}$$

This is a generalized version of (2.3). More precisely, (2.3) and (3.8) coincide for $\omega = 0$. The advantage of this more general situation is that for large ω the problem (3.8) is uniquely solvable.

Lemma 3.7. *Let $N \geqslant 3$. There exist $\omega \in \mathbb{R}$ and a constant c with the following property. If $f_0 \in L^{2N/(N+2)}(\Omega)$, $f_j \in L^2(\Omega)$, $j = 1, \ldots, N$, and $g \in L^{2(N-1)/N}(\partial\Omega)$, then problem (3.8) has a unique solution $u \in H^1(\Omega)$, and*

$$\|u\|_{H^1(\Omega)} \leqslant c\left(\|f_0\|_{L^{2N/(N+2)}(\Omega)} + \sum_{j=1}^{N} \|f_j\|_{L^2(\Omega)} + \|g\|_{L^{2(N-1)/N}(\partial\Omega)}\right). \tag{3.9}$$

Proof. By [11, Corollary 2.5] there exist $\eta > 0$ and $\omega \in \mathbb{R}$ such that

$$a^{\omega}_{L,\beta}(u, u) \geqslant \eta \|u\|^2_{H^1(\Omega)}.$$

Thus, by the Lax–Milgram theorem [18, Theorem 5.8] there exists a constant c_1 with the following property. For every $\psi \in H^1(\Omega)'$ there exists a unique function $u \in H^1(\Omega)$ that satisfies

$$a^{\omega}_{L,\beta}(u, v) = \psi(v) \quad \text{for all } v \in H^1(\Omega), \tag{3.10}$$

and for this u we have

$$\|u\|_{H^1(\Omega)} \leqslant c_1 \|\psi\|_{H^1(\Omega)'}. \tag{3.11}$$

Since $H^1(\Omega)$ embeds into $L^{2N/(N-2)}(\Omega)$ and the trace operator maps $H^1(\Omega)$ into $L^{2(N-1)/(N-2)}(\partial\Omega)$, there exists a constant c_2 with the following property. For $f_0 \in L^{2N/(N+2)}(\Omega)$, $f_j \in L^2(\Omega)$, $j = 1, \ldots, N$, and $g \in L^{2(N-1)/N}(\partial\Omega)$,

$$\psi(v) := \int_\Omega f_0 v \, \mathrm{d}\lambda + \sum_{j=1}^N \int_\Omega f_j D_j v \, \mathrm{d}\lambda + \int_{\partial\Omega} g v \, \mathrm{d}\sigma \tag{3.12}$$

defines a continuous linear functional ψ on $H^1(\Omega)$ that satisfies

$$\|\psi\|_{H^1(\Omega)'} \leqslant c_2 \left(\|f_0\|_{L^{2N/(N+2)}(\Omega)} + \sum_{j=1}^N \|f_j\|_{L^2(\Omega)} + \|g\|_{L^{2(N-1)/N}(\partial\Omega)} \right). \tag{3.13}$$

Now let $f_0 \in L^{2N/(N+2)}(\Omega)$, $f_j \in L^2(\Omega)$, $j = 1, \ldots, N$, and $g \in L^{2(N-1)/N}(\partial\Omega)$ be arbitrary. Define ψ as in (3.12), and let u be as in (3.10). Then u is a solution of (3.8). Since $\mathrm{C}^1(\bar{\Omega})$ is dense in $H^1(\Omega)$, every solution of (3.8) satisfies (3.10). Hence the solution of (3.8) is unique. Estimate (3.9) follows with $c := c_1 c_2$ by combining (3.11) and (3.13). □

Lemma 3.8. *Let $N = 2$ and $q > 1$. There exist $\omega \in \mathbb{R}$ and a constant c with the following property. If $f_0 \in L^q(\Omega)$, $f_j \in L^2(\Omega)$, $j = 1, \ldots, N$, and $g \in L^q(\partial\Omega)$, then problem* (3.8) *has a unique solution $u \in H^1(\Omega)$, and*

$$\|u\|_{H^1(\Omega)} \leqslant c \left(\|f_0\|_{L^q(\Omega)} + \sum_{j=1}^N \|f_j\|_{L^2(\Omega)} + \|g\|_{L^q(\partial\Omega)} \right). \tag{3.14}$$

Proof. The proof is similar to the proof of Lemma 3.7. Here, however, we use that $H^1(\Omega)$ embeds into $L^r(\Omega)$ for every $r < \infty$, and that the trace operator maps $H^1(\Omega)$ into $L^r(\partial\Omega)$ for every $r < \infty$. □

Remark 3.9. It should be noted that in Lemmata 3.7 and 3.8 we can take any $\omega \in \mathbb{R}$ such that (3.8) has a unique solution for some right-hand side. In fact, let A be the operator from $H^1(\Omega)$ to $H^1(\Omega)'$ defined by $\langle Au, v \rangle := a_{L,\beta}(u, v)$. Considering $H^1(\Omega)$ as a subspace of $H^1(\Omega)'$ via the scalar product in $L^2(\Omega)$, A is a densely defined, closed operator on $H^1(\Omega)'$. The resolvent of A is compact since $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$. For $\omega \in \mathbb{R}$, the Fredholm alternative asserts that either there exists $u \in H^1(\Omega)$ such that $(\omega + A)u = 0$, which means precisely that the solution of (3.8) is not unique, or $\omega + A$ is boundedly invertible, which implies estimate (3.9) or (3.14), respectively.

Now, as an interlude, we come back to the Neumann problem. Afterwards, the following lemmata will be generalized to cover Robin problems as well.

Lemma 3.10. *Let $p > N$, and let ω be as in Lemma* 3.7 *or Lemma* 3.8*, respectively. Assume $\beta = 0$. Then there exist $\gamma > 0$ and a constant c with the following property. If $f_0 \in L^{p/2}(\Omega)$, $f_j \in L^p(\Omega)$, $j = 1, \ldots, N$, and*

$g \in L^{p-1}(\partial\Omega)$, then the unique solution u of (3.8) *is in $C^{0,\gamma}(\Omega)$ and satisfies*

$$\|u\|_{C^{0,\gamma}(\Omega)} \leqslant c\left(\|f_0\|_{L^{p/2}(\Omega)} + \sum_{j=1}^{N}\|f_j\|_{L^p(\Omega)} + \|g\|_{L^{p-1}(\partial\Omega)}\right).$$

Proof. By Lemma 3.7 or Lemma 3.8, respectively, the solution is unique, and by (3.9) or (3.14) there exists a constant c_1 such that

$$\|u\|_{L^2(\Omega)} \leqslant c_1\left(\|f_0\|_{L^{p/2}(\Omega)} + \sum_{j=1}^{N}\|f_j\|_{L^p(\Omega)} + \|g\|_{L^{p-1}(\partial\Omega)}\right).$$

Thus the result follows from Proposition 3.6. □

Lemma 3.11. *Let $N \geqslant 3$, $\frac{2N}{N+2} \leqslant q < \frac{N}{2}$, $\varepsilon > 0$, and let ω be as in Lemma* 3.7. *Assume $\beta = 0$. Then there exists a constant c with the following property. If $f_0 \in L^{q+\varepsilon}(\Omega)$, $f_j \in L^{Nq/(N-q)+\varepsilon}(\Omega)$, $j = 1, \dots, N$, and $g \in L^{(N-1)q/(N-q)+\varepsilon}(\partial\Omega)$, then the unique solution u of* (3.8) *satisfies $u \in L^{Nq/(N-2q)}(\Omega)$ and $u|_{\partial\Omega} \in L^{(N-1)q/(N-2q)}(\partial\Omega)$, and*

$$\begin{aligned}&\|u\|_{L^{Nq/(N-2q)}(\Omega)} + \|u\|_{L^{(N-1)q/(N-2q)}(\partial\Omega)}\\ &\quad\leqslant c\left(\|f_0\|_{L^{q+\varepsilon}(\Omega)} + \sum_{j=1}^{N}\|f_j\|_{L^{Nq/(N-q)+\varepsilon}(\Omega)} + \|g\|_{L^{(N-1)q/(N-q)+\varepsilon}(\partial\Omega)}\right).\end{aligned}$$

Proof. Pick $p > N$. It will turn out at the end how close to N we have to pick p, but this condition will depend only on N and q, hence the argument is not circular.

To simplify the notation of the proof we introduce the Banach spaces

$$L^{r,s,t} := L^r(\Omega) \oplus L^s(\Omega)^N \oplus L^t(\partial\Omega) \quad \text{and} \quad L^{x,y} := L^x(\Omega) \oplus L^y(\partial\Omega).$$

Note that the complex interpolation spaces $[L^{r_0,s_0,t_0}, L^{r_1,s_1,t_1}]_\theta$ and $[L^{x_0,y_0}, L^{x_1,y_1}]_\theta$, $\theta \in [0,1]$, are in a natural way isomorphic to $L^{r,s,t}$ and $L^{x,y}$, respectively, where

$$\begin{aligned}&\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \qquad \frac{1}{s} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}, \qquad \frac{1}{t} = \frac{1-\theta}{t_0} + \frac{\theta}{t_1},\\ &\frac{1}{x} = \frac{1-\theta}{x_0} + \frac{\theta}{x_1}, \qquad \frac{1}{y} = \frac{1-\theta}{y_0} + \frac{\theta}{y_1}.\end{aligned} \tag{3.15}$$

This follows from [22, Section 1.18.4] and the observation that

$$[X_0 \oplus Y_0, X_1 \oplus Y_1]_\theta \cong [X_0, X_1]_\theta \oplus [Y_0, Y_1]_\theta$$

holds for all Banach spaces X_0, X_1, Y_0 and Y_1, which in turn is a direct consequence of the definition of the complex interpolation functor [22, Section 1.9].

For $f_0 \in L^{2N/(N+2)}(\Omega)$, $f_j \in L^2(\Omega)$, $j = 1, \dots, N$, and $g \in L^{2(N-1)/N}(\partial\Omega)$ we denote by $R(f_0, (f_j)_{j=1}^N, g)$ the unique solution $u \in H^1(\Omega)$ of (3.8). It is clear that R is a linear map. Let $\gamma > 0$ be as in Lemma 3.10. If we consider $H^1(\Omega)$ and $C^{0,\gamma}(\Omega)$ as subspaces of $L^{2,2}$ via the injection $u \mapsto (u, u|_{\partial\Omega})$, then the Sobolev embedding theorems and Lemmata 3.7 and 3.10 show that R maps $L^{2N/(N+2),2,2(N-1)/N}$ into $L^{2N/(N-2),2(N-1)/(N-2)}$ and $L^{p/2,p,p-1}$ into $L^{\infty,\infty}$.

Using [22, Theorem 1.9.3(a)] for

$$\theta := \frac{Nq + 2q - 2N}{q(N-2)},$$

we obtain that R maps L^{r_p, s_p, t_p} into $L^{Nq/(N-2q), (N-1)q/(N-2q)}$, where r_p, s_p and t_p are defined analogously to (3.15) for

$$r_0 = \frac{2N}{N+2}, \qquad s_0 = 2, \qquad t_0 = \frac{2(N-1)}{N},$$
$$r_1 = \frac{p}{2}, \qquad s_1 = p, \qquad t_1 = p - 1.$$

It is easy to see that the dependence of r_p, s_p and t_p on p is continuous and that

$$r_N = q, \qquad s_N = \frac{Nq}{N-q} \quad \text{and} \quad t_N = \frac{(N-1)q}{N-q}.$$

Thus there exists $p > N$ such that

$$r_p < q + \varepsilon, \qquad s_p < \frac{Nq}{N-q} + \varepsilon \quad \text{and} \quad t_p < \frac{(N-1)q}{N-q} + \varepsilon.$$

The result follows if we start the whole argument with such a p. □

Remark 3.12. We exclude $N = 2$ in Lemma 3.11 because the admissible range for q is empty in that case. However, if we take $N = 2$ and $q = 1$ and adopt the convention that $\frac{1}{0}$ be ∞, Lemma 3.11 is a trivial consequence of Lemma 3.10. More generally, this is true also for $N \geqslant 3$ in the boundary case $q = \frac{N}{2}$.

Now we come back to Robin boundary conditions. The following bootstrapping argument allows us to deduce regularity results for Robin problems from the corresponding results for Neumann problems.

Lemma 3.13. *Let $N \geqslant 3$, $\frac{2N}{N+2} \leqslant q \leqslant \frac{N}{2}$, and $\varepsilon > 0$. Then there exist $\tilde{\varepsilon} > 0$ and a constant c with the following property. If $f_0 \in L^{q+\varepsilon}(\Omega)$, $f_j \in L^{Nq/(N-q)+\varepsilon}(\Omega)$, $j = 1, \ldots, N$, and $g \in L^{(N-1)q/(N-q)+\varepsilon}(\partial\Omega)$, then every solution $u \in H^1(\Omega)$ of (2.3) satisfies $u \in L^{q+\tilde{\varepsilon}}(\Omega)$, $u|_{\partial\Omega} \in L^{(N-1)q/(N-q)+\tilde{\varepsilon}}(\partial\Omega)$ and*

$$\|u\|_{L^{q+\tilde{\varepsilon}}(\Omega)} + \|u\|_{L^{(N-1)q/(N-q)+\tilde{\varepsilon}}(\partial\Omega)} \leqslant c\left(\|u\|_{L^2(\Omega)} + \|f_0\|_{L^{q+\varepsilon}(\Omega)} + \sum_{j=1}^{N} \|f_j\|_{L^{Nq/(N-q)+\varepsilon}(\Omega)} + \|g\|_{L^{(N-1)q/(N-q)+\varepsilon}(\partial\Omega)}\right).$$

Proof. Define by induction

$$q_0 := \frac{2N}{N+2} \quad \text{and} \quad q_{n+1} := \min\left\{q, \frac{Nq_n}{N-2q_n}\right\},$$

where we adopt the convention that $1/0 := \infty$. Note that there exists $n_0 \in \mathbb{N}_0$ such that $q_n = q$, since otherwise we would have

$$q_n = \frac{Nq_{n-1}}{N-2q_{n-1}} \geqslant \frac{N}{N-2}q_{n-1} \geqslant \cdots \geqslant \left(\frac{N}{N-2}\right)^n q_0 \to \infty \quad (n \to \infty)$$

which is not possible since $q_n \leqslant q$ for all $n \in \mathbb{N}$ by definition.

For $n \in \mathbb{N}_0$, we say that (P_n) is fulfilled if there exist $\varepsilon_n > 0$ and a constant c_n with the following property. If $f_0 \in L^{q+\varepsilon}(\Omega)$, $f_j \in L^{Nq/(N-q)+\varepsilon}(\Omega)$, $j = 1, \ldots, N$, and $g \in L^{(N-1)q/(N-q)+\varepsilon}(\partial\Omega)$, then every solution $u \in H^1(\Omega)$ of (2.3) satisfies $u \in L^{q_n+\varepsilon_n}(\Omega)$, $u|_{\partial\Omega} \in L^{(N-1)q_n/(N-q_n)+\varepsilon_n}(\partial\Omega)$ and

$$\begin{aligned}&\|u\|_{L^{q_n+\varepsilon_n}(\Omega)} + \|u\|_{L^{(N-1)q_n/(N-q_n)+\varepsilon_n}(\partial\Omega)} \\ &\quad \leqslant c_n\left(\|u\|_{L^2(\Omega)} + \|f_0\|_{L^{q+\varepsilon}(\Omega)} + \sum_{j=1}^N \|f_j\|_{L^{Nq/(N-q)+\varepsilon}(\Omega)} + \|g\|_{L^{(N-1)q/(N-q)+\varepsilon}(\partial\Omega)}\right). \end{aligned} \tag{3.16}$$

The statement (P_0) is obviously true. So now assume that (P_n) is true for some $n \in \mathbb{N}_0$. If $q_n = q$, then (P_{n+1}) is trivially fulfilled since it is the same statement as (P_n). Thus we may assume $q_n < q$ without loss of generality. Let ω be as in Lemma 3.7 or Lemma 3.8, respectively, and note that every solution $u \in H^1(\Omega)$ of (2.3) satisfies

$$a_{L,0}^{\omega}(u, v) = \int_\Omega (f_0 + \omega u)v \, \mathrm{d}\lambda + \sum_{j=1}^N \int_\Omega f_j D_j v \, \mathrm{d}\lambda + \int_{\partial\Omega} (g - \beta u)v \, \mathrm{d}\sigma \tag{3.17}$$

for all $v \in \mathrm{C}^1(\bar{\Omega})$, i.e., u solves a Neumann problem for a different right-hand side that involves u. Thus Lemma 3.11 applied for a value $\tilde{q}$ such that $q_n < \tilde{q} < \min\{q, q_n + \varepsilon_n\}$ implies that there exist constants $\tilde{c}$ and $\varepsilon_{n+1} > 0$ such that

$$\begin{gathered} u \in L^{N\tilde{q}/(N-2\tilde{q})}(\Omega) \subset L^{q_{n+1}+\varepsilon_{n+1}}(\Omega), \\ u|_{\partial\Omega} \in L^{(N-1)\tilde{q}/(N-2\tilde{q})}(\partial\Omega) \subset L^{(N-1)q_{n+1}/(N-q_{n+1})+\varepsilon_{n+1}}(\partial\Omega), \end{gathered}$$

and

$$\begin{aligned}&\|u\|_{L^{q_{n+1}+\varepsilon_{n+1}}(\Omega)} + \|u\|_{L^{(N-1)q_{n+1}/(N-q_{n+1})+\varepsilon_n}(\partial\Omega)} \\ &\quad \leqslant \tilde{c}\Bigg(\|u\|_{L^{q_n+\varepsilon_n}(\Omega)} + \|u\|_{L^{(N-1)q_n/(N-q_n)+\varepsilon_n}(\partial\Omega)} + \|f_0\|_{L^{q+\varepsilon}(\Omega)} \\ &\qquad + \sum_{j=1}^N \|f_j\|_{L^{Nq/(N-q)+\varepsilon}(\Omega)} + \|g\|_{L^{(N-1)q/(N-q)+\varepsilon}(\partial\Omega)}\Bigg). \end{aligned}$$

Using (P_n) to estimate the norms of u on the right-hand side as in (3.16), we have proved (P_{n+1}). By induction, (P_n) is true for every $n \in \mathbb{N}_0$. Since the statement of the lemma is equivalent to (P_{n_0}) for some $n_0 \in \mathbb{N}_0$ such that $q_{n_0} = q$, this finishes the proof. □

The following theorem summarizes (and extends) the previous results of this section.

Theorem 3.14. *Let Ω be an open, bounded Lipschitz domain in $\mathbb{R}^N$, $N \geqslant 2$, and let L be a strictly elliptic differential operator as in* (1.1) *with bounded, measurable coefficients. Let $\frac{2N}{N+2} \leqslant q \leqslant \frac{N}{2}$ and $\varepsilon > 0$. Then there exist $\gamma > 0$ and a constant c with the following property. If $f_0 \in L^{q+\varepsilon}(\Omega)$, $f_j \in L^{Nq/(N-q)+\varepsilon}(\Omega)$, $j = 1, \ldots, N$, and $g \in L^{(N-1)q/(N-q)+\varepsilon}(\partial\Omega)$, then every function $u \in H^1(\Omega)$ satisfying* (2.3) *fulfills*:

(i) *if* $q < N/2$, *then* $u \in L^{Nq/(N-2q)}(\Omega)$, $u|_{\partial\Omega} \in L^{(N-1)q/(N-2q)}(\partial\Omega)$, *and*

$$\|u\|_{L^{Nq/(N-2q)}(\Omega)} + \|u\|_{L^{(N-1)q/(N-2q)}(\partial\Omega)} \leqslant c\left(\|u\|_{L^2(\Omega)} + \|f_0\|_{L^{q+\varepsilon}(\Omega)} + \sum_{j=1}^{N} \|f_j\|_{L^{Nq/(N-q)+\varepsilon}(\Omega)} + \|g\|_{L^{(N-1)q/(N-q)+\varepsilon}(\partial\Omega)}\right);$$

(ii) *if* $q = N/2$, *then* $u \in \mathrm{C}^{0,\gamma}(\Omega)$, *and*

$$\|u\|_{\mathrm{C}^{0,\gamma}(\Omega)} \leqslant c\left(\|u\|_{L^2(\Omega)} + \|f_0\|_{L^{N/2+\varepsilon}(\Omega)} + \sum_{j=1}^{N} \|f_j\|_{L^{N+\varepsilon}(\Omega)} + \|g\|_{L^{N-1+\varepsilon}(\partial\Omega)}\right).$$

Moreover, if the solution is unique, then it satisfies

(iii) *if* $q < N/2$, *then* $u \in L^{Nq/(N-2q)}(\Omega)$, $u|_{\partial\Omega} \in L^{(N-1)q/(N-2q)}(\partial\Omega)$, *and*

$$\|u\|_{L^{Nq/(N-2q)}(\Omega)} + \|u\|_{L^{(N-1)q/(N-2q)}(\partial\Omega)} \leqslant c\left(\|f_0\|_{L^{q+\varepsilon}(\Omega)} + \sum_{j=1}^{N} \|f_j\|_{L^{Nq/(N-q)+\varepsilon}(\Omega)} + \|g\|_{L^{(N-1)q/(N-q)+\varepsilon}(\partial\Omega)}\right);$$

(iv) *if* $q = N/2$, *then* $u \in \mathrm{C}^{0,\gamma}(\Omega)$, *and*

$$\|u\|_{\mathrm{C}^{0,\gamma}(\Omega)} \leqslant c\left(\|f_0\|_{L^{N/2+\varepsilon}(\Omega)} + \sum_{j=1}^{N} \|f_j\|_{L^{N+\varepsilon}(\Omega)} + \|g\|_{L^{N-1+\varepsilon}(\partial\Omega)}\right).$$

Proof. Let $u \in H^1(\Omega)$ be a solution of (2.3). Then there exists $\tilde{\varepsilon} > 0$ such that u is in $L^{q+\tilde{\varepsilon}}(\Omega)$ and $u|_{\partial\Omega}$ is in $L^{(N-1)q/(N-q)+\tilde{\varepsilon}}(\partial\Omega)$. Moreover, the norms in these space can be estimates by the right-hand sides in (i) and (ii), respectively. In fact, for $N = 2$ this is trivial, whereas for $N \geqslant 3$ this is the statement of Lemma 3.13.

Let ω be as in Lemma 3.7 or Lemma 3.8, respectively. By (2.3) the function u satisfies

$$a_{L,0}^{\omega}(u, v) = \int_{\Omega} (f_0 + \omega u) v \, \mathrm{d}\lambda + \sum_{j=1}^{N} \int_{\Omega} f_j D_j v \, \mathrm{d}\lambda + \int_{\partial\Omega} (g - \beta u) v \, \mathrm{d}\sigma$$

for all $v \in \mathrm{C}^1(\bar{\Omega})$, i.e., u is a weak solution of a Neumann problem with right-hand side as in Lemma 3.11 for $q < N/2$ or as in Proposition 3.6 for $q = N/2$. The respective conclusions of Lemma 3.11 and Proposition 3.6 yield (i) and (ii).

For (iii) and (iv) it only remains to show that $\|u\|_{L^2(\Omega)}$ can be estimated accordingly if the solution is unique. This follows from the Fredholm alternative, see Remark 3.9. □

4. Parabolic problems

Let Ω be an open, bounded Lipschitz domain in $\mathbb{R}^N$. Assume that L is a strictly elliptic differential operator as in (1.1) It has been shown in [25] that $-L = \Delta$ with Robin or Wentzell–Robin boundary conditions generates a C_0-semigroup on $\mathrm{C}(\bar{\Omega})$. Although the calculations contain a small mistake that oversimplifies the arguments, the proof can be saved with a minor modification. We employ the same idea to show the result is true in general.

4.1. Neumann and Robin boundary conditions

Let A be the operator on $L^2(\Omega)$ associated with the form $a_{L,\beta}$ defined in (2.2). It follows from the theory of forms that $-A$ generates a positive, compact, holomorphic C_0-semigroup $(T(t))_{t\geqslant 0}$ on $L^2(\Omega)$, cf. for example [21]. The trajectories of this semigroup are the unique mild solutions of the parabolic problem (1.3) with Robin boundary conditions, compare [13, Section VI.5].

It is known that each $T(t)$ is a kernel operator with a bounded kernel $k(t,\cdot,\cdot)$ which has Gaussian estimates [9, Corollary 6.1]. Thus $(T(t))_{t\geqslant 0}$ extrapolates to a family of holomorphic semigroups on $L^p(\Omega)$, $p \in [1,\infty]$, which have the same angle of holomorphy, and all operator $T(z)$ for $\operatorname{Re} z > 0$ are kernel operators satisfying a Gaussian estimate [2, Theorem 5.4].

We start this section by an investigation of the regularity of these kernels. In particular it follows from the next theorem that the kernels are jointly continuous in the time variable (away from $t = 0$) and in the space variables (up to the boundary of Ω).

Theorem 4.1. *The function $t \mapsto k(t,\cdot,\cdot)$ is analytic from $(0,\infty)$ to $C^{0,\gamma}(\Omega \times \Omega)$ for γ as in Theorem 3.14. In particular, $k \in C^{0,\gamma}([\tau_1,\tau_2] \times \Omega \times \Omega)$ for $0 < \tau_1 \leqslant \tau_2 < \infty$.*

Proof. Let ω be so large that $a^{\omega}_{L,\beta}$ is coercive, see Lemmata 3.7 and 3.8. Then $A + \omega$ is invertible, i.e., $\lambda := -\omega \in \varrho(A)$. By Theorem 3.14 there exists $m \in \mathbb{N}$ and $\gamma > 0$ such that

$$R(\lambda, A)^m L^2(\Omega) \subset C^{0,\gamma}(\Omega).$$

Since $(T(t))_{t\geqslant 0}$ is holomorphic, this implies that $T(t)$ maps $L^2(\Omega)$ boundedly to $C^{0,\gamma}(\Omega)$ for every $t > 0$.

Let φ_{hol} be the sector of holomorphy of $(T(t))_{t\geqslant 0}$ and fix $0 < \theta < \varphi_{\text{hol}}$. Let $k(z,\cdot,\cdot)$ denote the kernel of $T(z)$ for $z \in \Sigma_\theta$, and let $0 < \tau_1 < \tau_2$. Define

$$\Sigma_{\theta,\tau_1,\tau_2} := \bigl\{ z \in \mathbb{C}\colon\ z - \tau_1 \in \Sigma_\theta \text{ and } |z| < \tau_2 \bigr\}.$$

Since $k(t,\cdot,\cdot) \in L^\infty(\Omega \times \Omega)$, there exists a constant $K > 0$ that depends only on the semigroup and the set $\Sigma_{\theta,\tau_1,\tau_2}$ such that for all $z \in \Sigma_{\theta,\tau_1,\tau_2}$ and almost every $y \in \Omega$ we have

$$\begin{aligned}
\bigl\| k(z,\cdot,y) \bigr\|_{C^{0,\gamma}(\Omega)} &= \left\| T\left(\frac{\tau_1}{2}\right) T(z-\tau_1) k\left(\frac{\tau_1}{2},\cdot,y\right) \right\|_{C^{0,\gamma}(\Omega)} \\
&\leqslant \left\| T\left(\frac{\tau_1}{2}\right) \right\|_{\mathscr{L}(L^2(\Omega),C^{0,\gamma}(\Omega))} \bigl\| T(z-\tau_1) \bigr\|_{\mathscr{L}(L^2(\Omega))} \left\| k\left(\frac{\tau_1}{2},\cdot,y\right) \right\|_{L^2(\Omega)} \\
&\leqslant K.
\end{aligned}$$

Using a duality argument, we can estimate $\|k(z,x,\cdot)\|_{C^{0,\gamma}(\Omega)}$ in a similar manner, possibly increasing the value of K appropriately. Thus

$$\bigl| k(z,x,y) - k(z,\bar{x},\bar{y}) \bigr| \leqslant K|x-\bar{x}|^\gamma + K|y-\bar{y}|^\gamma \leqslant 2K \left| \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} \right|_\infty^\gamma \tag{4.1}$$

for almost every x, $\bar{x}$, y and $\bar{y}$ in Ω, which shows that $\{k(z,\cdot,\cdot)\colon\ z \in \Sigma_{\theta,\tau_1,\tau_2}\}$ is a bounded subset of $C^{0,\gamma}(\Omega \times \Omega)$.

Since $(T(z))_{z\in\Sigma_\theta}$ is holomorphic on $L^2(\Omega)$,

$$\bigl(T(z) \mathbb{1}_A \bigm| \mathbb{1}_B \bigr)_{L^2(\Omega)} = \int\limits_{\Omega\times\Omega} k(z,x,y) \mathbb{1}_{A\times B} \, \mathrm{d}(x,y)$$

is holomorphic for all measurable subsets A and B of Ω. Since the class of functionals on $\mathrm{C}^{0,\gamma}(\Omega\times\Omega)$ that arise from integration against $\mathbb{1}_{A\times B}$ is a separating subset, we thus obtain that the mapping $z\mapsto k(z,\cdot,\cdot)$ is holomorphic from $\Sigma_{\theta,\tau_1,\tau_2}$ to $\mathrm{C}^{0,\gamma}(\Omega\times\Omega)$ [4, Theorem A.7]. Since τ_1 and τ_2 are arbitrary, the first assertion follows. The second assertion is an easy consequence of the first. □

Next we show that $(T(t))_{t\geqslant 0}$ restricts to a C_0-semigroup on $\mathrm{C}(\bar{\Omega})$. For this we need the following density result.

Lemma 4.2. *Assume that $a_{L,\beta}$ is coercive and let γ be as in Theorem* 3.14. *For all $v\in\mathrm{C}^\infty(\bar{\Omega})$ and all $\varepsilon>0$ there exists $\psi\in\mathrm{C}^\infty(\bar{\Omega})$ such that the unique solution u of* (2.3) *for the right-hand side $f_0:=\psi$, $f_j:=0$, $j=1,\dots,N$, and $g:=0$ satisfies $\|u-v\|_{\mathrm{C}^{0,\gamma}(\Omega)}<\varepsilon$.*

Proof. Let $\tilde{\varepsilon}>0$ and $p>N$ be arbitrary. Let $h_d\in\mathrm{C}^\infty(\mathbb{R}^N;\mathbb{R}^N)$ be such that $h_d\cdot\nu\geqslant 1$ almost everywhere on $\partial\Omega$ [11, Lemma 3.2]. By the Stone–Weierstrass theorem we can find a smooth vector field $h\in\mathrm{C}^\infty(\mathbb{R}^N;\mathbb{R}^N)$ such that

$$\left\| h-\frac{\beta v h_d}{h_d\cdot\nu}\right\|_{L^p(\partial\Omega;\mathbb{R}^N)}<\tilde{\varepsilon}.$$

Hence $\tilde{g}:=h\cdot\nu-\beta v$ satisfies $\|\tilde{g}\|_{L^p(\partial\Omega)}<\tilde{\varepsilon}$. Since the test functions are dense in $L^p(\Omega)$ there exist $k_0,k_1,\dots,k_N$ in $\mathrm{C}_c^\infty(\Omega)$ such that the functions

$$\tilde{f}_0:=k_0-\sum_{i=1}^N c_iD_iv-dv,\qquad \tilde{f}_j:=-h_j-k_j-\sum_{i=1}^N a_{ij}D_iv-b_jv\quad (j=1,\dots,N),$$

satisfy $\|\tilde{f}_j\|_{L^p(\Omega)}<\tilde{\varepsilon}$ for $j=0,\dots,N$. Define $\psi\in\mathrm{C}^\infty(\bar{\Omega})$ by

$$\psi:=k_0+\sum_{j=1}^N D_jk_j+\operatorname{div}(h),$$

and let u be the unique solution of (2.3) as described in the claim. By the divergence theorem

$$\int_\Omega \psi\varphi\,\mathrm{d}\lambda=\int_\Omega k_0\varphi\,\mathrm{d}\lambda+\int_{\partial\Omega}(h\cdot\nu)\varphi\,\mathrm{d}\sigma-\sum_{j=1}^N\int_\Omega (h_j+k_j)D_j\varphi\,\mathrm{d}\lambda$$

for every $\varphi\in\mathrm{C}^1(\bar{\Omega})$, hence

$$\begin{aligned} a_{L,\beta}(u-v,\varphi)&=\int_\Omega\psi\varphi\,\mathrm{d}\lambda-a_{L,\beta}(v,\varphi)\\ &=\int_\Omega\tilde{f}_0\varphi\,\mathrm{d}\lambda+\sum_{j=1}^N\int_\Omega\tilde{f}_jD_j\varphi\,\mathrm{d}\lambda+\int_{\partial\Omega}\tilde{g}\varphi\,\mathrm{d}\sigma\end{aligned}$$

for all $\varphi\in\mathrm{C}^1(\bar{\Omega})$. Thus part (iv) of Theorem 3.14 implies that

$$\|u-v\|_{\mathrm{C}^{0,\gamma}(\Omega)}\leqslant c(N+2)\tilde{\varepsilon}$$

for some constant c that does not depend on v, ψ, ε or $\tilde{\varepsilon}$. Now if we pick $\tilde{\varepsilon}$ small enough such that $c(N+2)\tilde{\varepsilon} < \varepsilon$, the claim follows. □

Theorem 4.3. *The restriction of* $(T(t))_{t\geqslant 0}$ *to* $\mathrm{C}(\bar{\Omega})$ *is a positive, compact, holomorphic* C_0*-semigroup.*

Proof. Let ω be such that $a^{\omega}_{L,\beta}$ is coercive. Then $\lambda := -\omega \in \varrho(A)$. By Theorem 3.14 there exists $m \in \mathbb{N}$ and $\gamma > 0$ such that

$$R(\lambda, A)^m L^2(\Omega) \subset \mathrm{C}^{0,\gamma}(\Omega).$$

Since $(T(t))_{t\geqslant 0}$ is holomorphic, this implies that $T(t)$ maps $L^2(\Omega)$ boundedly to $\mathrm{C}^{0,\gamma}(\Omega)$ for every $t \geqslant 0$. In particular, the subspace $\mathrm{C}(\bar{\Omega})$ is invariant under $T(t)$, and factoring through $L^2(\Omega)$ we see that $T(t)$ is a compact operator on $\mathrm{C}(\bar{\Omega})$. Positivity follows from the positivity on $L^2(\Omega)$.

As was already remarked, the restriction of $(T(t))_{t\geqslant 0}$ to $L^\infty(\Omega)$ is a holomorphic semigroup in the sense of [4, Definition 3.7.1]. Its generator is the part of A in $L^\infty(\Omega)$. Since $\mathrm{C}^\infty(\bar{\Omega})$ is dense in $\mathrm{C}(\bar{\Omega})$, Lemma 4.2 shows that the part of $A+\omega$ in $\mathrm{C}(\bar{\Omega})$ and hence also the part of A in $\mathrm{C}(\bar{\Omega})$ is densely defined. Thus by [4, Proposition 3.7.4 and Remark 3.7.13] the restriction of $(T(t))_{t\geqslant 0}$ to $\mathrm{C}(\bar{\Omega})$ is a holomorphic C_0-semigroup, whose generator is the part of A in $\mathrm{C}(\bar{\Omega})$. □

4.2. Wentzell–Robin boundary conditions

Let A be the operator on the Hilbert space $\mathcal{H} := L^2(\Omega) \oplus L^2(\partial\Omega)$ that is associated with the form

$$\mathfrak{a}_{L,\beta}\big((u, u|_{\partial\Omega}), (v, v|_{\partial\Omega})\big) := a_{L,\beta}(u, v)$$

with the dense form domain

$$\mathcal{V} := \big\{(u, u|_{\partial\Omega}) \colon u \in H^1(\Omega)\big\} \subset \mathcal{H}.$$

It follows from the theory of forms that $-A$ generates a positive, compact, holomorphic C_0-semigroup $(T(t))_{t\geqslant 0}$ on $\mathcal{H}$. This semigroup, or more precisely its restriction to $\mathcal{V}$, describes the solutions of the evolution problem (1.4) with Wentzell–Robin boundary conditions, compare [5].

We want to show that $(T(t))_{t\geqslant 0}$ extrapolates to $\mathrm{C}(\bar{\Omega})$. An easy sufficient condition is quasi-L^∞-contractivity, i.e., to assume that the semigroup $(\mathrm{e}^{-\omega t}T(t))_{t\geqslant 0}$ is L^∞-contractive for some $\omega \in \mathbb{R}$. However, this cannot be expected in general, even if Ω is an interval and L is formally self-adjoint and has regular second order coefficients, as the following example shows.

Example 4.4. Consider the operator

$$(Lu)(x) = -\big(u'(x) + \operatorname{sgn}(x)u(x)\big)' + \operatorname{sgn}(x)u(x)$$

on $\Omega = (-1, 1)$ with Wentzell–Robin boundary conditions, i.e., $a = 1$, $b = c = \operatorname{sgn}$, $d = 0$, and β arbitrary. There exists no $\omega \in \mathbb{R}$ such that the semigroup $\mathrm{e}^{-\omega t}T(t)$ consists of contractions on $L^\infty(\Omega) \oplus L^\infty(\partial\Omega)$.

Proof. Assume that there exists $\omega \geqslant 0$ such that $\mathrm{e}^{-\omega t}T(t)$ is contractive on $L^\infty(\Omega) \oplus L^\infty(\partial\Omega)$ for all $t \geqslant 0$. This semigroup comes from the form $\mathfrak{a}^{\omega}_{L,\beta}$, which is defined by

$$\begin{aligned}
&\mathfrak{a}^{\omega}_{L,\beta}\big((u, u|_{\partial\Omega}), (v, v|_{\partial\Omega})\big) \\
&\quad := \int_{-1}^{1} \big(u'(x)v'(x) + \operatorname{sgn}(x)u(x)v'(x) + \operatorname{sgn}(x)u'(x)v(x) + \omega u(x)v(x)\big)\,\mathrm{d}x \\
&\qquad + \big(\beta(-1)+\omega\big)u(-1)v(-1) + \big(\beta(1)+\omega\big)u(1)v(1)
\end{aligned}$$

for $(u, u|_{\partial\Omega})$ and $(v, v|_{\partial\Omega})$ in $\mathcal{V}$. Thus by [21, Theorem 2.15]

$$\mathfrak{a}^{\omega}_{L,\beta}\big((v, v|_{\partial\Omega}), (w, w|_{\partial\Omega})\big) \geqslant 0 \quad \text{with } v := \big(1 \wedge |u|\big)\operatorname{sgn}(u) \text{ and } w := \big(|u|-1\big)^{+}\operatorname{sgn}(u)$$

for all $(u, u|_{\partial\Omega}) \in \mathcal{V}$, hence in particular

$$\int_{-1}^{1} \big(\operatorname{sgn}(x)u'(x) + \omega u(x)\big)\mathbb{1}_{\{u\geqslant 1\}}\,\mathrm{d}x \geqslant 0$$

for all $u \in H^1(-1,1)$ satisfying $u(-1) = u(1) = 0$ and $u \geqslant 0$. For $u_n(x) := 2(1-x^2)^n$ we obtain with $\alpha_n := (1-2^{-1/n})^{1/2}$

$$\begin{aligned}
0 &\leqslant \int_{-1}^{1} \big(\operatorname{sgn}(x)u_n'(x) + \omega u_n(x)\big)\mathbb{1}_{\{u_n\geqslant 1\}}\,\mathrm{d}x \\
&= -u_n|_{-\alpha_n}^{0} + u_n|_{0}^{\alpha_n} + \omega\int_{-\alpha_n}^{\alpha_n} u_n\,\mathrm{d}\lambda \leqslant -2 + 4\omega\alpha_n.
\end{aligned}$$

This is a contradiction since $\alpha_n \to 0$ as $n \to \infty$. □

However, if we assume some regularity of the coefficients, more precisely $b_j \in W^{1,\infty}(\Omega) = \operatorname{Lip}(\Omega)$ for $j = 1, \ldots, N$, then we obtain a quasi-submarkovian semigroup, i.e., a semigroup such that $(\mathrm{e}^{-\omega t}T(t))_{t\geqslant 0}$ is positive and L^∞-contractive for some $\omega \in \mathbb{R}$, as we show next.

Proposition 4.5. *If $b_j \in W^{1,\infty}(\Omega)$ for $j = 1, \ldots, N$, then the Wentzell–Robin semigroup $(T(t))_{t\geqslant 0}$ is quasi-submarkovian on $\mathcal{H}$.*

Proof. It follows from [21, Theorem 2.6] that $(T(t))_{t\geqslant 0}$ is positive. By assumption, there exists $k \geqslant 0$ such that $|b| \leqslant k$ and $\operatorname{div}(b) \leqslant k$ almost everywhere. Pick ω larger than $\|d\|_\infty + k$ and $\|\beta\|_\infty + k$. Since $D_j u^+ = D_j u \mathbb{1}_{\{u\geqslant 0\}}$, we obtain from the divergence theorem that for all $(u, u|_{\partial\Omega}) \in \mathcal{V}$ satisfying $u \geqslant 0$ we have

$$\begin{aligned}
&\mathfrak{a}^{\omega}_{L,\beta}\big((1\wedge u, 1\wedge u|_{\partial\Omega}), \big((u-1)^+, (u|_{\partial\Omega}-1)^+\big)\big) \\
&\quad = \int_{\Omega} b\nabla(u-1)^+\,\mathrm{d}\lambda + \int_{\Omega}(d+\omega)(u-1)^+\,\mathrm{d}\lambda + \int_{\partial\Omega}(\beta+\omega)(u-1)^+\,\mathrm{d}\sigma \\
&\quad = \int_{\partial\Omega}(u-1)^+ b\cdot\nu\,\mathrm{d}\sigma - \int_{\Omega}(u-1)^+\operatorname{div}(b)\,\mathrm{d}\lambda
\end{aligned}$$

$$+\int_\Omega (d+\omega)(u-1)^+\,\mathrm{d}\lambda + \int_{\partial\Omega} (\beta+\omega)(u-1)^+\,\mathrm{d}\sigma$$
$$\geqslant \int_\Omega (u-1)^+\bigl(\omega - \|d\|_\infty - k\bigr)\,\mathrm{d}\lambda + \int_{\partial\Omega} (u-1)^+\bigl(\omega - \|\beta\|_\infty - k\bigr)\,\mathrm{d}\sigma \geqslant 0.$$

It follows from [21, Corollary 2.17] that $(\mathrm{e}^{-\omega t}T(t))_{t\geqslant 0}$ is submarkovian. □

We need a density result that is similar to Lemma 4.2 in order to show that $(T(t))_{t\geqslant 0}$ restricts to a C_0-semigroup on $\mathrm{C}(\bar\Omega)$.

Lemma 4.6. *Assume that $a_{L,\beta}$ is coercive and let γ be as in Theorem* 3.14. *For all $v \in \mathrm{C}^\infty(\bar\Omega)$ and all $\varepsilon > 0$ there exists $\psi \in \mathrm{C}^\infty(\bar\Omega)$ such that the unique solution u of* (2.3) *for the right-hand side $f_0 := \psi$, $f_j := 0$, $j = 1,\dots,N$, and $g := \psi|_{\partial\Omega}$ satisfies $\|u - v\|_{\mathrm{C}^{0,\gamma}(\Omega)} < \varepsilon$.*

Proof. Let $p > N$ and $\tilde\varepsilon > 0$ be arbitrary. By the Stone–Weierstrass theorem there exists $\tilde k_0 \in \mathrm{C}^\infty(\bar\Omega)$ such that $\tilde g := (\tilde k_0 - \beta v)|_{\partial\Omega}$ satisfies $\|\tilde g\|_{L^p(\partial\Omega)} < \tilde\varepsilon$. Now pick test functions $k_j \in \mathrm{C}_c^\infty(\Omega)$, $j = 1,\dots,N$, such that

$$\tilde f_0 := \tilde k_0 + k_0 - \sum_{i=1}^N c_i D_i v - dv, \qquad \tilde f_j := k_j - \sum_{i=1}^N a_i D_i v - b_j v \quad (j = 1,\dots,N)$$

satisfy $\|\tilde f_j\|_{L^p(\Omega)} < \tilde\varepsilon$ for $j = 0,\dots,N$. Define

$$\psi := \tilde k_0 + k_0 + \sum_{j=1}^N k_j \in \mathrm{C}^\infty(\bar\Omega)$$

and let u be the unique solution of (2.3) as described in the claim. Then

$$a_{L,\beta}(u-v,\varphi) = \int_\Omega \psi\varphi\,\mathrm{d}\lambda + \int_{\partial\Omega} \psi\varphi\,\mathrm{d}\sigma - a_{L,\beta}(v,\varphi)$$
$$= \int_\Omega \tilde f_0\varphi\,\mathrm{d}\lambda + \sum_{j=1}^N \int_\Omega \tilde f_j D_j\varphi\,\mathrm{d}\lambda + \int_{\partial\Omega} \tilde g\varphi\,\mathrm{d}\sigma$$

for all $\varphi \in \mathrm{C}^1(\bar\Omega)$. Thus part (iv) of Theorem 3.14 implies that

$$\|u - v\|_{\mathrm{C}^{0,\gamma}(\Omega)} < c(N+2)\tilde\varepsilon$$

for some constant c that does not depend on v, ψ, ε or $\tilde\varepsilon$. If we pick $\tilde\varepsilon$ small enough such that $c(N+2)\tilde\varepsilon < \varepsilon$, the claim follows. □

Theorem 4.7. *Assume $b_j \in W^{1,\infty}(\Omega)$ for all $j = 1,\dots,N$. Then the restriction of $(T(t))_{t\geqslant 0}$ to $\mathscr{C} := \{(u, u|_{\partial\Omega})\colon u \in \mathrm{C}(\bar\Omega)\}$ is a positive, compact C_0-semigroup.*

Proof. Pick $\omega \geqslant 0$ large enough such that $a^{\omega}_{L,\beta}$ and hence in particular $\mathfrak{a}^{\omega}_{L,\beta}$ is coercive. Then $\lambda := -\omega$ is in $\varrho(A)$, where A denotes the generator of $(T(t))_{t \geqslant 0}$.

Using Theorem 3.14, one can show as in the proof of Lemma 3.13 that there exists $m \in \mathbb{N}$ such that

$$R(\lambda, A)^m \mathcal{H} \subset \mathscr{C}^{0,\gamma} := \left\{ (u, u|_{\partial\Omega}) \colon u \in \mathrm{C}^{0,\gamma}(\Omega) \right\}.$$

Since $(T(t))_{t \geqslant 0}$ is analytic, each $T(t)$, $t > 0$, is a bounded operator from $\mathcal{H}$ to $\mathscr{C}^{0,\gamma}$. In particular, $\mathscr{C}$ is invariant under each $T(t)$, $t > 0$.

By Proposition 4.5, the restriction of $(T(t))_{t \geqslant 0}$ to $\mathscr{C}$ is a semigroup in the sense of [4, Definition 3.2.5]. Its generator is the part of A in $\mathscr{C}$. Since $\mathrm{C}^{\infty}(\bar{\Omega})$ is dense in $\mathrm{C}(\bar{\Omega})$, the part of A in $\mathscr{C}$ is densely defined by Lemma 4.6. Thus by [4, Corollary 3.3.11] the restriction of $(T(t))_{t \geqslant 0}$ to $\mathscr{C}$ is a C_0-semigroup.

Since $T(t)$ is positive on $\mathcal{H}$, it is also positive on $\mathscr{C}$. Since $\mathcal{V}$ is compactly embedded into $\mathcal{H}$ by the Sobolev embedding theorems, $T(t)$ is compact on $\mathcal{H}$ for every $t > 0$. Compactness of the semigroup on $\mathscr{C}$ follows by factorization through $\mathcal{H}$. □

Remark 4.8. Typically one identifies the semigroup $(T(t))_{t \geqslant 0}$ on $\mathscr{C}$ of Theorem 4.7 via the isometric isomorphism

$$\mathscr{C} \to \mathrm{C}(\bar{\Omega}), \qquad (u, u|_{\partial\Omega}) \mapsto u$$

with a positive, compact C_0-semigroup on $\mathrm{C}(\bar{\Omega})$ and calls that one the Wentzell–Robin semigroup.

In the proof of the preceding theorem we used the regularity assumption on the coefficients only to ensure that the operator norm of $T(t)$ on $L^{\infty}(\Omega) \oplus L^{\infty}(\partial\Omega)$ is bounded for small t. There seems to be no simple argument that ensures the boundedness in the general case. For example, as we have seen in Example 4.4, we cannot expect the semigroup to be quasi-contractive.

However, the situation is different for the L^p-spaces, $1 < p < \infty$. By direct estimates, Daners [9] proved that under rather general regularity assumptions the Robin semigroup is quasi-L^p-contractive for every $p \in (1, \infty)$. Although a similar proof still works for the Wentzell–Robin semigroup in the product space $L^p(\Omega) \oplus L^p(\partial\Omega)$, as the last result in this article we show how such an estimate can be obtained by reduction to the Robin case, which extends the result in [16].

Proposition 4.9. *There exists a δ_0 that depends only on the coefficients of the differential operator L such that for every $p \in (1, \infty)$ we have*

$$\left\| T(t)u \right\|_p \leqslant \mathrm{e}^{\omega_p t} \|u\|_p$$

for all $t \geqslant 0$ and all $u \in \mathcal{H}$ that are in $L^p(\Omega) \oplus L^p(\partial\Omega)$. Here $\omega_p := \max\{p, p'\}\delta_0$, where p' denotes the dual exponent to p.

Proof. Let $p \in (1, \infty)$. Denote by $\mathcal{B}$ the intersection of $\mathcal{H}$ with the unit ball of $L^p := L^p(\Omega) \oplus L^p(\partial\Omega)$. By Fatou's lemma, $\mathcal{B}$ is closed in $\mathcal{H}$. Moreover, $\mathcal{B}$ is convex. Let $\mathcal{P}$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{B}$.

Since in the special case $L = -\Delta$ and $\beta = 0$ the corresponding semigroup $(S(t))_{t \geqslant 0}$ is quasi-submarkovian by Proposition 4.5, we obtain from the Riesz–Thorin interpolation theorem that there exists $\omega > 0$ such that $(\mathrm{e}^{-\omega t} S(t))_{t \geqslant 0}$ leaves $\mathcal{B}$ invariant. Thus

$$\mathcal{P}\mathcal{V} \subset \mathcal{V} \tag{4.2}$$

by [21, Theorem 2.2] since $\mathcal{V}$ is the form domain of $\mathfrak{a}^{\omega}_{-\Delta,0}$.

Let $(R(t))_{t\geqslant 0}$ denote the Robin semigroup for the form $a_{L,\beta}$. Let B denote the intersection of $L^2(\Omega)$ and the closed unit ball in $L^p(\Omega)$, and let P be the orthogonal projection of $L^2(\Omega)$ onto B. By [9, Theorem 5.1] the semigroup $(\mathrm{e}^{-\omega_p t}R(t))_{t\geqslant 0}$ maps B into itself. Thus

$$a_{L,\beta}^{\omega_p}(u, u - Pu) \geqslant 0 \quad \text{for all } u \in V := H^1(\Omega) \tag{4.3}$$

by [21, Theorem 2.2]. Since we already know that $\mathcal{P}$ maps $\mathcal{V}$ into $\mathcal{V}$, it is easy to see that

$$\mathcal{P}(u, u|_{\partial\Omega}) = \big(Pu, (Pu)|_{\partial\Omega}\big) \quad \text{for all } u \in H^1(\Omega), \tag{4.4}$$

since $\mathcal{P}(u, 0) = (Pu, 0)$ for all $u \in L^2(\Omega)$ and $\mathcal{P}$ is continuous. Now it follows from the definition of $\mathfrak{a}_{L,\beta}^{\omega_p}$ and from (4.3) and (4.4) that

$$\mathfrak{a}_{L,\beta}^{\omega_p}\big((u, u|_{\partial\Omega}), (I - \mathcal{P})(u, u|_{\partial\Omega})\big) \geqslant 0 \quad \text{for all } u \in H^1(\Omega). \tag{4.5}$$

Now theorem [21, Theorem 2.2], (4.2) and (4.5) imply that $\mathcal{B}$ is invariant under the semigroup $(\mathrm{e}^{-\omega_p t}T(t))_{t\geqslant 0}$. This is precisely the statement we wanted to prove. □

References

[1] J.-J. Alibert, J.-P. Raymond, Boundary control of semilinear elliptic equations with discontinuous leading coefficients and unbounded controls, Numer. Funct. Anal. Optim. 18 (3–4) (1997) 235–250.
[2] W. Arendt, A. ter Elst, Gaussian estimates for second order elliptic operators with boundary conditions, J. Operator Theory 38 (1) (1997) 87–130.
[3] W. Arendt, M. Warma, The Laplacian with Robin boundary conditions on arbitrary domains, Potential Anal. 19 (4) (2003) 341–363.
[4] W. Arendt, C. Batty, M. Hieber, F. Neubrander, Vector-Valued Laplace Transforms and Cauchy Problems, Monogr. Math., vol. 96, Birkhäuser Verlag, Basel, 2001.
[5] W. Arendt, G. Metafune, D. Pallara, S. Romanelli, The Laplacian with Wentzell–Robin boundary conditions on spaces of continuous functions, Semigroup Forum 67 (2) (2003) 247–261.
[6] R. Bass, P. Hsu, Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains, Ann. Probab. 19 (2) (1991) 486–508.
[7] T. Cazenave, A. Haraux, An Introduction to Semilinear Evolution Equations, Oxford Lecture Ser. Math. Appl., vol. 13, The Clarendon Press, Oxford University Press, New York, 1998, translated from the 1990 French original by Yvan Martel and revised by the authors.
[8] G. Coclite, A. Favini, G. Goldstein, J. Goldstein, S. Romanelli, Continuous dependence on the boundary conditions for the Wentzell Laplacian, Semigroup Forum 77 (1) (2008) 101–108.
[9] D. Daners, Heat kernel estimates for operators with boundary conditions, Math. Nachr. 217 (2000) 13–41.
[10] D. Daners, Robin boundary value problems on arbitrary domains, Trans. Amer. Math. Soc. 352 (9) (2000) 4207–4236.
[11] D. Daners, Inverse positivity for general Robin problems on Lipschitz domains, Arch. Math. (Basel) 92 (1) (2009) 57–69.
[12] K.-J. Engel, The Laplacian on $C(\bar{\Omega})$ with generalized Wentzell boundary conditions, Arch. Math. (Basel) 81 (5) (2003) 548–558.
[13] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Grad. Texts in Math., vol. 194, Springer-Verlag, New York, 2000, with contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.
[14] L. Evans, R. Gariepy, Measure theory and fine properties of functions, Stud. Adv. Math., CRC Press, Boca Raton, FL, 1992.
[15] A. Favini, G. Goldstein, J. Goldstein, S. Romanelli, The heat equation with generalized Wentzell boundary condition, J. Evol. Equ. 2 (1) (2002) 1–19.
[16] A. Favini, G. Goldstein, J. Goldstein, S. Romanelli, Wentzell boundary conditions in the nonsymmetric case, Math. Model. Nat. Phenom. 3 (7) (2008) 143–147.
[17] M. Fukushima, M. Tomisaki, Reflecting diffusions on Lipschitz domains with cusps — analytic construction and Skorohod representation, Potential Anal. 4 (4) (1995) 377–408.
[18] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics Math., Springer-Verlag, Berlin, 2001.
[19] J. Griepentrog, L. Recke, Linear elliptic boundary value problems with non-smooth data: normal solvability on Sobolev–Campanato spaces, Math. Nachr. 225 (2001) 39–74.
[20] P. Grisvard, Elliptic problems in nonsmooth domains, Monogr. Stud. Math., vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.
[21] E. Ouhabaz, Analysis of Heat Equations on Domains, London Math. Soc. Monogr. Ser., vol. 31, Princeton University Press, 2005.

[22] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, 2nd edition, Johann Ambrosius Barth, Heidelberg, 1995.
[23] G. Troianiello, Elliptic Differential Equations and Obstacle Problems, Univ. Ser. Math., Plenum Press, New York, 1987.
[24] F. Tröltzsch, Optimale Steuerung partieller Differentialgleichungen, Theorie, Verfahren und Anwendungen, Vieweg, Wiesbaden, 2005.
[25] M. Warma, The Robin and Wentzell–Robin Laplacians on Lipschitz domains, Semigroup Forum 73 (1) (2006) 10–30.
[26] W. Ziemer, Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation, Grad. Texts in Math., vol. 120, Springer-Verlag, New York, 1989.